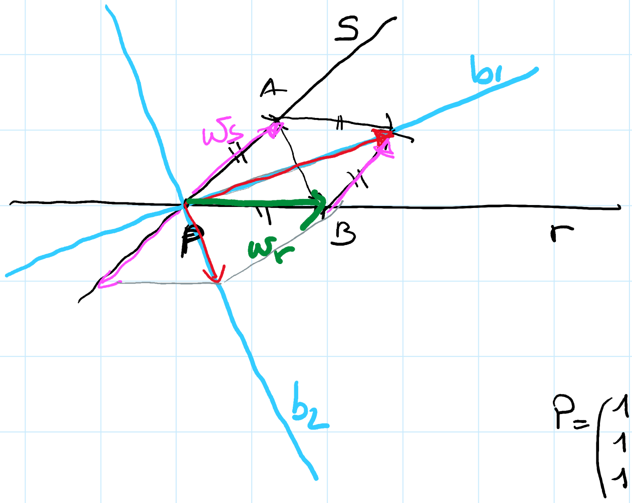
Bisettrici fra due rette incidenti

$r: \begin{cases} z=1 \\ x+y=2 \end{cases}$ $\quad\quad s: \begin{cases} z=1 \\ x-y=0 \end{cases}$

Verificare che $r \cap s = \{P\}$, determinare le coordinate di P e determinare le bisettrici b_1 e b_2



$r \cap s: \begin{cases} z=1 \\ x+y=2 \\ x-y=0 \end{cases}$

$\begin{cases} z=1 \\ y=x \\ 2x=2 \end{cases}$ $\quad \begin{cases} z=1 \\ y=1 \\ x=1 \end{cases}$

$P=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$b_1: P + \langle w_r + w_s \rangle$ $\quad w_r \in V_r$ $\quad \|w_r\| = \|w_s\|$

$b_2: P + \langle w_r - w_s \rangle$ $\quad w_s \in V_s$

$V_r: \begin{cases} z=0 \\ x+y=0 \end{cases}$ $\quad \begin{cases} z=0 \\ y=-x \end{cases}$ $\quad \begin{pmatrix} x \\ -x \\ 0 \end{pmatrix}$ $\quad V_r = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$ $\quad v_r = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$V_s: \begin{cases} z=0 \\ x-y=0 \end{cases}$ $\quad \begin{cases} z=0 \\ x=y \end{cases}$ $\quad \begin{pmatrix} y \\ y \\ 0 \end{pmatrix}$ $\quad V_s = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$ $\quad v_s = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

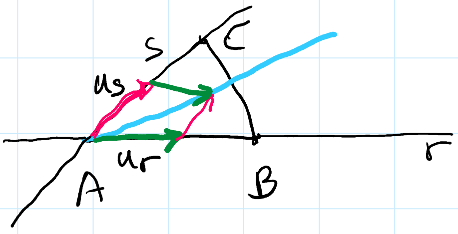
$\|v_r\| = \sqrt{2}$ $\quad \|v_s\| = \sqrt{2}$ $\quad$ posso prendere come $w_r = v_r$
come $w_s = v_s$

$b_1: \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \langle w_r + w_s \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ $\quad V_{b_1} \perp V_{b_2}$

$b_2: \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \langle w_r - w_s \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \right\rangle$

Esempio: dati i punti

A, B, C determinare la bisettrice dell'angolo $B\hat{A}C$



$r: A + \langle B-A \rangle$

$s: A + \langle C-A \rangle$

$r \cap s = A$

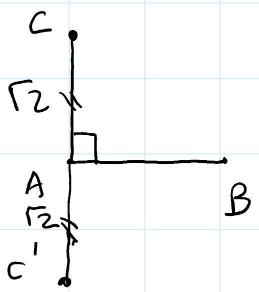
$$\boxed{\begin{array}{l} b_1: A + \left\langle \underbrace{\frac{B-A}{\|B-A\|}}_{u_r} \pm \underbrace{\frac{C-A}{\|C-A\|}}_{u_s} \right\rangle \\ b_2 \end{array}}$$

Esercizio: consideriamo i punti $A = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$.

Determinare C tale che il triangolo A, B, C sia contenuto nel piano $\pi: x - z = 0$, che $C\hat{A}B = \frac{\pi}{2}$ e $\|C-A\| = \sqrt{2}$.

