Proiezioni ortogonali

$\mathbb{R}^n$ $\quad T \quad$ $P_T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

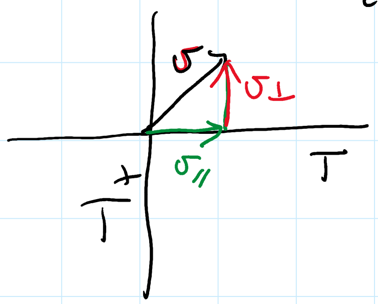
$v \in \mathbb{R}^n$ $\quad v = v_{\parallel} + v_{\perp}$ $\quad v_{\parallel} \in T$ $\quad v_{\perp} \in T^{\perp}$

$P_T(v) = v_{\parallel}$

$\sigma_T(v) = v_{\parallel} - v_{\perp}$

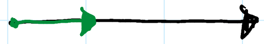
$v_{\parallel}$ Prendere una base ortonormale di T $\quad B_T = \{u_1, \dots, u_k\}$

$$\boxed{v_{\parallel} = \sum_{i=1}^{k} (v \cdot u_i)\, u_i} \in T$$



Se $T = \langle v \rangle$ $\quad v \in \mathbb{R}^n$ $\quad v \neq 0$ $\quad u = \frac{v}{\|v\|}$ è un versore

$\dim T = 1$ $\quad T = \langle u \rangle$ $\quad B_T = \{u\}$



$$w_{\parallel} = (w \cdot u)u = \left(w \cdot \frac{v}{\|v\|}\right) \frac{v}{\|v\|} = \frac{1}{\|v\|^2} (w \cdot v)\, v$$

$T = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$ $\quad v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \|v\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$ $\quad u = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$$P_T \overset{w}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = (w \cdot u)u = \frac{1}{\|v\|^2} (w \cdot v) v = \frac{1}{3} \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$= \frac{1}{3}(x+y+z) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{x+y+z}{3} \\ \frac{x+y+z}{3} \\ \frac{x+y+z}{3} \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$\operatorname{Im} P_T = T = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

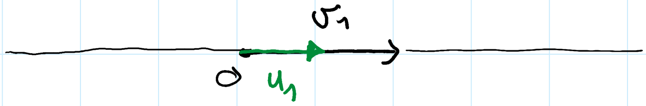
Procedimento di ortonormalizzazione di Gram-Schmidt GS

Proposizione: sia $T \leq \mathbb{R}^n$ e $B = \{v_1, \dots, v_k\}$ una base di T allora esiste $\{u_1, \dots, u_k\}$ base ortonormale di T tale che $\langle u_1, \dots, u_i \rangle = \langle v_1, \dots, v_i \rangle \ \forall i = 1, \dots, k$.

Dim:

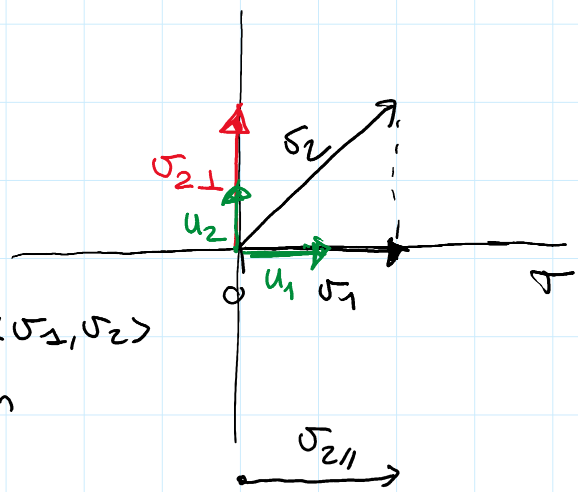
$i=1$ $\langle u_1 \rangle = \langle v_1 \rangle$ $\quad v_1 \neq 0_{\mathbb{R}^n}$

$\boxed{u_1 = \dfrac{v_1}{\|v_1\|}}$ è versore $\quad u_1 \neq 0 \quad \langle u_1 \rangle = \langle v_1 \rangle$



$i=2$ $\langle u_1 \rangle = \langle v_1 \rangle$

$\langle u_1, u_2 \rangle = \langle v_1, v_2 \rangle$



$v_{2\perp} = v_2 - (v_2 \cdot u_1)\, u_1 \in \langle u_1, v_2 \rangle = \langle v_1, v_2 \rangle$

