Esercizio:

$r_a: \begin{cases} x - y + az = a \\ ay - z = a+1 \end{cases}$ $\quad s: \begin{cases} x + y = 0 \\ y + z = 0 \end{cases}$

Pos. reciproca:

Sv:

$r_a: \left(\begin{array}{ccc|c} 1 & -1 & a & a \\ 0 & a & -1 & a+1 \end{array}\right)$ $\quad rg\begin{pmatrix} 1 & -1 & a \\ 0 & a & -1 \end{pmatrix} = 2$

$s: \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$ $\quad rg\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 2$

r_a ed s sono rette.

$r_a \cap s$ ha matrice completa $\left(\begin{array}{ccc|c} 1 & -1 & a & a \\ 0 & a & -1 & a+1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)\begin{array}{c} 1^\circ - 3^\circ \\ \\ 3^\circ \\ \\ \end{array}$

$\begin{array}{c} 3^\circ \\ 4^\circ \\ 1^\circ - 3^\circ \\ 2^\circ \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & a & a \\ 0 & a & -1 & a+1 \end{array}\right)\begin{array}{c} \\ \\ 3^\circ + 2 \cdot 2^\circ \\ 4^\circ - a\,2^\circ \end{array}$ $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & a+2 & a \\ 0 & 0 & -1-a & a+1 \end{array}\right)\begin{array}{c} \\ \\ 3^\circ + 4^\circ \\ \\ \end{array}$

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2a+1 \\ 0 & 0 & -(a+1) & a+1 \end{array}\right)\begin{array}{c} \\ \\ \\ 4^\circ + (a+1)3^\circ \end{array}$ $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2a+1 \\ 0 & 0 & 0 & \boxed{2(a+1)^2} \end{array}\right)$

$(a+1) + (a+1)(2a+1) = (a+1)(1 + 2a + 1) = (a+1)^2 2$

$rg(A) = 3 \quad rg(A|b) = 3$ se $a = -1 \quad rg(A|b) = 4$ se $a \neq -1$

Se $a = -1 \quad rg(A) = 3 = rg(A|b)$ per R-C il sistema ha soluzioni

$r_{-1} \cap s \neq \emptyset \qquad \dim(r_{-1} \cap s) = 3 - 3 = 0 \Rightarrow r_{-1} \cap s = \{P\}$

$r_{-1} \cap s \neq \emptyset \qquad \dim(r_{-1} \cap s) = 3-3=0 \Rightarrow r_{-1}\cap s = \{P\}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} x+y=0 \\ y+z=0 \\ z=-1 \\ 0=0 \end{cases} \quad \begin{cases} x=-y=-1 \\ y=-z=1 \\ z=-1 \end{cases}$$

$r_{-1} \cap s = \left\{ \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ rette incidenti in un punto.

Se $a \neq -1$ $\quad rg A = 3 \neq rg(A|b) = 4$ per R-C $\boxed{r_a \cap s = \emptyset}$

rette disgiunte $\quad \dim(V_{r_a} \cap V_s) = 3 - rg(A) = 3-3=0$

$A\underline{x} = \underline{0}$

$\boxed{V_{r_a} \cap V_s = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}}$ $\Rightarrow$ r_a ed s sono rette sghembe.

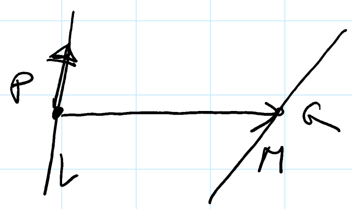
Definizione: date due sottovarietà lineari

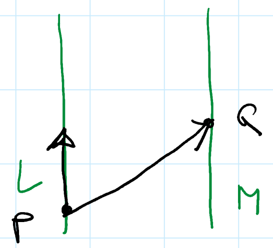
$L: P + V_L \qquad$ con $V_L, V_M \leq \mathbb{R}^n$; le più piccole

$M: Q + V_M$

sottovarietà lineare che contiene sia L che M viene detta sottovarietà lineare generata da L e M

$$\boxed{L \vee M := P + \langle V_L,\ Q-P,\ V_M \rangle}$$





Formule di Grassmann affini:

Siano L, M sottov. lineari di $\mathbb{R}^n$.

