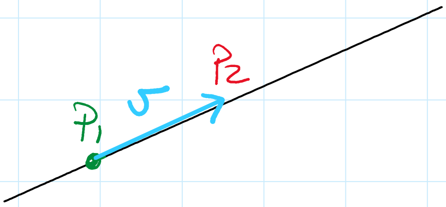
Sottovarietà lineari $\mathbb{R}^n$ $\dim L = K$

1° modo	2° modo eq. par.	3° modo
$L: P + \langle v_1, \dots, v_k \rangle$ con $\dim\langle v_1, \dots, v_k \rangle = K$	$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} + a_1 v_1 + \dots + a_k v_k$ con $a_1, \dots, a_k$ parametri reali	$A\underline{x} = \underline{b}$ sistema lineare $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\operatorname{rg} A = n - K$

$L_1: P_1 + \langle v_1, \dots, v_k \rangle$

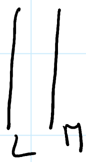
$L_2: P_2 + \langle w_1, \dots, w_h \rangle$

$$L_1 = L_2 \iff \begin{cases} \langle v_1, \dots, v_k \rangle = \langle w_1, \dots, w_h \rangle \\ P_2 - P_1 \in \langle v_1, \dots, v_k \rangle \end{cases}$$

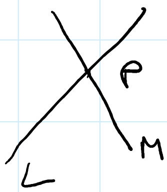


Def: L e M sottovarietà lineari di $\mathbb{R}^n$ si dicono

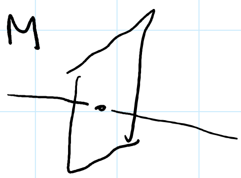
DISGIUNTE: se $L \cap M = \emptyset$



INCIDENTI: se $L \cap M \neq \emptyset$



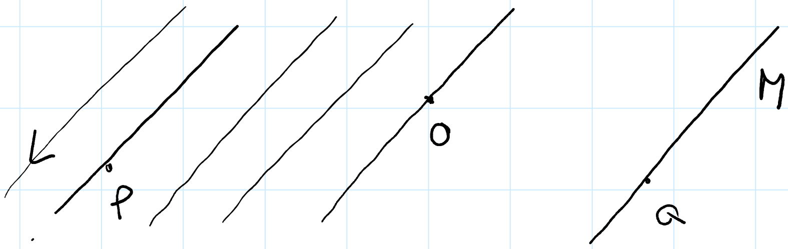
$P \in L \cap M$



Coincidenti o uguali se $L = M$

Parallele $L \parallel M$ $\quad L: P + V_L \quad M: Q + V_M$ $\quad \dim V_L \leq \dim V_M$ $\quad V_L, V_M \leq \mathbb{R}^n$

$$L \parallel M \iff V_L \leq V_M \iff O + V_L \subseteq O + V_M$$



Sghembe: L e M si dicono sghembe

$$\begin{cases} L \cap M = \emptyset \\ V_L \cap V_M = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\} \end{cases}$$

Esempi: in $\mathbb{R}^3$

$r: \begin{cases} x = 1 \\ y = 0 \end{cases}$ $\qquad$ $s: \begin{cases} x + y = 0 \\ z = 0 \end{cases}$

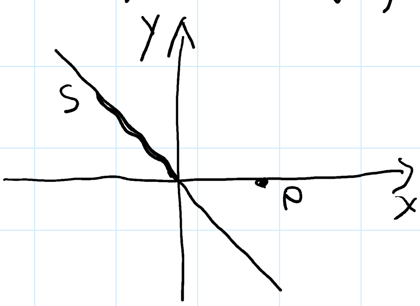
$r \cap s$ $\begin{cases} x = 1 \\ y = 0 \\ x + y = 0 \\ z = 0 \end{cases}$ $\quad \begin{cases} x = 1 \\ y = 0 \\ x = 0 \\ z = 0 \end{cases}$ impossibile $\Rightarrow$ r e s sono disgiunte

Giacitura di r è la sol. del sistema omogeneo associato

$\begin{cases} x = 0 \\ y = 0 \end{cases}$ $\quad \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$ $\quad \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = V_r$

$P = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$r = \begin{cases} x = 1 \\ y = 0 \end{cases}$



$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix}$

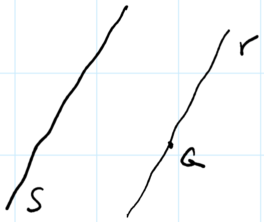
$r: \begin{cases} x = 1 \\ y = 0 \end{cases}$ $\quad \begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix}$ $\quad r: \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ $\quad V_r = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$s: \begin{cases} x + y = 0 \\ z = 0 \end{cases}$ $\quad \begin{cases} y = -x \\ z = 0 \end{cases}$ $\quad \begin{pmatrix} x \\ -x \\ 0 \end{pmatrix}$ $\quad s: \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$ $\quad V_s = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$

$V_r \cap V_s = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ $\Rightarrow$ r e s sono sghembe.

