Esercizio:

$r: \begin{cases} x = 1 \\ y + z = 0 \end{cases}$ $\quad s: \begin{cases} x + 2y = 2 \\ y - 5z = 5 \end{cases}$

$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad P = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}$

1) Verificare che $O \notin r$ e $O \notin s$.
Verificare che $P \notin r$ e $P \notin s$.
Dimostrare che r ed s sono sghembe.

2) Determinare, se esiste, una retta t passante per P ed incidente r ed s.

3) Determinare, se esiste, una retta ℓ passante per O ed incidente r ed s.

Soluzione:

1) $r: \begin{cases} x = 1 \\ y + z = 0 \end{cases}$ $\quad O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{cases} 0 = 1 \\ 0 + 0 = 0 \end{cases}$ impossibile $\Rightarrow O \notin r$

$s: \begin{cases} x + 2y = 2 \\ y - 5z = 5 \end{cases}$ $\quad O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \begin{cases} 0 + 0 = 2 \\ 0 - 0 = 5 \end{cases}$ impossibile $\Rightarrow O \notin s$

$r \begin{cases} x = 1 \\ y + z = 0 \end{cases}$ $\quad P = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}$ $\quad \begin{cases} 2 = 1 \\ 1 - 6 = 0 \end{cases}$ imp. $\Rightarrow P \notin r$

$s: \begin{cases} x + 2y = 2 \\ y - 5z = 5 \end{cases}$ $\quad P = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}$ $\quad \begin{cases} \underline{2 + 2 = 2} \\ 1 + 30 = 5 \end{cases}$ imp. $\Rightarrow P \notin s$

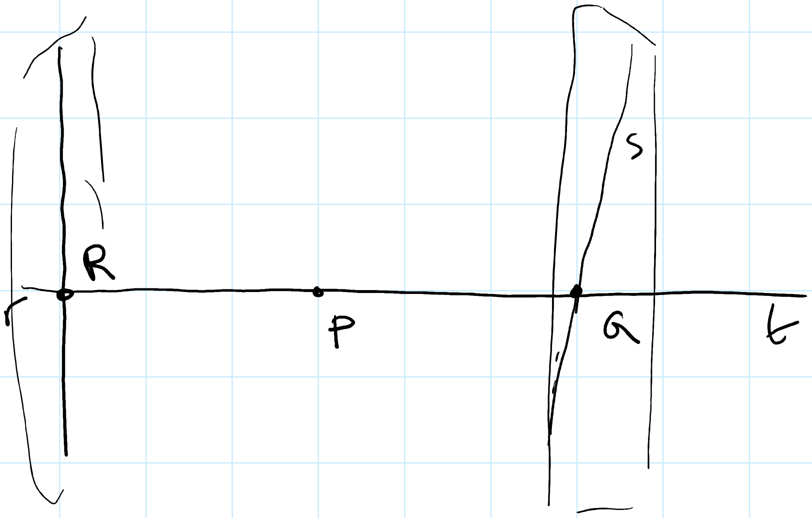
$r \cap s$ $\begin{cases} x = 1 \\ y+z=0 \\ x+2y=2 \\ y-5z=5 \end{cases}$ $A|b = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & 5 \end{array}\right)$ $3^\circ - 1^\circ$

$rg(A|b) = 4$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & -5 & 5 \end{array}\right)$ $\begin{array}{l} 3^\circ - 2 \cdot 2^\circ \\ 4^\circ - 2^\circ \end{array}$ $rg\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -6 & 5 \end{array}\right) = 4$

perché ha determinante non nullo

$rg(A) = 3 \neq rg(A|b) = 4 \Rightarrow$ r ed s sono sghembe.

2)



dato $P \notin r$ $\dim(P \vee r) = \pi_1$ t incidente r e passante per P

$\Rightarrow t \subseteq \pi_1$ $\dim(P \vee s) = \pi_2$ t incidente s e passante per P

$\Rightarrow t \subseteq \pi_2$

π_1 contenente r e passante per P: $\binom{\alpha}{\beta} \in \mathbb{R}^2$

$r: \begin{cases} x-1=0 \\ y+z=0 \end{cases}$ $\pi_{\alpha,\beta}: \alpha(x-1) + \beta(y+z) = 0$ con $\binom{\alpha}{\beta} \neq \binom{0}{0}$

cerchiamo $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ tale che $\pi_{\alpha\beta}$ passi per $P = \boxed{\begin{pmatrix}2\\1\\-6\end{pmatrix}}$

