Matrici di cambio di Base

V $\quad B_1 = \{v_1, \dots, v_n\}$ $\quad B_2 = \{w_1, \dots, w_n\}$ basi di V

$T^{B_2}_{B_1}$ matrice di cambio di base da B_1 a B_2

è A_{B_1, B_2, id_V}.

Esercizio:

$V = \mathbb{R}^3$ si considerino le basi

$$B_1 = \left\{ \begin{pmatrix} 5 \\ 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 8 \end{pmatrix} \right\} \qquad B_2 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right\}$$

determinare $T^{\mathcal{E}}_{B_1}$, $T^{\mathcal{E}}_{B_2}$, $T^{B_2}_{B_1}$.

Sia $v = \begin{pmatrix} 0 \\ -1 \\ 8 \end{pmatrix}$ verificare $(v)_{B_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ coordinate di v nella base B_1

e calcolare le coordinate di v nella base B_2.

Svolgimento:

$$B_1 = \left\{ \begin{pmatrix} 5 \\ 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 8 \end{pmatrix} \right\} \qquad T^{\mathcal{E}}_{B_1} = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 3 & -1 \\ 8 & 1 & 8 \end{pmatrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$B_2 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right\} \qquad T^{\mathcal{E}}_{B_2} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & -1 \\ 1 & 0 & 3 \end{pmatrix}$$

B_1 è base $\iff \det T^{\mathcal{E}}_{B_1} \neq 0$ $\quad \det T^{\mathcal{E}}_{B_1} = 5 \det \begin{pmatrix} 3 & -1 \\ 1 & 8 \end{pmatrix} = 5 \cdot 25 \overset{125}{\neq} 0$.

B_2 è base $\iff \det T^{\mathcal{E}}_{B_2} \neq 0$

$$\det \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & -1 \\ 1 & 0 & 3 \end{pmatrix} \underset{1^\circ + 2^\circ}{=} \det \begin{pmatrix} 0 & 0 & -1 \\ -1 & -1 & -1 \\ 1 & 0 & 3 \end{pmatrix} = -\det \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = -1 \neq 0$$

$$T^{B_2}_{B_1} \qquad \begin{tikzcd} & (\mathbb{R}^3, \mathcal{E}) \arrow[dr, "id"] & \\ (\mathbb{R}^3, B_1) \arrow[ur, "id"] \arrow[rr, "id"] & & (\mathbb{R}^3, B_2) \end{tikzcd} \qquad id \circ id = id$$

$$(\mathbb{R}^3, B_1) \xrightarrow{id} (\mathbb{R}^3, \mathcal{E}) \qquad T^{\mathcal{E}}_{B_1}$$

$$T^{B_2}_{\mathcal{E}} \cdot T^{\mathcal{E}}_{B_1} = T^{B_2}_{B_1}$$

$$B_1 \xrightarrow{f} \mathcal{E} \xrightarrow{g} B_2 \qquad g \circ f$$

$$\boxed{\underbrace{T^{B_2}_{\mathcal{E}}}_{g} \cdot \underbrace{T^{\mathcal{E}}_{B_1}}_{f} = T^{B_2}_{B_1}}$$

$$T^{\mathcal{E}}_{B_1} = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 3 & -1 \\ 8 & 1 & 8 \end{pmatrix}$$

$$T^{B_2}_{\mathcal{E}} = \left(T^{\mathcal{E}}_{B_2}\right)^{-1}$$ andiamo a calcolarla

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} 2^\circ + 1^\circ \\ 3^\circ - 1^\circ \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} \cdot -1 \\ \cdot -1 \end{array}$$ le scambio

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right) 2^\circ + 3 \cdot 3^\circ \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -3 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right) 1^\circ R - 2^\circ R$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & -2 & -3 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right) \qquad \boxed{T^{B_2}_{\mathcal{E}} = \begin{pmatrix} 3 & 3 & 1 \\ -2 & -3 & -1 \\ -1 & -1 & 0 \end{pmatrix}}$$

