Prodotto scalare usuale in $\mathbb{R}^n$ $n \geq 1$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i$$

$$v \cdot w = v^t \cdot w$$

Pr: $\forall a_1, a_2 \in \mathbb{R}$ $\forall v_1, v_2, w \in \mathbb{R}^n$

1) $(a_1 v_1 + a_2 v_2) \cdot w = a_1 v_1 \cdot w + a_2 v_2 \cdot w$ bilinearità

2) $v \cdot w = w \cdot v$ $\forall v, w \in \mathbb{R}^n$

3) Definito positivo:

$$v \cdot v = \sum_{i=1}^{n} x_i^2 \geq 0 \qquad v \cdot v \geq 0 \quad \forall v \in \mathbb{R}^n$$

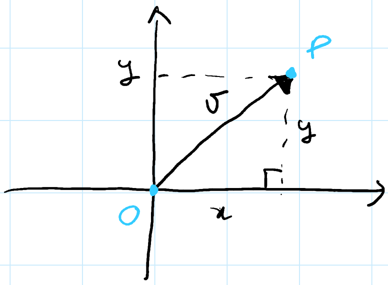
non degenere $v \cdot v = 0 \iff v = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Definizione:

$$\boxed{\|v\| = \sqrt{v \cdot v}} \qquad \|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}^{\geq 0}$$

Oss:

$\mathbb{R}^2$ $v = \begin{pmatrix} x \\ y \end{pmatrix}$ $\|v\| = \sqrt{x^2 + y^2}$



$\|v\|$ = lunghezza di OP

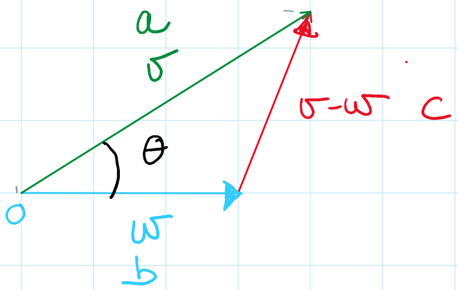
Proprietà della norma:

① $\|a v\| = |a| \, \|v\|$ $\forall a \in \mathbb{R}$ $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\|av\| = \left\| \begin{pmatrix} a x_1 \\ \vdots \\ a x_n \end{pmatrix} \right\| = \sqrt{(ax_1)^2 + (ax_2)^2 + \dots + (ax_n)^2} =$$

$$= \sqrt{a^2 (x_1^2 + \dots + x_n^2)} = |a| \sqrt{\sum_{i=1}^{n} x_i^2} = |a| \, \|v\|$$

② $\|v\| \geq 0$, $\|v\| = 0 \iff v = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$c^2 = \|v-w\|^2 = \left(\sqrt{(v-w)\cdot(v-w)}\right)^2 =$$

$$= (v-w)\cdot(v-w) = v\cdot(v-w) - w\cdot(v-w) =$$

$$= v\cdot v - v\cdot w - w\cdot v + w\cdot w =$$

$$= \|v\|^2 - 2v\cdot w + \|w\|^2 =$$

$$= \cancel{a^2} - 2v\cdot w + \cancel{b^2} \overset{?}{=} \cancel{a^2} + \cancel{b^2} - 2ab\cos\theta$$

$$v\cdot w = ab\cos\theta = \|v\|\,\|w\|\cos\theta$$

Se $v, w \neq 0_{\mathbb{R}^n}$

$$\boxed{\cos\theta := \frac{v\cdot w}{\|v\|\,\|w\|}}$$

$|\cos\theta| \leq 1$

$$\frac{|v\cdot w|}{\|v\|\,\|w\|} \leq 1$$

Disuguaglianza di Cauchy-Schwarz

$|v\cdot w| \leq \|v\|\,\|w\|$ $\quad \forall\, v, w \in \mathbb{R}^n$ e vale

$|v\cdot w| = \|v\|\,\|w\|$ se e solo i vettori v e w sono linearmente dipendenti.

Dimostrazione:

① Passo Se $v = 0_{\mathbb{R}^n}$ $\quad |v\cdot w| = \|v\|\,\|w\|$ $\qquad 0 = 0$

$0 = |0_{\mathbb{R}^n}\cdot w| = 0\cdot\|w\| = 0$

$0_{\mathbb{R}^n}$ e w sono lin. dip.

Se $w = 0_{\mathbb{R}^n}$ $\qquad 0 = 0$ e i vettori sono lin. dip.

