Lezione20

martedì 12 novembre 2024 13:56

Determinante

$$A=\begin{pmatrix} 2 & 1 & 5 & 1 \\ 8 & 0 & 1 & 0 \\ 3 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ calcolare det A

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$= -1 \det\begin{pmatrix} \boxed{\begin{matrix} 8 & 1 \\ 3 & -3 \end{matrix}} & \begin{matrix} 0 \\ 1 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} =$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$= -1\left[1(-24-3)\right] = 27$$

Teorema: il determinante non dipende dalla riga o dalla colonna usata per calcolarlo.

Esempi: matrici triangolari

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} = 1 \det\begin{pmatrix} 4 & 5 \\ 0 & 6 \end{pmatrix} =$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$= 1 \cdot 4 \cdot 6$$

$$\det\begin{pmatrix} 1 & 5 & 100 & 0 \\ 0 & -1 & 201 & 101 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -1$$

Osservazione: se $A \in M_{n,n}(\mathbb{R})$ $A^t = M_{n,n}(\mathbb{R})$

$$\boxed{\det(A) = \det(A^t)}$$

$$\det\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 8 & 100 & -3 & 1 \end{pmatrix} = 5 \cdot 2 \cdot 1 \cdot 1$$

$$\begin{array}{cc} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \\ v_1 \quad v_2 \end{array}$$

Proprietà:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ molt. per $a \in \mathbb{R}$ 1° riga

$\det A = 1$

$$A_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ $\det A_1 = a$

$$\det \begin{pmatrix} 2 & 8 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} = 2 \det \begin{pmatrix} 1 & 4 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= 4 \det \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 4$$

$$A_2 = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$$ 1° + a 3° riga

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$= 1 \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$$

Teorema di Binet

$\det(AB) = \det(A)\det(B)$ $\qquad A, B \in M_{n,n}(\mathbb{R})$

$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix} \quad \begin{matrix} 2^\circ R - 1^\circ R \\ 3^\circ R - 1^\circ R \\ 4^\circ R - 1^\circ R \end{matrix}$$

$$= \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = 6$$

Esercizio:

Per quali valori di $k \in \mathbb{R}$ l'endomorfismo di $\mathbb{R}^3$

$$f_k\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} kx - y + kz \\ x - 2y \\ y - kz \end{pmatrix}$$ è invertibile?

Dim: $A_{\mathcal{E},\mathcal{E},f_k} = \begin{pmatrix} k & -1 & k \\ 1 & -2 & 0 \\ 0 & 1 & -k \end{pmatrix} = A_k \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

f_k è invertibile $\iff \det A_k \neq 0$

$$\det A_k = k \det\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} - 0 - k\det\begin{pmatrix} k & -1 \\ 1 & -2 \end{pmatrix} =$$

$$= k - k(-2k+1) =$$

$$= k + 2k^2 - k = 2k^2 \neq 0 \iff k \neq 0$$

Come usare il determinante per calcolare le inverse

$A \in GL_n(\mathbb{R}) = \{ A \in M_{n,n}(\mathbb{R}) \mid \det A \neq 0 \}$ cioè invertibili

Costruiamo una matrice $S = (S_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}}$

$$S_{ij} = (-1)^{i+j} \det(A_{ij})$$

$B = S^t \qquad AB = \det(A)\, I_n$

$$\boxed{A^{-1} = \frac{1}{\det(A)} B}$$

Esempio: $A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ $\qquad \det A = 1 \det\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = 1$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$S_{ij} = (-1)^{i+j} \det A_{ij}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$S_{11} = + \det\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 0$ $\qquad S_{21} = - \det\begin{pmatrix} 3 & 5 \\ 0 & 0 \end{pmatrix} = 0$

$S_{12} = - \det\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = 2$ $\qquad S_{22} = + \det\begin{pmatrix} 1 & 5 \\ 1 & 0 \end{pmatrix} = -5$