$u_2 = \dfrac{v_{2\perp}}{\|v_{2\perp}\|} \in \langle v_1, v_2 \rangle \qquad v_{2\perp} \neq 0_{\mathbb{R}^n}$

perché $v_2 \notin \langle v_1 \rangle$.

u_1, u_2 sono ortogonali fra loro perché $v_{2/\!/} \in \langle u_1 \rangle$
$v_{2\perp} \in \langle u_1 \rangle^{\perp}$

u_1 e u_2 sono lin. ind. perché ortogonali non nulli $\langle u_1, u_2 \rangle = \langle v_1, v_2 \rangle$

i $\quad \langle u_1, \dots, u_{i-1} \rangle = \langle v_1, \dots, v_{i-1} \rangle$ per costruire u_i si pone

$$\boxed{\begin{aligned} w_i &:= v_i - \sum_{j=1}^{i-1} (v_i \cdot u_j)\, u_j \\ u_i &:= \frac{w_i}{\|w_i\|} \end{aligned}}$$

$w_i \neq 0_{\mathbb{R}^n}$ perché $v_i \notin \langle u_1, \dots, u_{i-1} \rangle = \langle v_1, \dots, v_{i-1} \rangle$

perché altrimenti $v_i \in \langle v_1, \dots, v_{i-1} \rangle$ è assurdo perché $v_1, \dots, v_k$ sono lin. indipendenti.

Essendo $P_{\langle u_1, \dots, u_{i-1} \rangle}(v_i) = \sum_{j=1}^{i-1} (v_i \cdot u_j)\, u_j$

$w_i \in \langle u_1, \dots, u_{i-1} \rangle^{\perp}$

T $\quad B = \{v_1, \dots, v_k\}$

1) $u_1 = \dfrac{v_1}{\|v_1\|}$

2) $w_2 = v_2 - (v_2 \cdot u_1) u_1 \qquad u_2 = \dfrac{w_2}{\|w_2\|}$

3) $w_3 = v_3 - (v_3 \cdot u_1) u_1 - (v_3 \cdot u_2) u_2 \qquad u_3 = \dfrac{w_3}{\|w_3\|}$

$\vdots$

k) $w_k = v_k - \sum_{i=1}^{k-1} (v_k \cdot u_i)\, u_i \qquad u_k = \dfrac{w_k}{\|w_k\|}$

Esempio: $B = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} \right\}$ procedimento di GS su B

$u_1 = \dfrac{v_1}{\|v_1\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \qquad \|v_1\| = \sqrt{3}$

$$w_2 = v_2 - [(v_2 \cdot u_1)]u_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} - \left[\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}\right]\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} =$$

$$= \begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{1}{3}\left[\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix}\right]\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$w_2 \cdot v_1 = 0 \quad \begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3} + \frac{1}{3} - \frac{2}{3} = 0$$

$$\|w_2\| = \left\| \frac{1}{3}\begin{pmatrix}1\\1\\-2\end{pmatrix}\right\| = \frac{1}{3}\sqrt{6}$$

$$= \left\|\begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix}\right\| = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}} = \frac{\sqrt{6}}{3}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix} \cdot \frac{1}{\frac{\sqrt{6}}{3}} \; \frac{3}{\sqrt{6}} =$$

$$= \begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix} \qquad \frac{1}{6} + \frac{1}{6} + \frac{4}{6} = \frac{6}{6} = 1$$

$$u_1 = \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} \qquad u_2 = \begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix}$$

$$w_3 = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2 =$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \left[\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}}_{1/\sqrt{3}+0+0}\right]\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} - \left[\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix}}_{\frac{1}{\sqrt{6}}}\right]\begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\-2/\sqrt{6}\end{pmatrix} =$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix} - \begin{pmatrix}1/6\\1/6\\-2/6\end{pmatrix} =$$

$$= \begin{pmatrix}\frac{6-2-1}{6}\\ \frac{-2-1}{6}\\ \frac{-2+2}{6}\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\\0\end{pmatrix}$$

$$u_3 = \frac{w_3}{\|w_3\|} = \begin{pmatrix}\sqrt{2}/2\\-\sqrt{2}/2\\0\end{pmatrix}$$

$$\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2} = 1$$

$$\|w_3\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$$

$$B_{orto} = \{u_1, u_2, u_3\} = \left\{ \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \right\}$$

Esercizio:

In $\mathbb{R}^4$ si consideri il sottospazio $T = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

1) Determinare la matrice rispetto alla base canonica della proiezione ortogonale su T.

