Distanze e Punti di minima distanza fra sottovarietà lineari in $\mathbb{R}^3$.

* **Punto-Punto:** P, Q $\boxed{d(P,Q) = \|Q-P\|}$

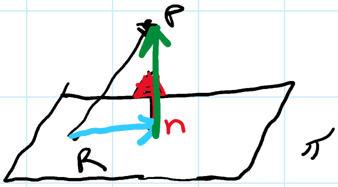
Coppia di punti di minima distanza $\boxed{(P,Q)}$

* **Punto-Piano:**

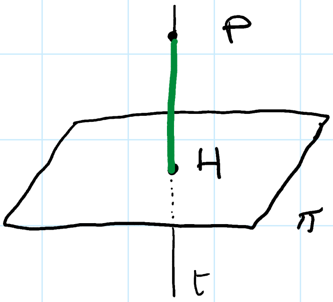
$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ $\pi: ax+by+cz=d$ $rg(a\ b\ c)=1$

$ax+by+cz-d=0$

$$\boxed{d(P,\pi) = \frac{|ap_1+bp_2+cp_3-d|}{\sqrt{a^2+b^2+c^2}}}$$



$n = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

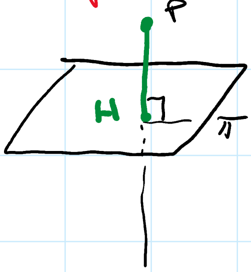


Cmd (P, H) con

$\{H\} = t \cap \pi$ con t retta $\perp$ a π passante per P.

$d(P,\pi) = d(P,H)$

Esempio: $\pi: x+y+z=6$ $P = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$



$$\boxed{d(P,\pi) =} \frac{|1+0+0-6|}{\sqrt{1^2+1^2+1^2}} = \boxed{\frac{5}{\sqrt{3}}}$$

Calcoliamo t retta $\perp$ a π e passante per P, cioè

$V_t \perp V_\pi$ cioè $V_t = V_\pi^\perp$ $\quad V_\pi : x+y+z=0$

$V_t = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$ $\quad t: P+V_t = \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle$

$\{H\} = t \cap \pi$ $\quad \begin{cases} x+y+z=6 \\ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + a\begin{pmatrix}1\\1\\1\end{pmatrix} \end{cases}$ $\quad \begin{cases} 1+a+a+a=6 \\ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1+a\\a\\a\end{pmatrix} = H \end{cases}$ $\quad 3a=5$, $a=\frac{5}{3}$

$H = \begin{pmatrix}\frac{8}{3}\\\frac{5}{3}\\\frac{5}{3}\end{pmatrix}$ $\quad d(P,\pi) = d(P,H) = \|H-P\| = \left\| \begin{pmatrix}\frac{8}{3}\\\frac{5}{3}\\\frac{5}{3}\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} \right\| = \left\| \begin{pmatrix}\frac{5}{3}\\\frac{5}{3}\\\frac{5}{3}\end{pmatrix} \right\|$

$= \left\| \frac{5}{3} \begin{pmatrix}1\\1\\1\end{pmatrix} \right\| = \frac{5}{3}\sqrt{3}$ $\qquad \|av\| = |a|\,\|v\|$

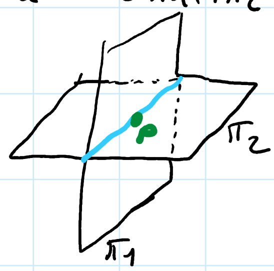
* Piano - Piano: π_1 e π_2

- Se $\pi_1 \cap \pi_2 \neq \emptyset$ $\boxed{d(\pi_1,\pi_2)=0}$

  Una c.m.d. (P,P)

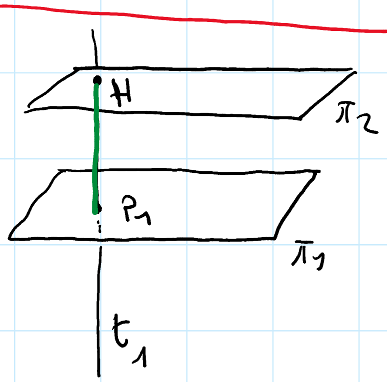
  Tutte le c.m.d $(P+v, P+v)$ con $v \in V_{\pi_1} \cap V_{\pi_2}$

Sia $P \in \pi_1 \cap \pi_2$



- Se $\pi_1 \cap \pi_2 = \emptyset$ allora $\pi_1 // \pi_2$ cioè $V_{\pi_1} = V_{\pi_2}$