Soluzione:

$A \in \pi$, $B \in \pi$ $\quad \pi: x = z \quad C = \begin{pmatrix} a \\ b \\ a \end{pmatrix} \leftarrow C \in \pi$



$$C - A = \begin{pmatrix} a \\ b \\ a \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a-1 \\ b-2 \\ a-1 \end{pmatrix}$$

$$B - A = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$(C-A) \cdot (B-A) = 0 \quad \begin{pmatrix} a-1 \\ b-2 \\ a-1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 0 \quad 2(b-2) = 0 \Rightarrow \boxed{b = 2}$

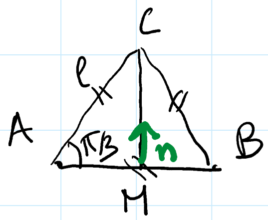
$C = \begin{pmatrix} a \\ 2 \\ a \end{pmatrix}$ imponiamo $\|C-A\| = \sqrt{2}$

$C - A = \begin{pmatrix} a-1 \\ 0 \\ a-1 \end{pmatrix} \quad \|C-A\| = \left\| \begin{pmatrix} a-1 \\ 0 \\ a-1 \end{pmatrix} \right\| =$

$= \left\| (a-1) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\| = |a-1| \sqrt{2} = \sqrt{2} \qquad \|a v\| = |a| \, \|v\|$

$|a-1|=1 \begin{cases} a-1=1 & a=2 & c=\begin{pmatrix}2\\2\\2\end{pmatrix} \\ a-1=-1 & a=0 & c'=\begin{pmatrix}0\\2\\0\end{pmatrix}\end{cases}$

Esempio:



$A, B \in \pi$

determinare $C \in \pi$ tale che il triangolo A, B, C sia equilatero.

$\ell = \|B-A\|$

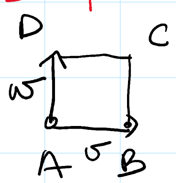
$C = A + (M-A) + (C-M)$

$M = \dfrac{A+B}{2}$

$\|C-M\| = \dfrac{\sqrt{3}}{2}\ell$

$\vec{n}$ è versore $\perp$ $(B-A)$ nel piano π $\qquad C-M = \dfrac{\sqrt{3}}{2}\ell\,\vec{n}$

Esempio: costruire i vertici di un quadrato $\subseteq \pi$ e avente come vertici A e B assegnati.



$C = B + w$ con $w \in V_\pi$ $w \perp v$ con

$D = A + w$ $\quad v = B - A$ e $\|v\| = \|w\|$.

Prodotto vettoriale e prodotto misto in $\mathbb{R}^3$

Prodotto vettoriale

$\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(v, w) \longrightarrow v \times w$

$v, w \in \mathbb{R}^3$ $\quad v \times w$ prodotto vettoriale, prodotto esterno

$v \wedge w$

$v \times w$ è un vettore che ha $\|v \times w\| =$ l'area del parallel. formato dai vettori v e w

Se $\|v \times w\| \neq 0$ la direzione di $v \times w$
è $\perp$ a v $\perp$ a w; ha come verso
$\det(v \;\; w \;\; v \times w) > 0$ cioè terna dx

Def: $v = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ $\quad w = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$$v \times w = \det \begin{pmatrix} x_1 & x_2 & e_1 \\ y_1 & y_2 & e_2 \\ z_1 & z_2 & e_3 \end{pmatrix} = \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$= e_1(y_1 z_2 - z_1 y_2) - e_2(x_1 z_2 - z_1 x_2) + e_3(x_1 y_2 - y_1 x_2) =$$

$$= \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{pmatrix} \qquad a e_1 + b e_2 + c e_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Esempio: calcolare $v \times w$ con $v = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ $\quad w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$v \times w = \det \begin{pmatrix} 1 & 1 & e_1 \\ 3 & 1 & e_2 \\ 0 & 1 & e_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

Proprietà del prodotto vettoriale:

1) $w \times v = - v \times w$

$\det(w \;\; v \;\; \mathcal{E}) = - \det(v \;\; w \;\; \mathcal{E})$

2) $(a_1 v_1 + a_2 v_2) \times w = a_1 v_1 \times w + a_2 v_2 \times w$

$v \times (a_1 w_1 + a_2 w_2) = a_1 \, v \times w_1 + a_2 \, v \times w_2$

3) $$\boxed{(v_1 \times v_2)\cdot v_3 = \det(v_1 \ v_2 \ v_3)}$$

Prodotto misto di v_1, v_2, v_3 $\quad (v_1 \times v_2)\cdot v_3$

$$v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \qquad v_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

$$(v_1 \times v_2)\cdot v_3 = \det\begin{pmatrix} x_1 & x_2 & e_1 \\ y_1 & y_2 & e_2 \\ z_1 & z_2 & e_3 \end{pmatrix} \cdot \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} =$$