Se L ed M sono incidenti oppure sghembe:

$\dim(L \vee M) = \dim L + \dim M - \dim(L \cap M)$

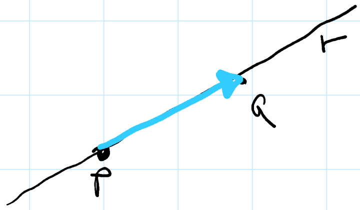
Se L ed M sono incidenti oppure sghembe:

$$\dim(L \vee M) = \dim L + \dim M - \dim(L \cap M)$$

Esempi:

Siano $P = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ $Q = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ determinare

$r = P \vee Q$

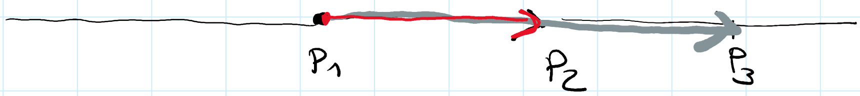


$r: \; P + \langle Q - P \rangle$ $\qquad r: \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ -2 \\ -5 \end{pmatrix} \right\rangle$

3 Punti allineati:

P_1, P_2, P_3 a due a due distinti, si dicono allineati se sono tutti contenuti in una retta $\Longleftrightarrow$

$P_1 \vee P_2 \vee P_3 : \; \boxed{P_1 + \langle P_2 - P_1, \; P_3 - P_1 \rangle}$ deve avere dimensione 1

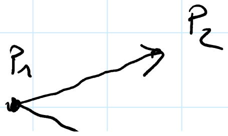


P_1, P_2, P_3 sono allineati $\Longleftrightarrow$ $\dim \langle P_2 - P_1, P_3 - P_1 \rangle = 1$

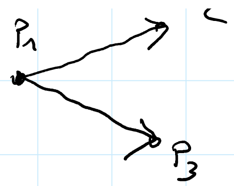
$P_1 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ $\quad P_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad P_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ sono allineati?

$P_2 - P_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ $\quad P_3 - P_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ $\quad \dim \langle P_2 - P_1, P_3 - P_1 \rangle = 2$

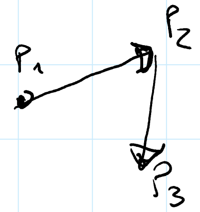
$\Rightarrow$ i punti non sono allineati



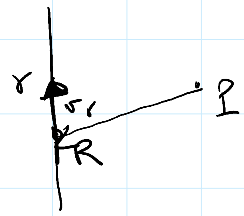
$P_1 \vee P_2 \vee P_3$: $\begin{pmatrix}0\\2\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\-3\\0\end{pmatrix}, \begin{pmatrix}1\\-2\\-1\end{pmatrix} \right\rangle$



π: $P_1 + \langle P_2 - P_1, P_3 - P_1 \rangle$



Esempio:



Sia r: $\begin{array}{cc} R & v_r \\ \begin{pmatrix}1\\0\\0\end{pmatrix} & + \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle \end{array}$

$P = \begin{pmatrix}0\\0\\1\end{pmatrix} \notin r$

Determinare il piano contenente r e P

π: $P + \langle R - P, v_r \rangle$

$\begin{pmatrix}x\\y\\z\end{pmatrix} - P \in \langle R - P, v_r \rangle$

π: $\begin{pmatrix}0\\0\\1\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} + a\begin{pmatrix}1\\0\\-1\end{pmatrix} + b\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}a+b\\b\\1-a\end{pmatrix}$ $\quad \begin{cases} x = a+b \\ \cancel{y = b} \\ z = 1-a \end{cases}$

$\begin{pmatrix}x\\y\\z-1\end{pmatrix} = a\begin{pmatrix}1\\0\\-1\end{pmatrix} + b\begin{pmatrix}1\\1\\0\end{pmatrix} \in \left\langle \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle$ $\quad \begin{cases} x = a+y \\ z = 1-a \end{cases} \Rightarrow z = 1-x-y$

$\det\begin{pmatrix} x & 1 & 1 \\ y & 0 & 1 \\ z-1 & -1 & 0 \end{pmatrix} = 0$ $\qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

$1 \cdot x - y + (z-1) = 0 \qquad x - y + z = 1$

$P + \langle v, w \rangle$

$$\det\left(\begin{pmatrix}x\\y\\z\end{pmatrix}-P \quad v \quad w\right)=0$$ eq. cartesiana del piano.