$S_{13} = + \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$ $\qquad S_{23} = - \det\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} = 3$

$S_{31} = + \det\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = 1$

$S_{32} = - \det\begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix} = -2$ $\qquad S = \begin{pmatrix} 0 & 2 & -1 \\ 0 & -5 & 3 \\ 1 & -2 & 1 \end{pmatrix}$

$S_{33} = + \det\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = 1$

$$B = S^t = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -5 & -2 \\ -1 & 3 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A} B \underset{\det A = 1}{=} B = \begin{pmatrix} 0 & 0 & 1 \\ 2 & -5 & -2 \\ -1 & 3 & 1 \end{pmatrix}$$

Verifica $A \cdot A^{-1} = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 2 & -5 & -2 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I}_3$

Formule di Cramer

Risolvere il sistema lineare

$$\begin{cases} ax + by = \alpha \\ cx + dy = \beta \end{cases} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$A\underline{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

ammette un'unica soluzione

$\iff \operatorname{rg} A = 2 \iff \det A \neq 0 \quad ad - bc \neq 0$

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad A^{-1} \cancel{A} \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}} =$$

$$= \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d\alpha - b\beta \\ -c\alpha + a\beta \end{pmatrix} =$$

$$= \begin{pmatrix} \dfrac{\begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \\[3ex] \dfrac{\begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \end{pmatrix}$$

con il simbolo $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$$\begin{cases} \underline{x} = \dfrac{\begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \boxed{\dfrac{d\alpha - b\beta}{ad - bc}} \\[4ex] \underline{y} = \dfrac{\begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \boxed{\dfrac{a\beta - c\alpha}{ad - bc}} \end{cases}$$

Esempio: risolvere con Cramer

$$\begin{cases} 2x - 3y = 1 \\ x + 5y = 0 \end{cases} \qquad A = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix} \qquad \det A = 10+3 = 13$$

$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{13} \begin{pmatrix} \begin{vmatrix} 1 & -3 \\ 0 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 5/13 \\ -1/13 \end{pmatrix}$$

$$\begin{cases} 2 \cdot \frac{5}{13} - 3 \cdot \left(-\frac{1}{13}\right) = 1 \\ \frac{5}{13} + 5\left(-\frac{1}{13}\right) = 0 \end{cases}$$

Matrici di cambio di base.

Sia V un $\mathbb{R}$-spazio vettoriale di $\dim V = n$ e B_1, B_2 due basi di V.

La matrice

$$A_{B_1, B_2, id_V} = T_{B_1}^{B_2}$$

matrice di cambio di base dalla base B_1 alla base B_2.

$v \in V$ con coordinate di v $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ rispetto alla base B_1

il vettore v ha coordinate $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = T_{B_1}^{B_2} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ nella base B_2.

Esempi:

$V = \mathbb{R}^3 \quad B_1 = \left\{ \begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{pmatrix}1\\1\\1\end{pmatrix}, & \begin{pmatrix}0\\2\\0\end{pmatrix}, & \begin{pmatrix}1\\0\\-1\end{pmatrix} \end{array} \right\} \quad \det\begin{vmatrix}1&0&1\\1&2&0\\1&0&-1\end{vmatrix} = 2(-2) = -4 \neq 0$

$B_2 = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} = \mathcal{E}$

$$T^{B_2}_{B_1} = T^{\mathcal{E}}_{B_1} = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix}1&0&1\\1&2&0\\1&0&-1\end{pmatrix} \end{array}$$

$$\boxed{T^{B_2}_{B_1} \, T^{B_1}_{B_2} = \mathbb{I}_n}$$

$$\left(\begin{array}{ccc|ccc}1&0&1&1&0&0\\1&2&0&0&1&0\\1&0&-1&0&0&1\end{array}\right)$$