  $\boxed{d(\pi_1,\pi_2) = d(P_1,\pi_2) \quad \text{con } P_1 \in \pi_1}$



Una cmd (P_1, H) con $P_1 \in \pi_1$

$\{H\} = t_1 \cap \pi_2$ con t_1 retta $\perp$ a π_2 e passante per P_1

Tutte le c.m.d sono $\boxed{(P_1+v,\ H+v) \quad \text{con} \quad v \in V_{\pi_1} \cap V_{\pi_2} = V_{\pi_1}}$

Esempio: $\pi_1 : x+y+z=6$ $\qquad d(\pi_2, \pi_1) = d(P, \pi_1)$

$\pi_2 : x+y+z=1$ $\qquad P \in \pi_2 \qquad P = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$d(\pi_2, \pi_1) = \frac{|1+0+0-6|}{\sqrt{3}} = \frac{5}{\sqrt{3}}$$

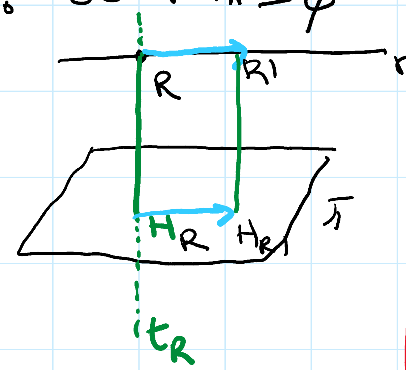
$H = t \cap \pi_1$ con $t \perp V_{\pi_1}$ passante per P

$H = \begin{pmatrix} \frac{8}{3} \\ \frac{5}{3} \\ \frac{5}{3} \end{pmatrix}$ $\qquad$ c.m.d tra π_1, π_2 $\quad (H, P)$ $\qquad$ tutte c.m.d

$(H+v, P+v)$ con $v \in V_\pi : x+y+z=0$

* Retta - Piano $\quad r, \pi$

- $\boxed{\text{se } r \cap \pi \neq \emptyset \quad d(r,\pi)=0 \quad \text{c.m.d} \quad P \in r \cap \pi \quad (P,P)}$
- se $r \cap \pi = \emptyset \implies r // \pi$ disgiunti

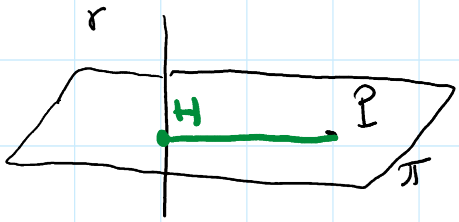


$d(r,\pi) = d(R,\pi)$ con $R \in r$

cmd (R, H_R) $\quad R \in r$ e

$\{H_R\} = t_R \cap \pi$ con t_R retta $\perp \pi$ e passante per R.

* Punto - Retta. $\quad P, r$

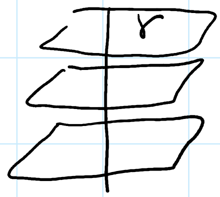


Cmd (P, H) con $\{H\} = r \cap \pi$ con π piano $\perp$ ad r passante per P.

$d(P, r) = d(P, H)$.

Esempio: $P=\begin{pmatrix}1\\0\\2\end{pmatrix}$ $\quad r: \begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}0\\-1\\3\end{pmatrix} \right\rangle$

Calcoliamo il piano $\pi \perp r$ e passante per P



$\pi_k: 0\cdot x - 1\cdot y + 3z = K$

$\pi_k: -y+3z=K \qquad P\in\pi_k \quad P=\begin{pmatrix}1\\0\\2\end{pmatrix}$

$-0+3\cdot 2 = K \qquad K=6$

$\pi \begin{cases} -y+3z=6 \\ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + a\begin{pmatrix}0\\-1\\3\end{pmatrix}\end{cases} \qquad \begin{cases} -1+a+3+9a=6 \\ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1-a\\1+3a\end{pmatrix}\end{cases} \qquad 10a=4 \quad a=\frac{4}{10}=\frac{2}{5}$

$H=\begin{pmatrix}1\\ \frac{3}{5} \\ 1+\frac{6}{5}\end{pmatrix} = \begin{pmatrix}1\\ \frac{3}{5}\\ \frac{11}{5}\end{pmatrix}$