$\mathbb{R}^3$ Fasci di piani paralleli (improprio)

$\pi: \ 2x-y+z=8$

Determinare tutti i piani $/\!/ \pi$. $\quad \sigma: ax+by+cz=d$

$$\begin{array}{c} \pi \\ \sigma \end{array}\left(\begin{array}{ccc|c} 2 & -1 & 1 & 8 \\ a & b & c & d \end{array}\right) \qquad rg(a \ \ b \ \ c)=1$$

$$\pi /\!/ \sigma \iff rg\begin{pmatrix} 2 & -1 & 1 \\ a & b & c \end{pmatrix}=1$$

$$(a \ \ b \ \ c) = k(2 \ \ -1 \ \ 1)$$

$\sigma: \ 2kx-ky+kz=d \qquad k\neq 0 \qquad k\in\mathbb{R}\setminus\{0\}$

Fascio improprio $\boxed{2x-y+z=h}$ $\qquad h=\frac{d}{k} \qquad d\in\mathbb{R}$

$h\in\mathbb{R}$

Es: det. piano $/\!/$ al piano $x+y-3z=5$ e passante per $C=\begin{pmatrix}1\\0\\0\end{pmatrix}$

$x+y-3z=h$ $\qquad$ passa per $C=\begin{pmatrix}1\\0\\0\end{pmatrix} \iff$

$1+0-3\cdot 0=h$

$\Rightarrow \boxed{x+y-3z=1}$

Fascio proprio di piani di sostegno una retta

$$r: \begin{cases} x+2y-8z=5 \\ 2x-z=1 \end{cases} \qquad \begin{cases} x+2y-8z-5=0 \\ 2x-z-1=0 \end{cases}$$

$$\pi: \ ax+by+cz=d$$

$$rg(a \ \ b \ \ c) = 1$$

Quando $r \subseteq \pi$?

$$\left(\begin{array}{ccc|c} 1 & 2 & -8 & 5 \\ 2 & 0 & -1 & 1 \\ a & b & c & d \end{array}\right) \qquad rg\, A = rg(A|b) = 2$$

$$r \subseteq \pi$$

$$\alpha \underbrace{(x+2y-8z-5)}_{eq_1} + \beta \underbrace{(2x-z-1)}_{eq_2} = 0 \qquad \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \neq \begin{pmatrix}0 \\ 0\end{pmatrix}$$

$$r \ \begin{cases} eq_1 = 0 \\ eq_2 = 0 \end{cases} \qquad \pi: \alpha\, eq_1 + \beta\, eq_2 = 0$$

$$\alpha(x+2y-8z-5) + \beta(2x-z-1) = 0$$

$$\alpha k(x+2y-8z-5) + \beta k(2x-z-1) = 0$$

$\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ $\quad k\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ individua lo stesso piano $k \neq 0$

$$\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \neq \begin{pmatrix}0 \\ 0\end{pmatrix}$$

Esempi:

1) Determinare l'equazione cartesiana di un piano π contenente la retta $r: \begin{cases} x+2y=1 \\ x-z=0 \end{cases}$ e passante per il punto $\begin{pmatrix}0 \\ 0 \\ a\end{pmatrix}$.

2) Determ. l'eq. cartesiana di un piano contenente la retta $r: \begin{cases} x+2y=1 \\ x-z=0 \end{cases}$ e parallelo alla retta

retta $r: \begin{cases} x+2y=1 \\ x-z=0 \end{cases}$ e parallelo alla retta

$s: \begin{pmatrix} 1\\3\\0 \end{pmatrix} + \left\langle \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right\rangle.$

Sv: $r: \begin{cases} x+2y=1 \\ x-z=0 \end{cases}$ $\begin{cases} x+2y-1=0 \\ x-z=0 \end{cases}$

$\pi_{\alpha,\beta}: \alpha(x+2y-1)+\beta(x-z)=0$ $\begin{pmatrix} \alpha\\ \beta \end{pmatrix} \in \mathbb{R}^2 \setminus \left\{ \begin{pmatrix} 0\\0 \end{pmatrix} \right\}$