$\alpha(2-1)+\beta(1-6)=0 \qquad \alpha-5\beta=0 \qquad \alpha=5\beta$

$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}5\\1\end{pmatrix} \qquad \pi_1=\pi_{5,1}: \ 5(x-1)+y+z=0$

$\pi_1: 5x+y+z=5$

Cerchiamo π_2 contenente s e passante per P

$\begin{cases} x+2y-2=0 \\ y-5z-5=0 \end{cases} \qquad \alpha(x+2y-2)+\beta(y-5z-5)=0 \qquad P=\begin{pmatrix}2\\1\\-6\end{pmatrix}$

$\alpha(\cancel{2}+2-\cancel{2})+\beta(1+30-5)=0$

$2\alpha+26\beta=0$

$\alpha+13\beta=0 \qquad \alpha=-13\beta \qquad \begin{matrix}\alpha=-13\\ \beta=1\end{matrix}$

$\pi_2: \ -13(x+2y-2)+y-5z-5=0$

$-13x-26y+26+y-5z-5=0$

$\pi_2: -13x-25y-5z=-21 \qquad \pi_2: 13x+25y+5z=21$

$t: \begin{cases} 5x+y+z=5 \\ 13x+25y+5z=21 \end{cases}$ è certamente complanare con r

è certamente complanare con s

$P\in t$ ma $P\notin r$ $\quad t\neq r \quad r: \begin{cases} x=1 \\ y+z=0 \end{cases} \quad V_r: \begin{cases} x=0 \\ y+z=0 \end{cases}$

$V_r=\left\langle \begin{pmatrix}0\\-1\\1\end{pmatrix}\right\rangle \qquad V_r\notin V_t \ \begin{cases} 5x+y+z=0 \\ 13x+25y+5z=0 \end{cases} \qquad \begin{cases} 0-1+1=0 \\ 0-25+5\neq 0 \end{cases}$

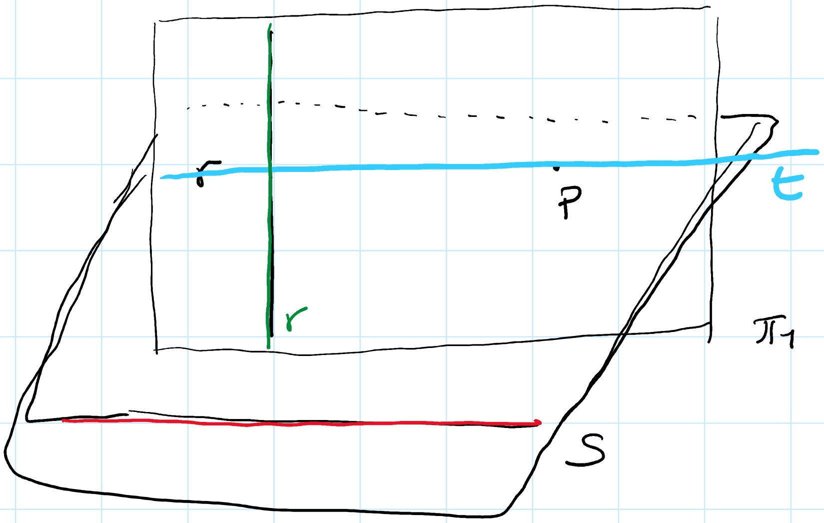
quindi r e t non sono parallele $\Rightarrow$ sono incidenti.

$V_s = \left\langle \begin{pmatrix} -10 \\ 5 \\ 1 \end{pmatrix} \right\rangle$ $\qquad V_s : \begin{cases} x+2y=0 \\ y-5z=0 \end{cases}$ $\quad \begin{cases} x=-2y=-10z \\ y=5z \end{cases}$

$s \parallel t \iff \begin{pmatrix} -10 \\ 5 \\ 1 \end{pmatrix} \in V_t \; ; \begin{cases} 5x+y+z=0 \\ 13x+25y+5z=0 \end{cases}$ $\quad -50+5+1 \neq 0$

$\Rightarrow$ la retta s non è parallela alla retta t

$\Rightarrow$ è incidente.