$d(P,r)=d(P,H)=\|H-P\|=$

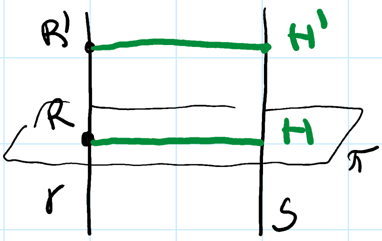
$=\left\|\begin{pmatrix}1\\ \frac{3}{5}\\ \frac{11}{5}\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix}\right\| = \left\|\begin{pmatrix}0\\ 3/5\\ 1/5\end{pmatrix}\right\| =$

$= \left\|\frac{1}{5}\begin{pmatrix}0\\3\\1\end{pmatrix}\right\| = \frac{1}{5}\sqrt{9+1} = \frac{\sqrt{10}}{5}$

* **Retta - Retta** $\quad r,s$

- $\boxed{\text{Se } r\cap s\neq\emptyset \quad d(r,s)=0 \quad P\in r\cap s \quad \text{cmd } (P,P)}$

Rette parallele: $r /\!/ s \qquad r\cap s=\emptyset$



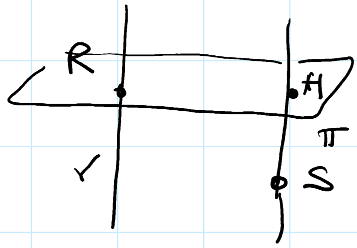
$d(r,s)=d(R,s)=d(R,H)$ con

$\{H\}=s\cap\pi$ con π piano $\perp s$ e passante per R

Tutte cmd $\boxed{(R+v, H+v) \text{ con } v\in V_r\cap V_s = V_r = V_s}$ perché $r/\!/s$.

Esempio:

$$r: \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \overset{v_r}{\begin{pmatrix}0\\1\\2\end{pmatrix}} \right\rangle \qquad s: \overset{S}{\begin{pmatrix}2\\-1\\-2\end{pmatrix}} + \left\langle \overset{v_s}{\begin{pmatrix}0\\1\\2\end{pmatrix}} \right\rangle$$



$$d(r,s) = d(R,s) \qquad R = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$= d(R,H) \qquad \{H\} = s \cap \pi$$

con $\pi \perp s$ e passante per R

$\pi_K: y + 2z = K$ passante per $R = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\qquad \pi: y+2z=0$

$$0 + 2\cdot 0 = K \quad \Rightarrow K = 0$$

$$H \begin{cases} y + 2z = 0 \\ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\-1\\-2\end{pmatrix} + a\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}2\\-1+a\\-2+2a\end{pmatrix} \end{cases}$$

$$-1+a-4+4a=0 \qquad 5a=5 \quad a=1$$

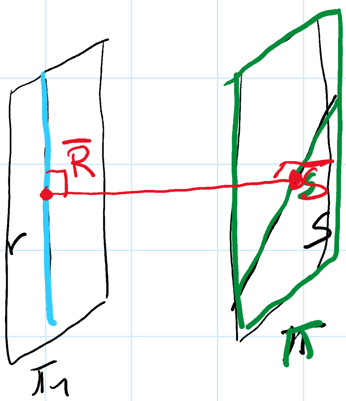
$$H = \begin{pmatrix}2\\0\\0\end{pmatrix}$$

$$d(r,s) = d(R,H) = \|H-R\| = \left\| \begin{pmatrix}2\\0\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} \right\| = \left\| \begin{pmatrix}1\\0\\0\end{pmatrix} \right\| = 1$$

Rette sghembe.

$r: R + \langle v_r \rangle$ $\qquad$ sono sghembe $\Longleftrightarrow \dim \langle S-R, v_r, v_s \rangle = 3$

$s: S + \langle v_s \rangle$



$\boxed{d(r,s) = d(r,\pi) \text{ con } \pi \text{ piano contenente } s \text{ e parallelo ad } r}$

La cmd $(\bar{R}, \bar{S})$ con $\bar{R} \in r$ e $\bar{S} \in s$

tale che $(\bar{S} - \bar{R}) \perp r$

$(\bar{S} - \bar{R}) \perp s$

$$R_a = R + a v_r$$
$$S_b = S + b v_s$$

$$S_b - R_a = S + b v_s - R - a v_r$$

$$\begin{cases} (S_b - R_a)\cdot v_r = 0 \\ (S_b - R_a)\cdot v_s = 0 \end{cases}$$ ha sempre un'unica soluzione $(\bar a, \bar b) \Rightarrow$ le cmd $(R_{\bar a}, S_{\bar b})$

Esempio: $r: \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rangle \qquad s: \begin{pmatrix}0\\1\\5\end{pmatrix} + \left\langle \begin{pmatrix}0\\0\\-1\end{pmatrix} \right\rangle$

Verificare che r ed s sono sghembe calcolare $d(r,s)$ e cmd.