$\begin{pmatrix} 0\\0\\0 \end{pmatrix} \in \pi_{\alpha,\beta}$ se e solo se $\begin{matrix} x=0\\ y=0 \\ z=0 \end{matrix}$ $\alpha(\cancel{0}+2\cdot\cancel{0}-1)+\beta(\cancel{0}-\cancel{0})=0$

$-\alpha=0$

$\begin{pmatrix} 0\\ \beta \end{pmatrix} \in \mathbb{R}^2 \setminus \left\{ \begin{pmatrix} 0\\0 \end{pmatrix} \right\}$ $\beta \neq 0$ $\beta=1$ $\pi_{0,1}: x-z=0$

Per cose $\pi_{\alpha,\beta}$ passante per $P=\begin{pmatrix} 1\\1\\-1 \end{pmatrix}$

2) $r: \begin{cases} x+2y-1=0 \\ x-z=0 \end{cases}$

$\pi_{\alpha,\beta} \parallel s: \begin{pmatrix} 1\\3\\0 \end{pmatrix} + \boxed{\left\langle \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right\rangle}$ $V_s = \left\langle \begin{pmatrix} 1\\1\\1 \end{pmatrix} \right\rangle$

$\boxed{\pi_{\alpha,\beta}: \alpha(x+2y-1)+\beta(x-z)=0}$

$s \parallel \pi_{\alpha,\beta}$ se e solo se $V_s \subseteq V_{\pi_{\alpha,\beta}}$ $\begin{pmatrix} 1\\1\\1 \end{pmatrix} \in V_{\pi_{\alpha,\beta}}$

$V_{\pi_{\alpha,\beta}}: \alpha(x+2y)+\beta(x-z)=0 \leftarrow \begin{matrix} x=1\\ y=1\\ z=1 \end{matrix}$

$\alpha(1+2)+\beta(\cancel{1-1})=0$

$3\alpha=0$ $\alpha=0$ $\begin{pmatrix} 0\\ \beta \end{pmatrix}$ $\boxed{\begin{cases} \alpha=0 \\ \beta=1 \end{cases}}$

$\pi_{0,1}: x-z=0$

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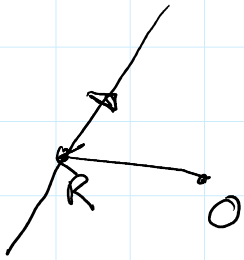
1) $r: \begin{cases} x+2y-1=0 \\ x-z=0 \end{cases}$ $\quad \cdot\, r \subseteq \pi$ $\quad \begin{pmatrix}0\\0\\0\end{pmatrix} \in \pi$

$r \begin{cases} x=-2y+1 \\ z=-2y+1 \end{cases}$

$\begin{pmatrix}-2y+1\\y\\-2y+1\end{pmatrix}$ $\quad r: \overset{R}{\begin{pmatrix}1\\0\\1\end{pmatrix}} + \langle \overset{v_r}{\begin{pmatrix}-2\\1\\-2\end{pmatrix}} \rangle$

$y=0$



$\pi: \begin{pmatrix}0\\0\\0\end{pmatrix} + \langle R-O, v_r \rangle \quad \pi: \begin{pmatrix}0\\0\\0\end{pmatrix} + \langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-2\\1\\-2\end{pmatrix} \rangle$

$x-z=0$

---

2) $r: \begin{pmatrix}1\\0\\1\end{pmatrix} + \langle \begin{pmatrix}-2\\1\\-2\end{pmatrix} \rangle$ $\quad s: \begin{pmatrix}1\\3\\0\end{pmatrix} + \langle \begin{pmatrix}1\\1\\1\end{pmatrix} \rangle$

$r \subseteq \pi \quad s /\!/ \pi$ $\quad V_s \subseteq V_\pi$

$\pi: \begin{pmatrix}1\\0\\1\end{pmatrix} + \langle \begin{pmatrix}-2\\1\\-2\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \rangle$

Es: det. piano // $r: \begin{pmatrix}1\\3\\0\end{pmatrix} + \langle \overset{v_r}{\begin{pmatrix}1\\2\\5\end{pmatrix}} \rangle$