Es:

Determinare una retta r // ad $s: \begin{pmatrix}1\\2\\3\end{pmatrix} + \langle \overbrace{\begin{pmatrix}1\\3\\5\end{pmatrix}}^{V_s} \rangle$ e passante per il punto $Q = \begin{pmatrix}-1\\0\\3\end{pmatrix}$.



$r // s \iff V_r = V_s$

$$r: \begin{pmatrix}-1\\0\\3\end{pmatrix} + \langle \begin{pmatrix}1\\3\\5\end{pmatrix} \rangle$$

Det. un piano π parallelo a $r: \begin{pmatrix}1\\0\\2\end{pmatrix} + \langle \overbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}^{V_r} \rangle$ a

$s: \begin{pmatrix}1\\1\\1\end{pmatrix} + \langle \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{V_s} \rangle$

e passante per $P = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$\pi: P + V_\pi$

$r // \pi \Rightarrow V_r \subseteq V_\pi \quad \langle \begin{pmatrix}1\\0\\1\end{pmatrix} \rangle \subseteq V_\pi$

$s // \pi \Rightarrow V_s \subseteq V_\pi \quad \langle \begin{pmatrix}1\\1\\0\end{pmatrix} \rangle \subseteq V_\pi$

$\Rightarrow \langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \rangle = V_\pi \qquad \pi: \begin{pmatrix}1\\1\\1\end{pmatrix} + \langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \rangle$

Determinare il piano $\pi // \sigma: x+y+z=3$ passante per il punto $P = \begin{pmatrix}1\\1\\1\end{pmatrix}$.

$\sigma: S + V_\sigma$

$\pi: P + V_\sigma$

$V_\sigma: x+y+z=0$ tutti i piani paralleli a questo piano

$x+y+z=K$ cerchiamo il piano passante per $P = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$x+y+z=k$ cerchiamo il piano passante per $P=\begin{pmatrix}11\\1\\1\end{pmatrix}$

$11+1+1=k$

$\boxed{\pi: x+y+z=13}$

Posizione reciproca di 2 rette in $\mathbb{R}^2$ (nel piano affine)

$r: \quad ax+by=c \qquad \text{rg}(a\ b)=1$

$s: \quad a'x+b'y=c' \qquad \text{rg}(a'\ b')=1$

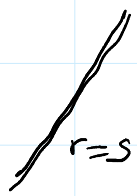
$r\cap s \quad \begin{cases} ax+by=c \\ a'x+b'y=c' \end{cases} \qquad A|\underline{b} = \left(\begin{array}{cc|c} a & b & c \\ a' & b' & c' \end{array}\right)$

Studiamo i ranghi $\text{rg}A \quad \text{rg}(A|\underline{b})$

1° caso: $\text{rg}A=1=\text{rg}(A|\underline{b})$ per R-C il sistema $r\cap s$ ha soluzioni con $\dim(r\cap s) = \text{num.di inc.} - \text{rg}(A) = 2-1=1$

$\Longrightarrow \quad r=s=r\cap s$

coincidenti



$r: 2x+2y=2$

$s: x+y=1$

$\left(\begin{array}{cc|c} 2 & 2 & 2 \\ 1 & 1 & 1 \end{array}\right)$

2° caso $\text{rg}A=1\neq \text{rg}(A|b)=2$ per R-C il sistema $r\cap s$ non ha soluzione $\Rightarrow r\cap s=\emptyset$ disgiunte

$V_r: ax+by=0 \qquad V_r\cap V_s \quad \left(\begin{array}{cc|c} a & b & 0 \\ a' & b' & 0 \end{array}\right) \qquad (A|\underline{0})$

$V_s: a'x+b'y=0$

$\dim V_r\cap V_s = 2-\text{rg}(A) = 2-1=1$ quindi abbiamo

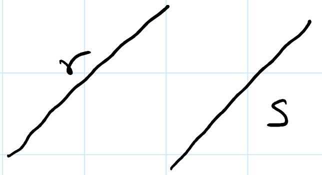
$\dim V_r = 1$

$\dim(V_r\cap V_s)=1 \quad \Longrightarrow \quad V_r=V_s=V_r\cap V_s$

$\dim V_r = 1$

$\dim V_s = 1$

$\dim(V_r \cap V_s) = 1 \implies V_r = V_s = V_r \cap V_s$



$r: \; x + y = 1$

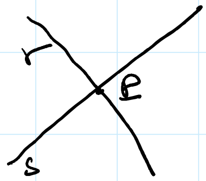
$s: \; x + y = 2$

$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right)$

3° caso: $\mathrm{rg}\, A = 2 = \mathrm{rg}(A|b)$ per R-C $r \cap s$ ha soluzioni