$$= \det\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

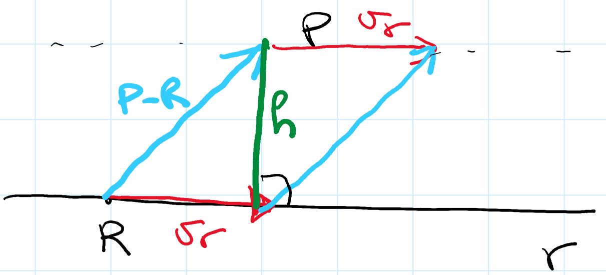
$$(v_1 \times v_2)\cdot v_1 = \det(v_1 \ v_2 \ v_1) = 0 \qquad v_1 \times v_2 \perp v_1$$

$$(v_1 \times v_2)\cdot v_2 = \det(v_1 \ v_2 \ v_2) = 0 \qquad v_1 \times v_2 \perp v_2$$

$\|v_1 \times v_2\| = 0 \iff$ i vettori v_1 e v_2 sono lin. dipendenti

$rg \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{pmatrix} < 2 \iff$ Tutti i minori 2×2 hanno det. 0.

Uso del prodotto vettoriale e del prodotto misto nel calcolo di distanze.



Distanza P, r

$r: R + \langle v_r \rangle$

$$\boxed{d(P,r) = \frac{\text{Area}\langle v_r,\ P-R\rangle}{\|v_r\|}} = \frac{\|v_r \times (P-R)\|}{\|v_r\|}$$

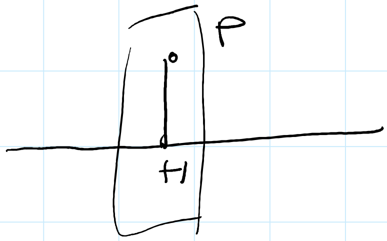
Es:

$$P = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad r: \overset{R}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

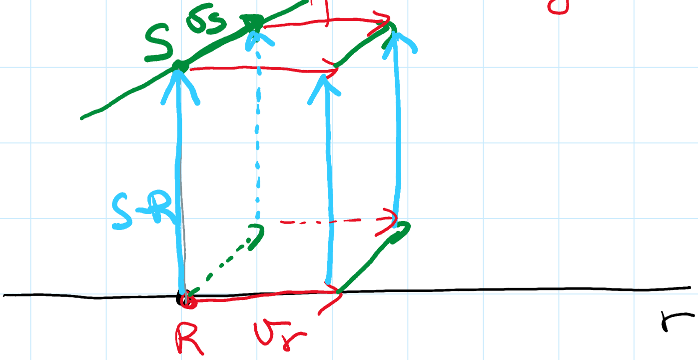
$$d(P,r) = \frac{\text{Area}\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \overset{P-R}{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}} \right\rangle}{\sqrt{3}} = \frac{\left\| \det\begin{pmatrix} 1 & 0 & e_1 \\ 1 & 2 & e_2 \\ 1 & 0 & e_3 \end{pmatrix} \right\|}{\sqrt{3}} =$$

$$\sqrt{3} = \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|$$

$$= \frac{\left\| \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \right\|}{\sqrt{3}} = \frac{\left\| 2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\|}{\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{3}}$$



Distanza fra rette sghembe



$r: R + \langle v_r \rangle$

$s: S + \langle v_s \rangle$

$$d(r,s) = \frac{\text{Volume P.}\langle v_r\ v_s\ S-R\rangle}{\text{Area par.}\langle v_r, v_s\rangle} =$$

$$\boxed{= \frac{|\det(v_r\ v_s\ S-R)|}{\|v_r \times v_s\|} = d(r,s)}$$

Es:

$$r: \overset{R}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

$$s: \overset{S}{\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}} + \left\langle \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\rangle$$

sono sghembe $\dim\langle v_r\ v_s\ S-R\rangle = 3$

$$S-R = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$$

$$d(r,s) = \frac{\dfrac{\left|\det\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 5 \end{pmatrix}\right|}{\left\|\det\begin{pmatrix} 0 & 0 & e_1 \\ 1 & 0 & e_2 \\ -1 & -1 & e_3 \end{pmatrix}\right\|}}{} = \frac{1}{\left\|\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\right\|} = 1$$