$s: \begin{pmatrix}0\\1\\0\end{pmatrix} + \langle \underset{v_s}{\begin{pmatrix}2\\1\\1\end{pmatrix}} \rangle$ $\quad$ passante

per $T = \begin{pmatrix}1\\0\\-1\end{pmatrix}$

$\pi: T + \langle v_r, v_s \rangle$

$\begin{pmatrix}1\\0\\\cdot\end{pmatrix} + \langle \begin{pmatrix}1\\2\\\cdot\end{pmatrix}, \begin{pmatrix}2\\1\\\cdot\end{pmatrix} \rangle$

$$\begin{pmatrix}1\\0\\-1\end{pmatrix} + \left\langle \begin{pmatrix}1\\2\\5\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix} \right\rangle$$

Esercizio: $\pi_1: x+y=1$

$\pi_2: \begin{pmatrix}1\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle$

determinare le pos. rec. di π_1 e π_2.

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + a\begin{pmatrix}1\\-1\\1\end{pmatrix} + b\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1+a+b\\1-a+b\\a\end{pmatrix}$$

$$\begin{cases} x = 1+a+b \\ y = 1-a+b \\ \cancel{z=a} \end{cases} \qquad \begin{cases} \cancel{x=1+z+b} \\ y = 1-z+b \end{cases} \qquad b = x-1-z$$

$a = z$

$y = \cancel{1} - z + x - \cancel{1} - z \qquad \pi_2: x - y - 2z = 0$

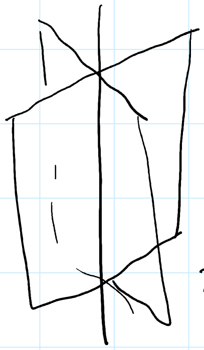
$r: \pi_1 \cap \pi_2 \begin{cases} x+y=1 \\ x-y-2z=0 \end{cases}$ piani incidenti in una retta

Determinare una retta t parallela sia a π_1 che a π_2 ed incidente la retta $s: \begin{pmatrix}1\\-1\\2\end{pmatrix} + \left\langle \begin{pmatrix}5\\0\\-1\end{pmatrix} \right\rangle$

$r: R + \langle v_r \rangle$

$\pi_1: R + \langle v_r, v_1 \rangle$

$\pi_2: R + \langle v_r, v_2 \rangle$



π_1

$t: T + \langle v_t \rangle$

$$\left.\begin{array}{l} \langle v_t \rangle \subseteq V_{\pi_1} \\ \langle v_t \rangle \subseteq V_{\pi_2} \end{array}\right\} \Rightarrow \langle v_t \rangle \subseteq V_{\pi_1} \cap V_{\pi_2}$$

$$\langle v_t \rangle = V_{\pi_1} \cap V_{\pi_2} = V_r \qquad r: \begin{cases} x+y=1 \\ x-y-2z=0 \end{cases}$$

$$V_r: \begin{cases} x+y=0 \\ x-y-2z=0 \end{cases} \quad \begin{cases} y=-x \\ x+x-2z=0 \end{cases} \quad \begin{cases} y=-x \\ z=x \end{cases} \quad \begin{pmatrix} x \\ -x \\ x \end{pmatrix}$$

$$V_r = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle \qquad t: \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

$\pi_1: P_1 + V_{\pi_1}$ $\qquad r // \pi_1$ $\qquad$ r incidente $\qquad s: S + \langle v_s \rangle$

$\pi_2: P_2 + V_{\pi_2}$ $\qquad r // \pi_2$

$t: S + \langle v_r \rangle$. $\qquad \langle v_r \rangle = V_{\pi_1} \cap V_{\pi_2}$

Es: Posizione reciproca di 2 piani in $\mathbb{R}^4$.

$$\left(\begin{array}{cccc|c} a & b & c & d & e \\ a' & b' & c' & d' & e' \\ a'' & b'' & c'' & d'' & e'' \\ a''' & b''' & c''' & d''' & e''' \end{array}\right) \qquad \begin{array}{c} A|b \\ A \in M_{4,4}(\mathbb{R}) \\ A|b \in M_{4,5}(\mathbb{R}) \end{array}$$

2	2
2	3
3	3
3	4
4	4

$$\begin{cases} x=0 \\ y=0 \end{cases}$$

$$\begin{cases} z=0 \\ t=0 \end{cases}$$