② Passo se $w \neq 0_{\mathbb{R}^n}$ $\quad$ w e v sono lin. dipendenti se e solo

$v \neq 0_{\mathbb{R}^n}$ $\quad$ se esiste un $x \in \mathbb{R}$ tale che

$v = xw$ $\quad \boxed{v - xw = 0_{\mathbb{R}^n}}$ $\quad \boxed{z_x = v - xw}$

$\|z_x\|^2 = (v - xw)\cdot(v - xw) =$

$= v\cdot v - 2x\, v\cdot w + (xw)\cdot(xw) =$

$= \|v\|^2 - 2x\, v\cdot w + x^2\|w\|^2$

$$x^2\|w\|^2 - 2x(v\cdot w) + \|v\|^2 \geq 0 \qquad \forall\, x \in \mathbb{R}$$

$$\Rightarrow \frac{\Delta}{4} \leq 0 \qquad (v\cdot w)^2 - \|v\|^2\|w\|^2 \leq 0$$

$$\boxed{\begin{array}{c}(v\cdot w)^2 \leq \|v\|^2\,\|w\|^2 \\ |v\cdot w| \leq \|v\|\,\|w\|\end{array}}$$

$|v\cdot w| = \|v\|\,\|w\| \iff \frac{\Delta}{4} = 0$ quindi l'equazione

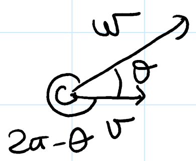
$x^2\|w\|^2 - 2x(v\cdot w) + \|v\|^2 = 0$ ha soluzione quindi

esiste $\bar{x} \in \mathbb{R}$ soluzione $\Rightarrow \|z_{\bar{x}}\|^2 = 0 \Rightarrow z_{\bar{x}} = v - \bar{x}w = 0_{\mathbb{R}^n}$

$\iff$ v e w sono lin. dipendenti. $\square$

Definizione: dati $v, w \in \mathbb{R}^n$ non nulli si

$$\cos\theta = \frac{v\cdot w}{\|v\|\,\|w\|}$$



Conseguenza

diremo che v è ortogonale a w $v \perp w$ se

$\cos\theta = 0 \iff v \cdot w = 0$

Esempio:

$\mathbb{R}^n$ $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ $e_1 \cdot e_n = \sum_{i=1}^{n} x_i y_i = 0 + \dots + 0 = 0$

$i \neq j$ $e_i \cdot e_j = 0$ $e_i \perp e_j$ $\forall i \neq j$

$\|e_i\|^2 = e_i \cdot e_i = \sum_{j=1}^{n} x_j^2 = 0 + \dots + 0 + 1 + 0 \dots + 0 = 1$ $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Definizione: una base $B = \{u_1, \dots, u_n\}$ di $\mathbb{R}^n$ si dice ortonormale se e solo $\|u_i\| = 1$ cioè u_i è versore $\forall i = 1, \dots, n$ e $\forall i \neq j$ $u_i \perp u_j$.

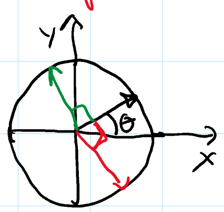
Esempio: $\mathcal{E}$ base canonica di $\mathbb{R}^n$ è base ortonormale.

Nota bene: $B = \{u_1, \dots, u_n\}$ è ortonormale $\iff$

$$u_i \cdot u_j \begin{cases} 0 & \text{se } i \neq j \\ 1 & \text{se } i = j \end{cases}$$

$1 = \|u_i\|^2 = \|u_j\|^2$

Esempio: Tutte le basi ortonormali di $\mathbb{R}^2$:



$u_1 = \begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}$ $u_2 = \begin{pmatrix} -\operatorname{sen}\theta \\ \cos\theta \end{pmatrix}$

$u_1 = \begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}$ $u_2' = -u_2 = \begin{pmatrix} \operatorname{sen}\theta \\ -\cos\theta \end{pmatrix}$

$B = \left\{ \begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}, \begin{pmatrix} -\operatorname{sen}\theta \\ \cos\theta \end{pmatrix} \right\}$ Basi ortonormali dx di $\mathbb{R}^2$

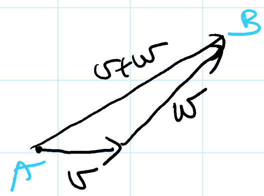
$\det T_B^{\mathcal{E}} = 1 = \cos^2\theta + \operatorname{sen}^2\theta$