$\dim r \cap s = 2 - 2 = 0 \implies$ r e s si intersecano in un solo punto

$r \cap s = \{P\}$ incidenti



	$\mathrm{rg}\, A$	$\mathrm{rg}(A\|b)$	
$r = s$	1	1	Rette coincidenti
$r // s$, $r \cap s = \emptyset$	1	2	Rette parallele disgiunte
$r \cap s = \{P\}$	2	2	Rette incidenti in un punto

Posizione reciproca di due piani in $\mathbb{R}^3$

$\pi_1: \; ax + by + cz = d \qquad \mathrm{rg}\,(a \;\; b \;\; c) = 1$

$\pi_2: \; a'x + b'y + c'z = d' \qquad \mathrm{rg}\,(a' \;\; b' \;\; c') = 1$

$\pi_1 \cap \pi_2: \begin{cases} ax + by + cz = d \\ a'x + b'y + c'z = d' \end{cases} \qquad A|\underline{b} = \left(\begin{array}{ccc|c} a & b & c & d \\ a' & b' & c' & d' \end{array}\right)$

	$rg(A)$	$rg(A\|b)$	
$\pi_1 = \pi_2$	1	1	Piani coincidenti
$\pi_1 // \pi_2,\ \pi_1 \cap \pi_2 = \emptyset$	1	2	Piani paralleli disgiunti
$\pi_1 \cap \pi_2 = r$	2	2	Piani incidenti in una retta

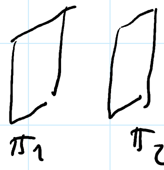
1° caso: $rg(A) = rg(A|b) = 1$ per R-C $\pi_1 \cap \pi_2 \neq \emptyset$ $\dim(\pi_1 \cap \pi_2) = 3-1 = 2$

$\dim \pi_1 = 2$

$\dim \pi_2 = 2$ $\quad \dim(\pi_1 \cap \pi_2) = 2 \implies \pi_1 = \pi_2 = \pi_1 \cap \pi_2$ sono piani uguali.

2° caso $rg A = 1 \neq rg(A|b) = 2$ per R-C $\pi_1 \cap \pi_2 = \emptyset$ cioè π_1 e π_2 sono disgiunti

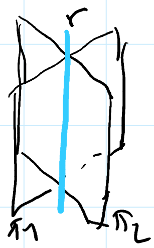
$$\dim(\underbrace{V_{\pi_1} \cap V_{\pi_2}}_{A\underline{x} = \underline{0}}) = 3 - rg(A) = 3 - 1 = 2$$



$\dim V_{\pi_1} = \dim V_{\pi_2} = \dim(V_{\pi_1} \cap V_{\pi_2}) = 2 \implies V_{\pi_1} = V_{\pi_2}$ cioè $\pi_1 // \pi_2$

Sono piani paralleli disgiunti.

3° caso: $rg(A) = 2 = rg(A|b)$ per R-C $\pi_1 \cap \pi_2 \neq \emptyset$ quindi π_1 e π_2 sono incidenti $\dim(\pi_1 \cap \pi_2) = 3-2 = 1 \implies \pi_1 \cap \pi_2 = r$ retta.



Posizione reciproca di un piano e una retta in $\mathbb{R}^3$

$\pi: ax + by + cz = d$ $\qquad rg\begin{pmatrix} a & b & c \end{pmatrix} = 1$

$$r: \begin{cases} a'x + b'y + c'z = d' \\ a''x + b''y + c''z = d'' \end{cases} \qquad rg\begin{pmatrix} a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} = 2$$

$\pi \cap r$ $\begin{cases} ax+by+cz=d \\ a'x+b'y+c'z=d' \\ a''x+b''y+c''z=d'' \end{cases}$

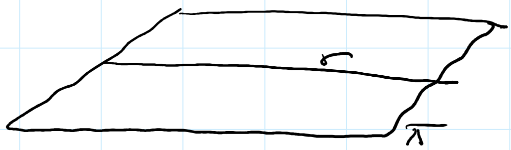
$$\boxed{A|\underline{b} = \left(\begin{array}{ccc|c} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \end{array}\right)}$$

$A \in M_{3,3}(\mathbb{R})$ $\qquad A|\underline{b} \in M_{3,4}(\mathbb{R})$ $\qquad rg(A) \geq 2$

1° caso: $\boxed{rg\,A = 2 = rg(A|\underline{b})}$ per R-C il sistema $\pi \cap r$ ha soluzioni $\dim(\pi \cap r) = 3-2 = 1$

essendo $\dim(\pi) = 2$ $\quad \dim(\pi \cap r) = 1 \Rightarrow \pi \cap r = r$ cioè