2) Determinare la matrice rispetto alla base canonica della simmetria ortogonale di asse T.

3) Determinare una base ortonormale di T e completarla a base ortonormale di $\mathbb{R}^4$.

4) Determinare un vettore $v \in \langle e_1, e_3, e_4 \rangle$ tale che $P_T(v) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

Svolgimento:

$P_T(v) = (v \cdot u_1)\, u_1 + (v \cdot u_2)\, u_2$ con $\{u_1, u_2\}$ base ortonormale di T

$T = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ $\quad B = \Big\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{v_1}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Big\}$ procedimento di G.S.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{1^2+1^2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$w_2 = v_2 - (v_2 \cdot u_1)\, u_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\cancel{2}}{\cancel{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 1-0 \\ 0-1 \\ 0-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad w_2 \cdot v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 1+0-1+0=0$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad B_{T\,orton} = \Big\{ \underbrace{\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \end{pmatrix}}_{u_2} \Big\}$$

$$P_T(v) = (v \cdot u_1)\, u_1 + (v \cdot u_2)\, u_2 \qquad v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right] \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}\right] \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} =$$

$$= \frac{3}{6}(x_1 + x_3) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{2}{6}(x_1 + x_2 - x_3) \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} =$$

$$= \frac{1}{6} \begin{pmatrix} 3x_1 + 3x_3 \\ 0 \\ 3x_1 + 3x_3 \\ 0 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 2x_1 + 2x_2 - 2x_3 \\ 2x_1 + 2x_2 - 2x_3 \\ -2x_1 - 2x_2 + 2x_3 \\ 0 \end{pmatrix} =$$

$$= \frac{1}{6} \begin{pmatrix} 5x_1 + 2x_2 + x_3 \\ 2x_1 + 2x_2 - 2x_3 \\ x_1 - 2x_2 + 5x_3 \\ 0 \end{pmatrix} \qquad \frac{1}{6} \begin{pmatrix} 5 & 2 & 1 & 0 \\ 2 & 2 & -2 & 0 \\ 1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = P$$

verificare che $P = P^t$.

Notiamo che data la base ortonormale

$$\underset{u_1}{\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \end{pmatrix}} \qquad P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 0 \end{pmatrix} \in M_{4,4}(\mathbb{R})$$

$$\underset{4 \times 2}{(u_1 \ \ u_2)} \underset{2 \times 4}{\begin{pmatrix} u_1^t \\ u_2^t \end{pmatrix}} \qquad \text{perché}$$

$$P_T(v) = (v \cdot u_1) u_1 + (v \cdot u_2) u_2 = u_1 (u_1 \cdot v) + u_2 (u_2 \cdot v) =$$

$$= u_1 u_1^t v + u_2 u_2^t v = (u_1 \ u_2) \begin{pmatrix} u_1^t \\ u_2^t \end{pmatrix} v$$ quindi se

$B_T = \{u_1, u_2\}$ è base ortonormale di T

$$\boxed{A_{\mathcal{E},\mathcal{E},P_T} = (u_1 \ u_2) \begin{pmatrix} u_1^t \\ u_2^t \end{pmatrix}}$$

2) $A_{\mathcal{E},\mathcal{E},\sigma_T} = 2P - \mathbb{I}_4$ $\quad v = v_{\parallel} + v_{\perp}$ $\quad P_T(v) = v_{\parallel}$ $\quad \sigma_T(v) = v_{\parallel} - v_{\perp} =$

$= 2v_{\parallel} - (v_{\parallel} + v_{\perp}) = 2P_T - id$

$$= \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 & 0 \\ 2 & -1 & -2 & 0 \\ 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} = S$$

è matrice simmetrica

$S^2 = \mathbb{I}_4$

3) Determinare una base ortonormale di T e completarla a base ortonormale di $\mathbb{R}^4$.

Base $\quad B_{T\,ortorm.} = \{u_1, u_2\} = \left\{ \underset{u_1}{\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \end{pmatrix}} \right\}$

$T = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

$B = \{u_1, u_2, u_3, u_4\}$ base ortonormale di $\mathbb{R}^4$ come determinare u_3 e u_4?