Svolg:

r ed s sono sghembe $\iff \dim\langle S-R, v_r, v_s\rangle = 3$

$$S-R = \begin{pmatrix}0\\1\\5\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\5\end{pmatrix} \qquad v_r = \begin{pmatrix}0\\1\\-1\end{pmatrix} \qquad v_s = \begin{pmatrix}0\\0\\-1\end{pmatrix}$$

$$\det\begin{pmatrix}-1 & 0 & 0\\ 1 & 1 & 0\\ 5 & -1 & -1\end{pmatrix} = 1 \neq 0 \quad \Rightarrow \quad \text{r ed s sono sghembe.}$$

$d(r,s) = d(R,\pi) \qquad R = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \pi$ piano contenente s e $/\!/ r$

$$\pi: \begin{pmatrix}0\\1\\5\end{pmatrix} + \left\langle \begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rangle \qquad \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\1\\5\end{pmatrix} + \begin{pmatrix}0\\0\\-a\end{pmatrix} + \begin{pmatrix}0\\b\\-b\end{pmatrix} = \begin{pmatrix}0\\1+b\\5-a-b\end{pmatrix}$$

$$\pi: x = 0 \qquad R = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{cases} x = 0 \\ y = 1+b \\ z = 5-a-b \end{cases} \qquad d(R,\pi) = \frac{|1|}{\sqrt{1}} = 1$$

C.m.d. $\quad r: \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rangle \qquad s: \begin{pmatrix}0\\1\\5\end{pmatrix} + \left\langle \begin{pmatrix}0\\0\\-1\end{pmatrix} \right\rangle$

$$R_a = \begin{pmatrix}1\\0\\0\end{pmatrix} + a\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\a\\-a\end{pmatrix}$$

$$S_b = \begin{pmatrix}0\\1\\5\end{pmatrix} + b\begin{pmatrix}0\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\5-b\end{pmatrix}$$

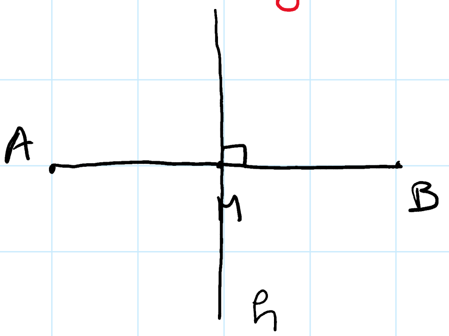
$$S_b - R_a = \begin{pmatrix}0\\1\\5-b\end{pmatrix} - \begin{pmatrix}1\\a\\-a\end{pmatrix} = \begin{pmatrix}-1\\1-a\\5-b+a\end{pmatrix}$$

$$\begin{cases}(S_b - R_a)\cdot v_r = 0\\ (S_b - R_a)\cdot v_s = 0\end{cases} \qquad \begin{cases}\begin{pmatrix}-1\\ 1-a\\ 5-b+a\end{pmatrix}\cdot\begin{pmatrix}0\\ 1\\ -1\end{pmatrix} = 0\\[2ex] \begin{pmatrix}-1\\ 1-a\\ 5-b+a\end{pmatrix}\cdot\begin{pmatrix}0\\ 0\\ -1\end{pmatrix} = 0\end{cases} \qquad \begin{cases}1-a\,\boxed{-5+b-a}=0\\[2ex] \boxed{-5+b-a}=0\end{cases}$$

$$\begin{cases}a=1\\ -5+b-1=0\end{cases} \qquad \begin{cases}a=1\\ b=6\end{cases} \qquad \text{cmd}\,(R_1, S_6) = \Big(\underbrace{\begin{pmatrix}1\\ 1\\ -1\end{pmatrix}}_{R_1}, \underbrace{\begin{pmatrix}0\\ 1\\ -1\end{pmatrix}}_{S_6}\Big)$$

$$d(r,s) = d(R_1, S_6) = \|S_6 - R_1\| = \left\|\begin{pmatrix}0\\ 1\\ -1\end{pmatrix} - \begin{pmatrix}1\\ 1\\ -1\end{pmatrix}\right\| = \left\|\begin{pmatrix}-1\\ 0\\ 0\end{pmatrix}\right\| = 1.$$

Asse di un segmento in $\mathbb{R}^2$?