Date $\quad r: R + \langle v_r \rangle \quad$ sghembe $\quad \dim \langle R-S, v_r, v_s \rangle = 3$

$s: S + \langle v_s \rangle$

$P \notin r$ e $P \notin s$ esiste sempre una retta t complanare con r, t complanare con s e t passante per P:

$t = \pi_1 \cap \pi_2 \qquad \pi_1 : P \vee r : P + \langle R-P, v_r \rangle$

$\qquad \pi_2 : P \vee s : P + \langle S-P, v_s \rangle$

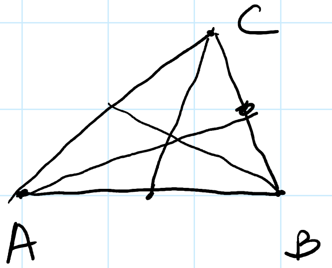
Punto medio di un segmento.

$M = A + \left(\frac{B-A}{2} \right) =$

$= A + \frac{B}{2} - \frac{A}{2} = \frac{A+B}{2}$

$hA + kB \qquad h+k=1$

Es: $A = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \qquad B = \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \qquad M = \begin{pmatrix} \frac{4+2}{2} \\ \frac{1+3}{2} \\ \frac{3+1}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$



$\frac{A+B+C}{3}$ Baricentro del Triangolo

$A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} \qquad C = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$$T = \frac{A+B+C}{3} = \begin{pmatrix} \frac{2}{3} \\ 1/3 \\ 1 \end{pmatrix}$$

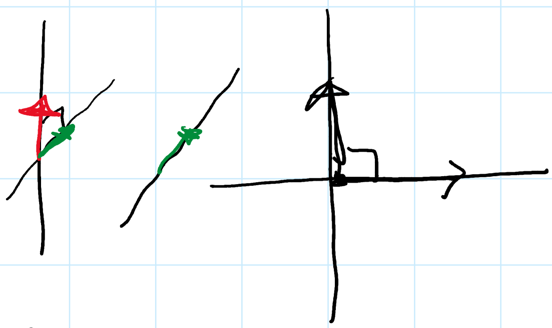
$\mathbb{A}^n(\mathbb{R})$

Spazi Euclidei $\mathbb{E}^n(\mathbb{R})$ $\qquad \mathbb{R}^n$

È lo spazio affine $\mathbb{R}^n$ con la giacitura $V = \mathbb{R}^n$ dotato del prodotto scalare euclideo

$$v \cdot w = v^t w = \langle v, w \rangle = \langle v | w \rangle$$

Definizione:



Due rette $r: R + \langle v_r \rangle$

$s: S + \langle v_s \rangle$

(con $v_r \neq \underline{0}$, $v_s \neq \underline{0}$) si dicono ortogonali se $v_r \cdot v_s = 0$

Esempio: $r: \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$ $\quad s: \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$ $\quad r \perp s$

$s \parallel t$

$r: \overset{R}{\begin{pmatrix}1\\0\\0\end{pmatrix}} + \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$ $\quad t: \overset{O}{\begin{pmatrix}0\\0\\0\end{pmatrix}} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$ $\quad r \perp t$

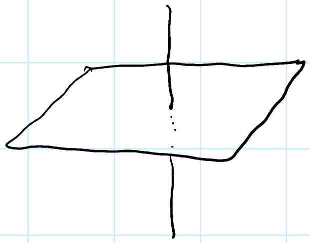
$$r \vee t: \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle R-O, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle = \mathbb{R}^3$$

$\Rightarrow$ r e t sono sghembe

Definizione: $r: R + \langle v_r \rangle$ e un piano $\left(\begin{matrix} \dim \langle v_r \rangle = 1, \\ \dim V_\pi = 2 \end{matrix} \right)$

$\pi: P + V_\pi$

in $\mathbb{R}^3$ si dicono ortogonali $\langle v_r \rangle \perp V_\pi$

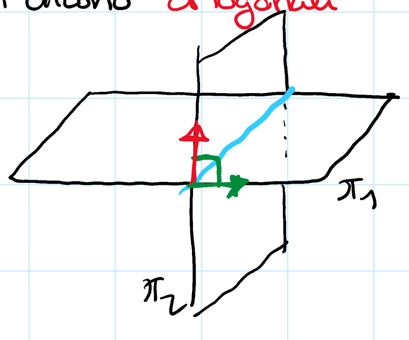


Esempio: $r: \begin{pmatrix}0\\1\\5\end{pmatrix} + \left\langle \begin{pmatrix}1\\3\\-8\end{pmatrix} \right\rangle$ $\quad \pi: x+3y-8z=11$