Esercizio: Se $W = \langle v_1, v_2 \rangle$ in $\mathbb{R}^3$ $\dim W = 2$ allora $\dim W^\perp = 1$ $\quad W^\perp = \langle v_1 \times v_2 \rangle$

$$W = \langle \underset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}} \rangle \qquad W^\perp = \langle v_1 \times v_2 \rangle$$

$$\det\begin{pmatrix} 1 & 2 & e_1 \\ 0 & 1 & e_2 \\ 1 & 0 & e_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \qquad W^\perp = \langle \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \rangle$$

$$W^\perp \begin{cases} x+z=0 \\ 2x+y=0 \end{cases} \quad \begin{cases} z=-x \\ y=-2x \end{cases} \quad \begin{pmatrix} x \\ -2x \\ -x \end{pmatrix} \quad \langle \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \rangle$$

W ha eq. cart. $-x+2y+z=0$

Se

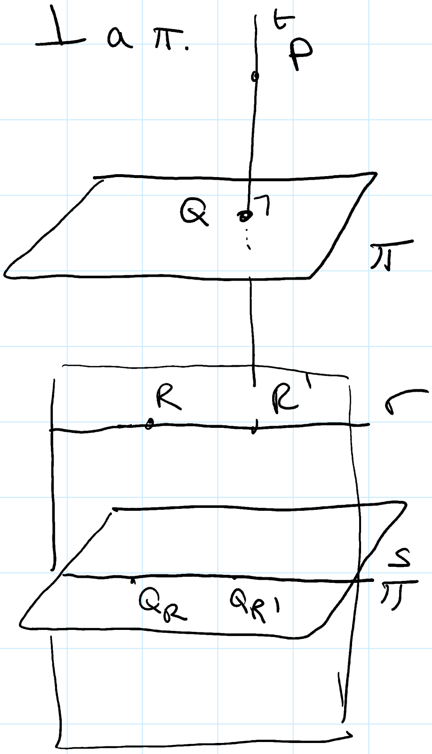
$$W: \begin{cases} 2x-y+z=0 \\ x+y=0 \end{cases} \qquad \dim W = 1$$

$$W^\perp = \langle \underset{v_1}{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \rangle$$

determinare l'eq. cartesiana di $W^\perp$ $\quad ax+by+cz=0$ $\quad \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ vettore ortogonale a v_1 e a v_2 $\quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = v_1 \times v_2$.

Definizione:

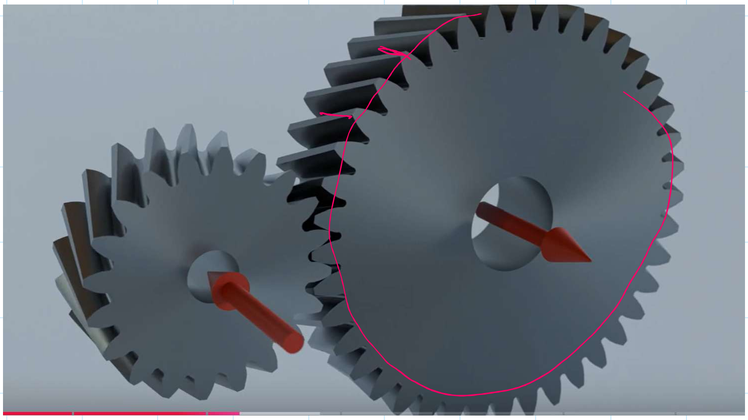
La proiezione ortogonale del punto P in $\mathbb{R}^3$ su un piano π è l'intersezione fra il piano π e la retta t passante per P e $\perp$ a π.

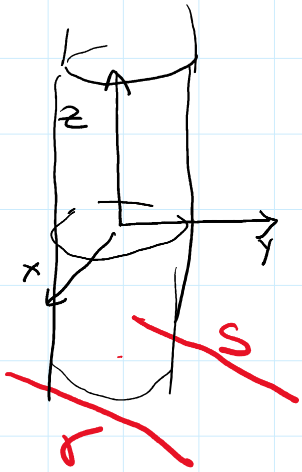


Q è la pr. ort. di P su π.

$$\begin{cases} \pi: ax+by+cz=d \\ t = \begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} + \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle \end{cases} \qquad P = \begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix}$$