Per cosa $\begin{pmatrix} 3 & 3 & 1 \\ -2 & -3 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & -1 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\boxed{T^{B_2}_{B_1}} = T^{B_2}_{\mathcal{E}} \, T^{\mathcal{E}}_{B_1} = \begin{pmatrix} 3 & 3 & 1 \\ -2 & -3 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 1 & 3 & -1 \\ 8 & 1 & 8 \end{pmatrix} =$$

$$= \boxed{\begin{pmatrix} 26 & 10 & 5 \\ -21 & -10 & -5 \\ -6 & -3 & 1 \end{pmatrix}}$$

$-2 \cdot 5 - 3 \cdot 1 - 1 \cdot 8 =$

$= -10 - 3 - 8 = -21$

$$B_1 = \left\{ \begin{pmatrix} 5 \\ 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 8 \end{pmatrix} \right\}$$

$v = \begin{pmatrix} 0 \\ -1 \\ 8 \end{pmatrix} \qquad (v)_{B_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad v = 0 \cdot \begin{pmatrix} 5 \\ 1 \\ 8 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ -1 \\ 8 \end{pmatrix} =$

$= \begin{pmatrix} 0 \\ -1 \\ 8 \end{pmatrix}$ sì

$$\boxed{(v)_{B_2} = T_{B_1}^{B_2} (v)_{B_1}} = \begin{pmatrix} 26 & 10 & 5 \\ -21 & -10 & -5 \\ -6 & -3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}$$

$\boxed{(v)_{B_2} = \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}}$ Verifica $B_2 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right\}$

$$v = 5 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - 5 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5-5+0 \\ -5+5-1 \\ 5-0+3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 8 \end{pmatrix}$$

$$v = a \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \qquad (v)_{B_2} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Uso delle matrici di cambio di base

Esercizio: $f: \mathbb{R}^3 \longrightarrow M_{2,2}(\mathbb{R})$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2z & y+z \\ y-2z & x+5z \end{pmatrix}$$

1) Determinare la matrice associata ad f rispetto alle basi $\underline{B} = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ e

$$\underline{\mathcal{C}} = \left\{ \overset{v_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{v_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \overset{v_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \right\}$$

$\underline{T_B^{\mathcal{E}}} \qquad \underline{T_{\mathcal{E}_M}^{\mathcal{C}}}$

$\mathcal{E} = \{e_1, e_2, e_3\}$ base canonica di $\mathbb{R}^3$

$\mathcal{E}_M = \left\{ v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

2) $B_{\ker f}$ $\quad B_{\operatorname{Im} f}$

3) Determinare, se esiste,

$g: M_{2,2}(\mathbb{R}) \longrightarrow \mathbb{R}^3$ tale che $g \circ f = id_{\mathbb{R}^3}$

4) Determinare, se esiste, $h: M_{2,2}(\mathbb{R}) \longrightarrow \mathbb{R}^3$ tale che $f \circ h = id_{M_{2,2}(\mathbb{R})}$.

Svolgimento:

$$f: \underset{\substack{\uparrow \\ 3}}{\mathbb{R}^3} \longrightarrow \underset{\substack{\uparrow \\ 4}}{M_{2,2}(\mathbb{R})}$$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2z & y+z \\ y-2z & x+5z \end{pmatrix}$$

$$= \begin{pmatrix} x+2z \\ y+z \\ y-2z \\ x+5z \end{pmatrix} \qquad M_{4,3}(\mathbb{R})$$

$$A_{\mathcal{E},\mathcal{E}_M,f} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 5 \end{pmatrix} \in M_{4,3}(\mathbb{R})$$

$$T_B^{\mathcal{E}} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} = B$$

$$T_{\mathcal{E}_M}^{\mathcal{C}} = \left(T_{\mathcal{C}}^{\mathcal{E}_M}\right)^{-1} \qquad \{\sigma_2, \sigma_1, \sigma_4, \sigma_3\} = \mathcal{C}$$

$$\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \quad \mathcal{E}_M$$

$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$T_{\mathcal{C}}^{\mathcal{E}_M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\boxed{T_{\mathcal{E}_M}^{\mathcal{C}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{array}{ccc} \mathbb{R}^3, \mathcal{E} & \xrightarrow{\ f\ } & M_{2,2}(\mathbb{R}), \mathcal{E}_M \\ \uparrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ \mathbb{R}^3, B & \xrightarrow{\ f\ } & M_{2,2}(\mathbb{R}), \mathcal{C} \end{array}$$

$$T^{\mathcal{C}}_{\mathcal{E}_M} \; A_{\mathcal{E},\mathcal{E}_M,f} \; T^{\mathcal{E}}_{B} =$$