$\langle u_1, u_2 \rangle \oplus \langle u_3, u_4 \rangle = \mathbb{R}^4 \qquad \langle u_3, u_4 \rangle = T^{\perp}$

$T \oplus T^{\perp} = \mathbb{R}^4 \qquad T = \left\langle \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \end{pmatrix} \right\rangle$

Calcoliamo $T^{\perp}$

$$T^{\perp}: \begin{cases} \frac{1}{\cancel{\sqrt{2}}} x_1 + \frac{1}{\cancel{\sqrt{2}}} x_3 = 0 \\ \frac{1}{\cancel{\sqrt{3}}} x_1 + \frac{1}{\cancel{\sqrt{3}}} x_2 - \frac{1}{\cancel{\sqrt{3}}} x_3 = 0 \end{cases} \qquad \begin{cases} x_1 + x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \end{cases}$$

$$\begin{cases} x_3 = -x_1 \\ x_1 + x_2 + x_1 = 0 \end{cases} \qquad \begin{cases} x_3 = -x_1 \\ x_2 = -2x_1 \end{cases} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ -2x_1 \\ -x_1 \\ x_4 \end{pmatrix}$$

$T^{\perp} = \left\langle \underset{v_3}{\begin{pmatrix} 1 \\ -2 \\ -1 \\ 0 \end{pmatrix}}, \underset{v_4}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \right\rangle$ $\qquad$ fare GS su v_3, v_4

$v_3 \cdot v_4 = 0$

$$\boxed{u_3 = \frac{v_3}{\|v_3\|} = \frac{v_3}{\sqrt{1+4+1}} = \begin{pmatrix} 1/\sqrt6 \\ -2/\sqrt6 \\ -1/\sqrt6 \\ 0 \end{pmatrix}}$$

$$w_4 = v_4 - (v_4 \cdot u_3)u_3 = v_4 - \Bigg[\underbrace{\begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt6 \\ -2/\sqrt6 \\ -1/\sqrt6 \\ 0 \end{pmatrix}}_{=0} \Bigg] \begin{pmatrix} 1/\sqrt6 \\ -2/\sqrt6 \\ -1/\sqrt6 \\ 0 \end{pmatrix} = v_4 = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}$$

$$w_4 = v_4 \Rightarrow \boxed{u_4 = \frac{v_4}{\|v_4\|} = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}}.$$

2° metodo: T $\{u_1, u_2, z_3, z_4\}$ cioè completare $\{u_1, u_2\}$

a base di $\mathbb{R}^4$ e poi fare GS su $\{u_1, u_2, z_3, z_4\}$.

4) $\boxed{v \in \langle e_1, e_3, e_4 \rangle}$: $\boxed{P_T(v) = \begin{pmatrix} 1\\0\\1\\0 \end{pmatrix}}$

$$v = x_1 e_1 + x_3 e_3 + x_4 e_4 = \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \end{pmatrix}$$

$$P_T(v)\ \frac{1}{6} \begin{pmatrix} 5 & 2 & 1 & 0 \\ 2 & 2 & -2 & 0 \\ 1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 5x_1 + x_3 \\ 2x_1 - 2x_3 \\ x_1 + 5x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\\0\\1\\0 \end{pmatrix}$$

$$\begin{cases} \frac{5}{6}x_1 + \frac{1}{6}x_3 = 1 \quad \cdot 6 \\ \frac{\cancel{2}}{6}x_1 - \frac{\cancel{2}}{6}x_3 = 0 \\ \frac{x_1}{\cancel{6}} + \frac{5}{\cancel{6}}x_3 = 1 \quad \cdot 6 \\ 0 = 0 \end{cases} \quad \begin{cases} 5x_3 + x_3 = 6 \\ x_1 = x_3 \\ x_3 + 5x_3 = 6 \end{cases} \quad \begin{cases} 6x_3 = 6 \\ x_1 = x_3 \end{cases} \quad \begin{cases} x_3 = 1 \\ x_1 = 1 \end{cases}$$

$$v = \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \end{pmatrix} \qquad \left\{ v = \begin{pmatrix} 1 \\ 0 \\ 1 \\ x_4 \end{pmatrix} \;\middle|\; x_4 \in \mathbb{R} \right\}$$

sono tutti i vettori $v \in \mathbb{R}^4$: $P_T(v) = \begin{pmatrix} 1\\0\\1\\0 \end{pmatrix}$.

Determinare $P_T(v) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\qquad T = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

$$P_T(v) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \notin T$$ $\Rightarrow$ non esistono vettori

$v \in \mathbb{R}^4$: $P_T(v) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ perché $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \notin \operatorname{Im} P_T = T$