$B' = \left\{ \begin{pmatrix} \cos\theta \\ \operatorname{sen}\theta \end{pmatrix}, \begin{pmatrix} \operatorname{sen}\theta \\ -\cos\theta \end{pmatrix} \right\}$ Basi ortonormali sx di $\mathbb{R}^2$

$\det T_{B'}^{\mathcal{E}} = -\cos^2\theta - \operatorname{sen}^2\theta = -1$

Disuguaglianza triangolare

$\forall\, v, w \in \mathbb{R}^n \quad \|v+w\| \leq \|v\| + \|w\|$



Dim:

$\|v+w\| \leq \|v\| + \|w\|$

$\|v+w\|^2 \leq (\|v\| + \|w\|)^2$ perché sono tutti numeri positivi o nulli

$\boxed{CS: |v \cdot w| \leq \|v\|\,\|w\|}$

$\|v+w\|^2 = (v+w)\cdot(v+w) = \|v\|^2 + 2\, v \cdot w + \|w\|^2 \leq$

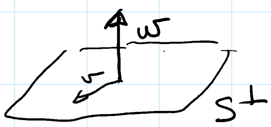
$\leq \|v\|^2 + 2\,|v\cdot w| + \|w\|^2 \underset{CS}{\leq} \|v\|^2 + 2\|v\|\,\|w\| + \|w\|^2 =$

$= (\|v\| + \|w\|)^2$

Definizione: dato $S \subseteq \mathbb{R}^n$ si dice **ortogonale di S**

$$S^\perp = \{ v \in \mathbb{R}^n \mid v \cdot w = 0 \quad \forall\, w \in S \}$$

$S = \{w\}$



Proprietà:

Sia $S \subseteq \mathbb{R}^n$:

1) $S^\perp \leqslant \mathbb{R}^n$

$(\mathbb{R}^n)^\perp = \{0_{\mathbb{R}^n}\}$ $\qquad \{0_{\mathbb{R}^n}\}^\perp = \mathbb{R}^n$

2) $S_1 \subseteq S_2 \qquad S_1^\perp \supseteq S_2^\perp$

3) Se $T = \langle w_1, \dots, w_k \rangle$ $\quad T^{\perp} = \{ v \in \mathbb{R}^n \mid v \cdot w_i = 0 \ \forall i = 1, \dots, k \}$.

Dim.:

① $S^{\perp} \leq \mathbb{R}^n$ $\qquad S^{\perp} = \{ v \in \mathbb{R}^n \mid v \cdot w = 0 \ \forall w \in S \}$

$v_1, v_2 \in S^{\perp} \quad a_1, a_2 \in \mathbb{R}$

$a_1 v_1 + a_2 v_2 \in S^{\perp} \iff (a_1 v_1 + a_2 v_2) \cdot w = 0$

$$a_1 \underbrace{v_1 \cdot w}_{0} + a_2 \underbrace{v_2 \cdot w}_{0} = a_1 \cdot 0 + a_2 \cdot 0 = 0$$

$\{0_{\mathbb{R}^n}\}^{\perp} = \{ v \in \mathbb{R}^n \mid v \cdot 0_{\mathbb{R}^n} = 0 \} = \mathbb{R}^n$

$(\mathbb{R}^n)^{\perp} = \{ v \in \mathbb{R}^n \mid v \cdot w = 0 \ \forall w \in \mathbb{R}^n \}$ $\qquad$ Se $w = v$

$\|v\|^2 = v \cdot v = 0$

$\|v\| = 0 \Rightarrow v = 0_{\mathbb{R}^n}$

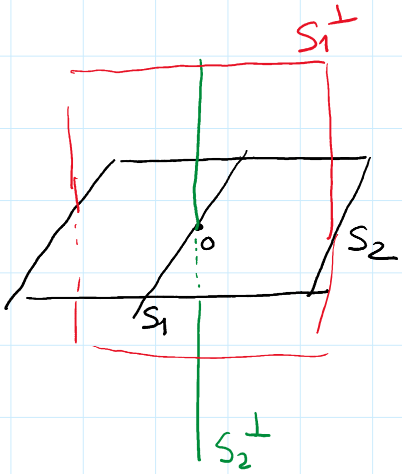
$= \{0_{\mathbb{R}^n}\}$

2) $\boxed{S_1 \subseteq S_2}$ $\qquad \boxed{S_1^{\perp} \supseteq S_2^{\perp}}$