$4\times4 \qquad 4\times3 \qquad 3\times$

$$= \left[\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 5 \end{pmatrix}\right]\begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 5 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & 1 & 0 \\ 3 & 3 & 1 \\ 6 & 6 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

2) $\mathrm{Ker} f$ $\qquad$ " $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ "

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2z \\ y+z \\ y-2z \\ x+5z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x+2z=0 \\ y+z=0 \\ y-2z=0 \\ x+5z=0 \end{cases} \qquad \begin{cases} x=0 \\ z=0 \\ y=0 \\ x=0 \end{cases} \implies \mathrm{Ker} f = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\} = \{0_{\mathbb{R}^3}\}$$

$$\dim \mathrm{Ker} f = 0 \qquad B_{\mathrm{Ker} f} = \emptyset$$

$$\dim V = \dim \mathrm{Ker} f + \dim \mathrm{Im} f$$

$$3 = 0 + \dim \mathrm{Im} f \implies \dim \mathrm{Im} f = 3$$

è iniettiva perché $\mathrm{Ker} f = \{0_{\mathbb{R}^3}\}$

non è suriettiva perché $\dim \mathrm{Im} f = 3 \neq 4$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2z \\ y+z \\ y-2z \\ x+5z \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 5 \end{pmatrix}$$

$$\mathrm{Im} f \leq M_{2,2}(\mathbb{R}) \qquad B_{\mathrm{Im} f} = \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{f(e_1)}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{f(e_2)}, \underbrace{\begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix}}_{f(e_3)} \right\}$$

3) Determinare, se esiste,

$g: M_{2,2}(\mathbb{R}) \longrightarrow \mathbb{R}^3$ tale che $\boxed{g \circ f = id_{\mathbb{R}^3}}$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "f"] \arrow[rr, bend right, "id_{\mathbb{R}^3}"'] & M_{2,2}(\mathbb{R}) \arrow[r, "g"] & \mathbb{R}^3 \end{tikzcd}$$

$g \circ f(e_1) = e_1 \qquad g(f(e_1)) = g\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$g \circ f(e_2) = e_2 \qquad g(f(e_2)) = g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$g \circ f(e_3) = e_3 \qquad g(f(e_3)) = g\begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Esiste $g : \underset{V}{M_{2,2}(\mathbb{R})} \longrightarrow \underset{W}{\mathbb{R}^3}$ tale che

$$g\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{v_1} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{w_1} \qquad g\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{v_2} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{w_2} \qquad g\underbrace{\begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix}}_{v_3} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{w_3} \ ?$$

$$B_{M_{2,2}(\mathbb{R})} = \left\{ \underset{v_1}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 2 & 1 \\ -2 & 5 \end{pmatrix}}, \underset{v_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$$

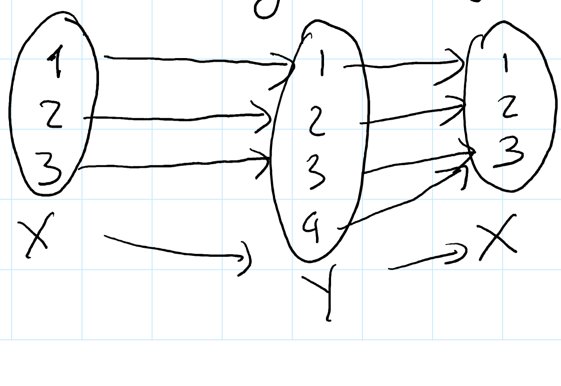
$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & -1 & 0 \\ & 1 & 2 & 5 \end{pmatrix} \quad 3^\circ - 2 \cdot 1^\circ \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & 3 \end{pmatrix} \quad 3^\circ - 2^\circ \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{g\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{v_4} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{w_4}}$$

esiste un'unica applicazione lineare tale che $g(v_i) = w_i \ \forall i = 1, \dots, 4$

e la funzione g verifica $g \circ f = id_{\mathbb{R}^3}$

Questa g non è unica perché se cambiamo $g(v_4)$ verifica ancora $g \circ f = id_{\mathbb{R}^3}$.