Def: due piani $\pi_1: a_1x+b_1y+c_1z=d_1$ $\quad rg(a_1\ b_1\ c_1)=1$

e $\pi_2: a_2x+b_2y+c_2z=d_2$ $\quad rg(a_2\ b_2\ c_2)=1$

si dicono ortogonali



$$\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix} \cdot \begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix} = 0$$

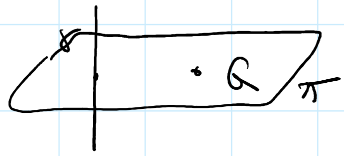
Es: $\pi_1: x+y=0$ $\quad \begin{pmatrix}1\\1\\0\end{pmatrix}$ $\quad \begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0 \quad \pi_1 \perp \pi_2$

$\pi_2: z=0$ $\quad \begin{pmatrix}0\\0\\1\end{pmatrix}$

Esercizio: determinare l'equazione cartesiana del piano ortogonale a

$r: \begin{cases} x-y=1 \\ x+y-z=2 \end{cases}$ e passante per $Q=\begin{pmatrix}1\\1\\1\end{pmatrix}$.

Sv:



$V_r = \begin{cases} x-y=0 \\ x+y-z=0 \end{cases} \quad \begin{cases} x=y \\ 2y-z=0 \end{cases} \quad \begin{cases} x=y \\ z=2y \end{cases}$

$\begin{pmatrix}y\\y\\2y\end{pmatrix} \qquad V_r = \left\langle \begin{pmatrix}1\\1\\2\end{pmatrix} \right\rangle$

$\pi_k: x+y+2z=k$ $\quad$ cerchiamo $k\in\mathbb{R}$ tale che $Q\in\pi_k$: $Q=\begin{pmatrix}1\\1\\1\end{pmatrix}$

$1+1+2=k \qquad k=4$

$$\boxed{\pi: \ x+y+2z=4}$$

Distanze in $\mathbb{R}^n$

Distanza fra due punti

$A \longrightarrow B$

$d(A,B) := \| B-A \|$

Es:

$d(A,B) = \left\| \begin{pmatrix}2\\4\end{pmatrix} \right\| = \qquad A=\begin{pmatrix}1\\1\end{pmatrix} \qquad B=\begin{pmatrix}3\\5\end{pmatrix}$

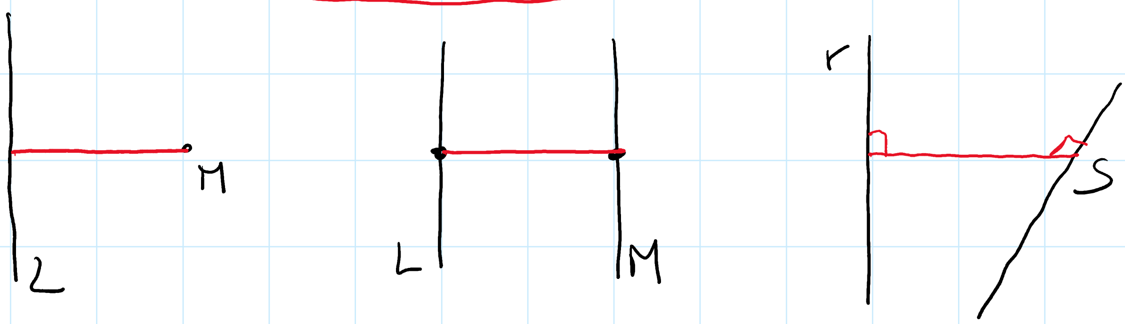
$= \sqrt{4+16} = \sqrt{20}$

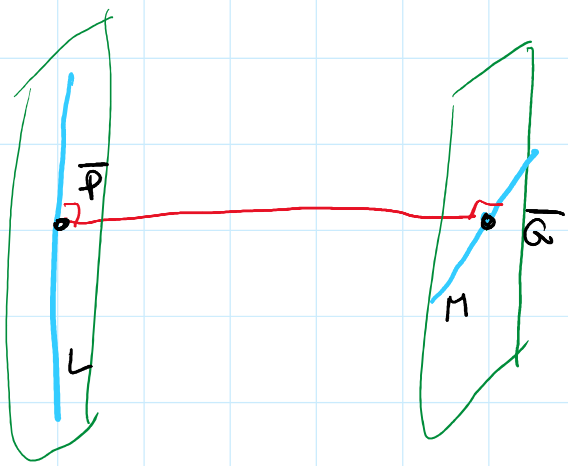
$d(A,B) = \left\| \begin{pmatrix}1\\1\\3\end{pmatrix} \right\| =$ $\quad A=\begin{pmatrix}1\\0\\5\end{pmatrix} \quad B=\begin{pmatrix}2\\1\\8\end{pmatrix}$