Ruote dentate con assi paralleli



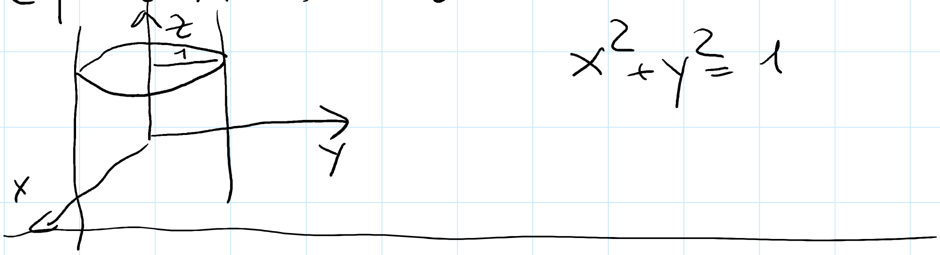


$x^2+y^2=1$

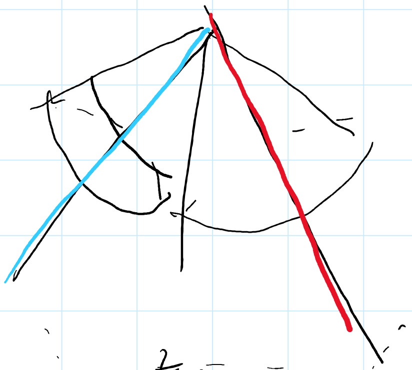
Se $r // s$ possiamo usare dei cilindri

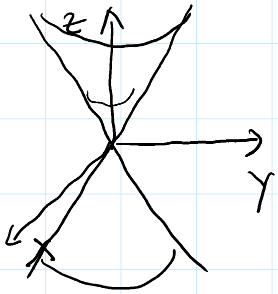
L'equazione del cilindro di asse di rotazione $\begin{cases} x=0 \\ y=0 \end{cases}$

$$x^2+y^2=1$$



Ruote dentate con assi concorrenti



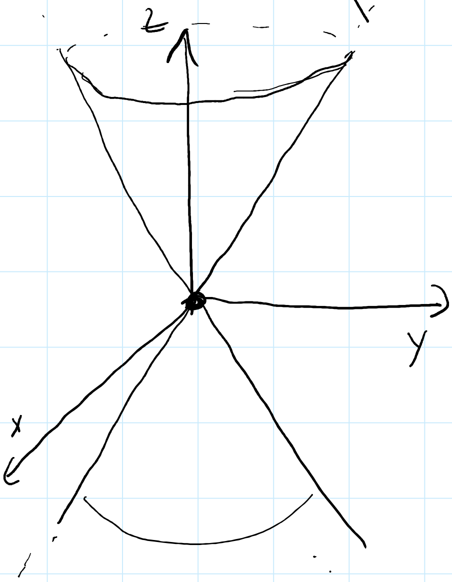


$$z^2=x^2+y^2$$

$$\begin{cases} y=0 \\ z^2=x^2 \end{cases}$$

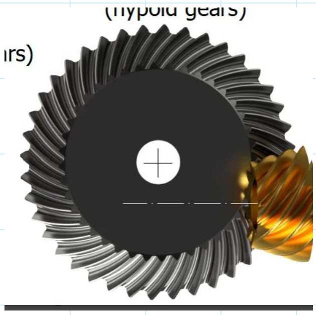
Equazione di un cono:

$$z^2=x^2+y^2$$

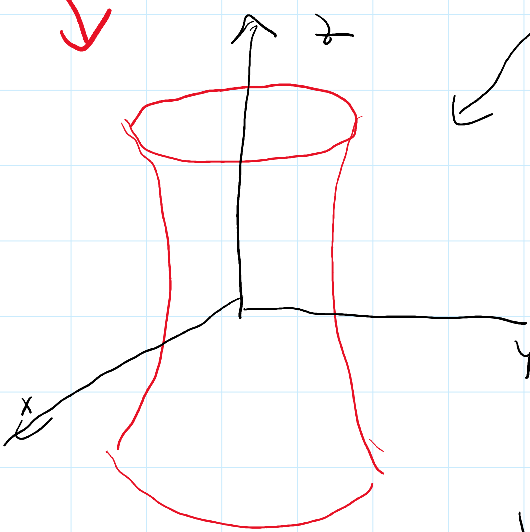


Ruote dentate con sezioni coniche

Ruote dentati con assi sghembi:



Iperboloide iperbolico



è una sup. rigata

$e: \; z^2 = x^2 + y^2 - 1$

quindi

$$z^2 - y^2 = x^2 - 1$$

$$t(z^2 - y^2) = t(x^2 - 1)$$

$$\boxed{t(z-y)}(z+y) = t\boxed{(x-1)}(x+1)$$

$$\begin{cases} t(z-y) = x-1 \\ z+y = t(x+1) \end{cases}$$ I° schiera di rette

$$\begin{cases} t(z-y) = x+1 \\ z+y = t(x-1) \end{cases}$$ II° schiera di rette