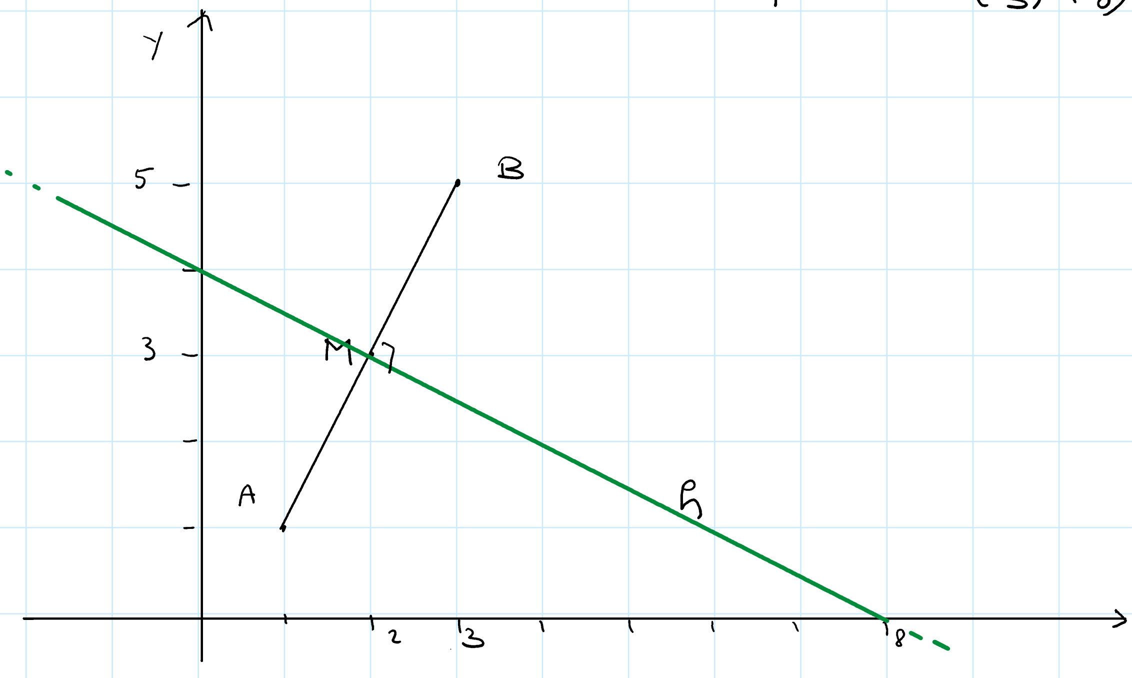
$$h: \frac{A+B}{2} + \langle B-A\rangle^{\perp}$$

Esempio: $A = \begin{pmatrix}1\\ 1\end{pmatrix}$ $\quad B = \begin{pmatrix}3\\ 5\end{pmatrix}$ $\qquad \left\langle \begin{pmatrix}2\\ 4\end{pmatrix}\right\rangle^{\perp} = \left\langle \begin{pmatrix}1\\ 2\end{pmatrix}\right\rangle^{\perp}$

$$h: \overset{M}{\begin{pmatrix}2\\ 3\end{pmatrix}} + \left\langle \begin{pmatrix}2\\ 4\end{pmatrix}\right\rangle^{\perp}$$

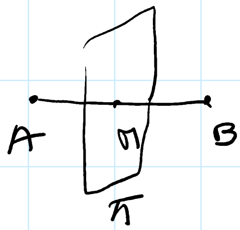
$x + 2y = 2 + 2\cdot 3 = 8$

passa per $\begin{pmatrix}2\\ 3\end{pmatrix}$ $\begin{pmatrix}8\\ 0\end{pmatrix}$



Asse di A, B in $\mathbb{R}^3$

$$\pi \quad \frac{A+B}{2} + \langle B-A \rangle^{\perp}$$



$$A = \begin{pmatrix}1\\0\\2\end{pmatrix} \qquad B = \begin{pmatrix}3\\1\\4\end{pmatrix}$$

$$\pi : \underbrace{\begin{pmatrix}2\\1/2\\3\end{pmatrix}}_{M} + \langle \begin{pmatrix}2\\1\\2\end{pmatrix} \rangle^{\perp}$$

$$2x + y + 2z = 2 \cdot 2 + \frac{1}{2} + 2 \cdot 3 =$$

$$= 4 + \frac{1}{2} + 6 = 10 + \frac{1}{2} = \frac{21}{2}$$

Come si calcolano le bisettrici di due rette incidenti in $\mathbb{R}^3$?