$= \sqrt{1+1+9} = \sqrt{11}$

Definizione: date L, M sottovarietà lineari non vuote di $\mathbb{R}^n$

$$\boxed{d(L,M) = \min_{\substack{P\in L\\ Q\in M}} d(P,Q)}$$





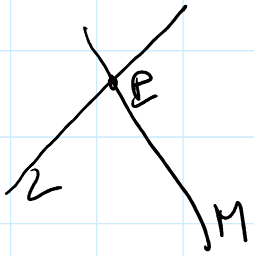
Fatto: esistono sempre $\bar{P}\in L$ e $\bar{Q}\in M$ tali che

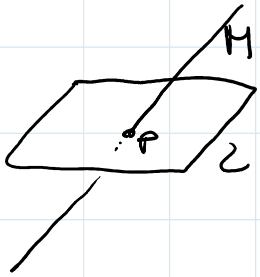
$d(L,M) = d(\bar{P},\bar{Q})$ $\quad (\bar{P},\bar{Q})$ è detta coppia di punti di **minima distanza**

Esempi:

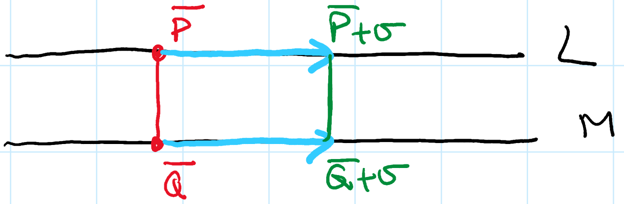
1) Se $L\cap M \neq \emptyset$ cioè sono incidenti

$P\in L\cap M \quad d(L,M) = d(P,P) = 0$

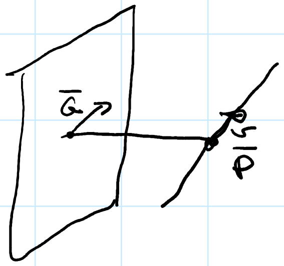




Fatto: Le coppie di punti di minima distanza sono uniche $\iff$ $V_L \cap V_M = \{\underline{0}\}$. Altrimenti se $v \in V_L \cap V_M$ e $(\bar{P}, \bar{Q})$ è coppia di minima distanza anche $(\bar{P}+v, \bar{Q}+v)$ è coppia di minima distanza



$V_L = V_M = \langle v \rangle$



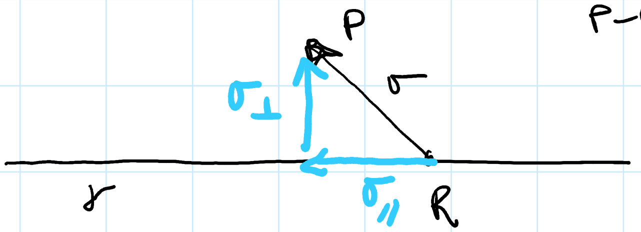
Distanze in $\mathbb{R}^2$: L, M

- Punto - Punto: $d(L, M) = \|M - L\|$
- Punto - retta: P, r

Se $P \in r$ $\quad d(P, r) = 0$

Se $P \notin r$ $\quad P = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad r: ax + by = c$

$$\boxed{d(P, r) = \frac{|ap_1 + bp_2 - c|}{\sqrt{a^2 + b^2}}}$$



$P - R = v$

$d(P, r) = \|v_\perp\|$

$R = \begin{pmatrix} x \\ y \end{pmatrix}$

$ax + by = c$ $u = \begin{pmatrix} a \\ b \end{pmatrix} \frac{1}{\sqrt{a^2+b^2}}$ $ax + by = c$

$$v = P - R = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p_1 - x \\ p_2 - y \end{pmatrix}$$

$$v_\perp = (v\ u)\, u = \left[\frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} p_1 - x \\ p_2 - y \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} \right] u$$

$\|u\| = 1$

$$\|v_\perp\| = \left| \frac{(p_1 - x)a + (p_2 - y)b}{\sqrt{a^2+b^2}} \right| = \frac{|ap_1 + bp_2 - ax - by|}{\sqrt{a^2+b^2}} =$$

$$= \boxed{\frac{|ap_1 + bp_2 - c|}{\sqrt{a^2+b^2}}}$$