$\dim(r) = 1$ $\qquad \boxed{r \subseteq \pi}$



2° caso: $\boxed{rg\,A = 2 \neq rg(A|\underline{b}) = 3}$ per R-C $\pi \cap r = \emptyset$ quindi

r e π sono disgiunti $\qquad V_r$ giaciture di r

$\qquad V_\pi$ " di π

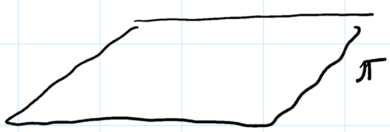
$\dim(V_r \cap V_\pi) = 3 - rg(A)$

$A\underline{x} = \underline{0}$ $\qquad = 3-2 = 1$

$\dim V_r = 1$

$\dim V_\pi = 2$ $\qquad \dim(V_r \cap V_\pi) = 1 \Rightarrow V_r = V_r \cap V_\pi$ cioè

$V_r \subseteq V_\pi$ cioè $\boxed{\begin{array}{l} r \parallel \pi \\ r \cap \pi = \emptyset \end{array}}$



3° caso $\boxed{rg(A) = 3 = rg(A|\underline{b})}$ per R-C $r \cap \pi \neq \emptyset$ sono incidenti $\dim(r \cap \pi) = 3 - rg(A) = 3-3 = 0$ quindi

$r \cap \pi = \{P\}$

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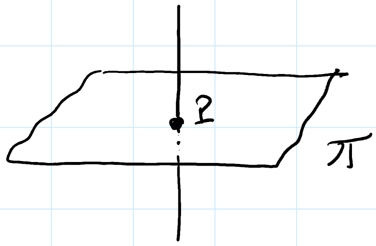

	rg(A)	rg(A|b)
$r \subseteq \pi$	2	2
$r // \pi$ $r \cap \pi = \emptyset$	2	3
$r \cap \pi = \{P\}$	3	3

Posizione reciproca di 2 rette in $\mathbb{R}^3$

$r: \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$ $\quad rg\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$

$s: \begin{cases} a_3x + b_3y + c_3z = d_3 \\ a_4x + b_4y + c_4z = d_4 \end{cases}$ $\quad rg\begin{pmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{pmatrix} = 2$

$r \cap s: \begin{cases} a_1x + b_1y + c_1z = d_1 \\ \vdots \\ a_4x + b_4y + c_4z = d_4 \end{cases}$ $\quad A|b = \left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{array}\right)$

$A \in M_{4,3}(\mathbb{R}) \qquad A|b \in M_{4,4}(\mathbb{R})$

1° caso: $\boxed{rg A = 2 = rg(A|b)}$ per R-C $r \cap s \neq \emptyset$

$\dim(r \cap s) = 3 - 2 = 1 \Rightarrow \dim r = \dim s = \dim(r \cap s) = 1$

$\Rightarrow r = s$ rette uguali.

2° caso $\boxed{rg A = 2 \neq rg(A|b) = 3}$ per R-C $\boxed{r \cap s = \emptyset}$ rette

2° caso $\boxed{rg A = 2 \neq rg(A|b) = 3}$ per R-C $\boxed{r \cap s = \emptyset}$ rette disgiunte ma

$$\dim(V_r \cap V_s) = 3 - rg(A) = 3 - 2 = 1$$

$\dim V_r = \dim V_s = \dim(V_r \cap V_s) = 1 \Rightarrow V_r = V_s$ cioè $\boxed{r // s}$.

3° caso: $\boxed{rg(A) = 3 = rg(A|b)}$ per R-C $\boxed{r \cap s \neq \emptyset}$ rette incidenti $\dim(r \cap s) = 3 - rg(A) = 3 - 3 = 0$ cioè $\boxed{r \cap s = \{P\}}$

4° caso $\boxed{rg(A) = 3 \neq rg(A|b)}$ per R-C $\boxed{r \cap s = \emptyset}$ sono disgiunte

$\dim(V_r \cap V_s) = 3 - rg(A) = 3 - 3 = 0 \Rightarrow \boxed{V_r \cap V_s = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}}$

$\Rightarrow$ r e s sono sghembe.

	$rg(A)$	$rg(A\|b)$	
$r = s$	2	2	Coincidenti
$r \cap s = \emptyset$, $r // s$	2	3	Parallele disgiunte
$r \cap s = \{P\}$	3	3	Incidenti in un punto
$r \cap s = \emptyset$ $V_r \cap V_s = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$	3	4	rette sghembe

Esercizio:

$$r_a : \begin{cases} x - y + az = a \\ ay - z = a + 1 \end{cases} \qquad s : \begin{cases} x + y = 0 \\ y + z = 0 \end{cases}$$

Studiare le posizione reciproca di r_a, s al variare di $a \in \mathbb{R}$.