1 $\quad$ 2

$S_1^{\perp} \supseteq S_2^{\perp}$

2 $\quad$ 1



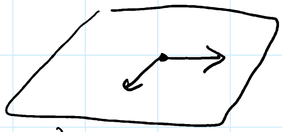
$S_1^{\perp} = \{ v \in \mathbb{R}^n \mid v \cdot w = 0 \ \forall w \in S_1 \}$

$\cup\!\mathsf{I}$

$S_2^{\perp} = \{ v \in \mathbb{R}^n \mid v \cdot w = 0 \ \forall w \in S_2 \} =$ $\qquad S_2^{\perp} \subseteq S_1^{\perp}$

$= \{ v \in \mathbb{R}^n \mid v \cdot w = 0 \ \ \forall w \in S_1, \ \forall w \in S_2 \setminus S_1 \}$

3) $T = \langle w_1, \dots, w_k \rangle \leq \mathbb{R}^n$



$$T^\perp = \{ v \in \mathbb{R}^n \mid v \cdot w = 0 \quad \forall w \in T \} = \left\{ v \in \mathbb{R}^n \,\middle|\, \begin{cases} v \cdot w_1 = 0, \\ v \cdot w_2 = 0 \\ \vdots \\ v \cdot w_k = 0 \end{cases} \right\}$$

$$\begin{cases} v \cdot w_1 = 0 \\ \vdots \\ v \cdot w_k = 0 \end{cases} \qquad v \cdot \left(\sum_{i=1}^{k} a_i w_i \right) = 0 \qquad \forall a_i \in \mathbb{R}$$

Se $\quad S \leq \mathbb{R}^n \quad (S^\perp)^\perp = S$

Esempi: $T = \langle \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{w_1} \rangle$ cerchiamo

$$T^\perp = \left\{ v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, v \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0 \right\} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x + 2y + 3z = 0 \right\} = \qquad x = -2y - 3z$$

$$= \left\{ \begin{pmatrix} -2y-3z \\ y \\ z \end{pmatrix} \,\middle|\, y, z \in \mathbb{R} \right\} = \langle \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}_{z_1}, \underbrace{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}_{z_2} \rangle \qquad \begin{matrix} z_1 \perp w_1 \\ z_2 \perp w_2 \end{matrix}$$

θ l'angolo fra z_1 e z_2

$$\cos\theta = \frac{z_1 \cdot z_2}{\|z_1\| \, \|z_2\|} = \frac{6+0+0}{\sqrt{5}\,\sqrt{10}} = \frac{6}{\sqrt{50}}$$

$\mathbb{R}^n \qquad V = \langle \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{v} \rangle \qquad v \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$$V^\perp : \; a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0 \iff \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = 0$$

$\dim V = 1 \qquad \dim V^{\perp} = n-1$

Prop: se $\{v_1, \dots, v_k\}$ sono vettori non nulli e $v_i \perp v_j$ se $i \neq j$ allora $\{v_1, \dots, v_k\}$ è linearmente indipendenti.

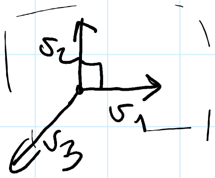
Dim:

$$\sum_{i=1}^{k} a_i v_i = 0_{\mathbb{R}^n} \qquad v_j \cdot \left(\sum_{i=1}^{k} a_i v_i \right) = v_j \cdot 0_{\mathbb{R}^n}$$

$$a_j (v_j \cdot v_j) = \sum_{i=1}^{k} a_i (v_i \cdot v_j) \qquad = 0$$

$$a_j \cancel{\|v_j\|^2} = 0 \qquad \Rightarrow a_j = 0 \qquad \forall j = 1, \dots, n.$$

ma $v_j \neq 0_{\mathbb{R}^n}$



$$\underset{v_1}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}$$

$\|v_1\| = \sqrt{3} \qquad \|v_2\| = \sqrt{2} \qquad \|v_3\| = \sqrt{}$

$v_1 \cdot v_2 = 0$

$v_1 \cdot v_3 = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot (-2) = 0$

$v_2 \cdot v_3 = 1 \cdot 1 + (-1) \cdot 1 + 0 \cdot (-2) =$
$= 1 - 1 = 0$

$$\underset{u_1}{\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}} \quad \underset{u_2}{\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}} \quad \underset{u_3}{\begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}}$$

$B = \{u_1, u_2, u_3\}$ è base ortonormale di $\mathbb{R}^3$.