Ridurre con Gauss

$$\left(\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&2&-1&-1&1&0\\0&0&-2&-1&0&1\end{array}\right)$$

$$\left(\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&-1/2&-1/2&1/2&0\\0&0&1&1/2&0&-1/2\end{array}\right)$$

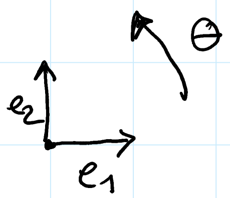
$$\left(\begin{array}{ccc|ccc}1&0&0&1/2&0&1/2\\0&1&0&-1/4&1/2&-1/4\\0&0&1&1/2&0&-1/2\end{array}\right)$$

$$T^{B_1}_{\mathcal{E}} = \left(T^{\mathcal{E}}_{B_1}\right)^{-1}$$

$$= \begin{pmatrix}1/2&0&1/2\\-1/4&1/2&-1/4\\1/2&0&-1/2\end{pmatrix}$$

Esempio:

Se $V = \mathbb{R}^2$



$$R_\theta = \begin{array}{c} \begin{pmatrix}\cos\theta & -\operatorname{sen}\theta\\ \operatorname{sen}\theta & \cos\theta\end{pmatrix} \\ \begin{array}{cc} v_1 & v_2 \end{array} \end{array} = A_{\mathcal{E},\mathcal{E},\, \rho_\theta}$$

Se $T^{\mathcal{E}}_{B} = \begin{pmatrix}\cos\theta & -\operatorname{sen}\theta\\ \operatorname{sen}\theta & \cos\theta\end{pmatrix} \Rightarrow B = \left\{ \begin{array}{cc} \begin{pmatrix}\cos\theta\\ \operatorname{sen}\theta\end{pmatrix}, & \begin{pmatrix}-\operatorname{sen}\theta\\ \cos\theta\end{pmatrix} \\ v_1 & v_2 \end{array} \right\}$

Notiamo $A \in GL_n(\mathbb{R})$ può essere sempre vista

come $A_{\mathcal{E},\mathcal{E},f}$ $\quad f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$A = (v_1 \dots v_n)$

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$f(e_1) = v_1$
$\vdots$
$f(e_n) = v_n$

oppure

$$A = T^{\mathcal{E}}_{B} \qquad B = \{v_1, \dots, v_n\}$$

Scopo:

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad A \in M_{n,n}(\mathbb{R})$$

$$A = A_{\mathcal{E},\mathcal{E},f}$$

$$A = A_{B,B,f}$$

Più in generale $f: V \longrightarrow W$

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad A_{\mathcal{E}_n,\mathcal{E}_m,f}$$

$$A_{B_1,B_2,f} \ ?$$

Proposizione:

$f: V \longrightarrow W$ siano B_1 e B_2 basi di V

$\mathcal{C}_1$ e $\mathcal{C}_2$ basi di W

e_1 e e_2 basi di W

$$\begin{tikzcd} V_{B_1} \arrow[r, "f"] & W_{e_1} \arrow[d] \\ V_{B_2} \arrow[u] \arrow[r, "f"] & W_{e_2} \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, "f"] & W \arrow[d, "id_W"] \\ V \arrow[u, "id_V"] \arrow[r, "f"'] & W \end{tikzcd}$$

$$id_W \circ f \circ id_V = f$$

$$T_W \; A \; T_V = B$$

$$B = A_{B_2, e_2, f}$$

$$T_V = T_{B_2}^{B_1} \qquad A = A_{B_1, e_1, f}$$

$$T_W = T_{e_1}^{e_2}$$

$$\boxed{T_{e_1}^{e_2} \, A_{B_1, e_1, f} \, T_{B_2}^{B_1} = A_{B_2, e_2, f}}$$

$$\begin{tikzcd} V_{B_1} \arrow[r, "f"] & W_{e_1} \arrow[d, "id_W"] \\ V_{B_2} \arrow[u, "id_V"] \arrow[r, "f"] & W_{e_2} \end{tikzcd}$$