4) Esiste $h: M_{2,2}(\mathbb{R}) \longrightarrow \mathbb{R}^3$ tale che $\boxed{f \circ h = id_{M_{2,2}(\mathbb{R})}}$? No

$f \circ h(v) = v \quad \forall v \in M_{2,2}(\mathbb{R})$

$\boxed{f(h(v)) = v} \quad \forall v \in M_{2,2}(\mathbb{R})$

$$f: \mathbb{R}^3 \longrightarrow M_{2,2}(\mathbb{R})$$
$$h(v) \longmapsto v$$

se esistesse $z = h(v)$ avremmo che
$\forall v \in M_{2,2}(\mathbb{R})\ \exists z \in \mathbb{R}^3 : f(z) = v$ cioè f suriettiva, ma f non era suriettiva

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Esercizio: $V = \mathbb{R}^3 \quad B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} \right\}$

$\mathcal{E}$ base canonica di $\mathbb{R}^3$ e f, g endomorfismi di $\mathbb{R}^3$

$$A_{B,\mathcal{E},f} = \begin{pmatrix} 0 & 2 & 1 \\ 3 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} = A_1 \qquad (\mathbb{R}^3, B) \xrightarrow[A_1]{f} (\mathbb{R}^3, \mathcal{E})$$

$$A_{\mathcal{E},B,g} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 4 & 1 & -5 \end{pmatrix} = A_2 \qquad (\mathbb{R}^3, \mathcal{E}) \xrightarrow[A_2]{g} (\mathbb{R}^3, B)$$

determinare $A_{\mathcal{E},\mathcal{E},f\circ g} \quad A_{\mathcal{E},\mathcal{E},g\circ f}$.

Svolg:
$$(\mathbb{R}^3, \mathcal{E}) \xrightarrow{g} (\mathbb{R}^3, B) \xrightarrow{f} (\mathbb{R}^3, \mathcal{E})$$

$$\begin{tikzcd} (\mathbb{R}^3, B) \arrow[r, "f", "A_1"'] & (\mathbb{R}^3, \mathcal{E}) \\ (\mathbb{R}^3, \mathcal{E}) \arrow[u, "A_2", "g"'] \arrow[ur, "f\circ g"'] & \end{tikzcd}$$

$$A_{B,\mathcal{E},f} \cdot A_{\mathcal{E},B,g} = A_{\mathcal{E},\mathcal{E},f\circ g}$$

$$\boxed{A_1 \cdot A_2 = A_{\mathcal{E},\mathcal{E},f\circ g}}$$

$$(\mathbb{R}^3, B) \xrightarrow{f} (\mathbb{R}^3, \mathcal{E})$$

$$g \circ f$$

$$(\mathbb{R}^3, \mathcal{E}) \xrightarrow{g} (\mathbb{R}^3, B)$$

$$A_2 \cdot A_1$$

$$(\mathbb{R}^3, B) \xrightarrow[A_1]{f} (\mathbb{R}^3, \mathcal{E}) \xrightarrow[A_2]{g} (\mathbb{R}^3, B)$$

$$A_2 \cdot A_1 = A_{B,B,g\circ f}$$

$$T_B^{\mathcal{E}} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 5 & 0 & 6 \end{pmatrix} = H$$

$$T_B^{\mathcal{E}} \; A_2 \cdot A_1 \; T_{\mathcal{E}}^{B}$$

$$H \; A_2 A_1 \; H^{-1}$$

Definizione: due matrici quadrate $A, B \in M_{n,n}(\mathbb{R})$ si dicono **simili** se esiste una matrice invertibile $H \in GL_n(\mathbb{R})$ tale che

$$\boxed{H^{-1} A H = B}$$

$A \sim B$ $\qquad\qquad$ $\sim$ relazione di equivalenza

$A \sim A$

$A \sim B \iff B \sim A$

$A \sim B$ e $B \sim C \Rightarrow A \sim C$