Esercizio:

1) determinare i valori del parametro $a \in \mathbb{R}$ tale che esista un'applicazione lineare

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 :$$

$$f\underbrace{\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}}_{v_1} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{w_1}, \quad f\underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}_{v_2} = \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{w_2}, \quad f\underbrace{\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}}_{v_3} = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{w_3}$$

$$f\underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{v_4} = \underbrace{\begin{pmatrix} 10 \\ a \end{pmatrix}}_{w_4}.$$

2) Dimostrare che $B = \left\{ \underbrace{\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}}_{v_3} \right\}$ è base di $\mathbb{R}^3$, $B' = \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{w_1}, \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{w_2} \right\}$ è base di $\mathbb{R}^2$ e calcolare la matrice associata $A_{B,B',f}$.

3) Determinare $B_{\mathrm{Ker} f}$ e $B_{\mathrm{Im} f}$.

4) Esistono una base B_1 di $\mathbb{R}^3$ e una base B_2 di $\mathbb{R}^2$ tali che $A_{B_1,B_2,f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$?

Svolgimento:

1)
$$\left(\begin{array}{c|c} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \\ v_4 & w_4 \end{array}\right) \qquad \begin{array}{c} \\ \\ \\ \\ \end{array}\left(\begin{array}{ccc|cc} 2 & 3 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 \\ 2 & 2 & -1 & 2 & 1 \\ 1 & 2 & 3 & 10 & a \end{array}\right) \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$$\begin{array}{c} R_2 \\ R_1 \\ R_3 \\ R_4 \end{array} \left(\begin{array}{ccc|cc} 1 & 1 & -1 & -1 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ 2 & 2 & -1 & 2 & 1 \\ 1 & 2 & 3 & 10 & a \end{array}\right) \qquad \begin{array}{c} R_2 \\ R_1 - 2R_2 \\ R_3 - 2R_2 \\ R_4 - R_2 \end{array} \left(\begin{array}{ccc|cc} 1 & 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 1 & 4 & 11 & a \end{array}\right)$$

$4^{\circ} - 2^{\circ}$

$$\begin{array}{l} R_2 \\ R_1-2R_2 \\ R_3-2R_2 \\ \underbrace{R_4-R_2-R_1+2R_2}_{= R_4+R_2-R_1} \end{array} \qquad \left(\begin{array}{ccc|cc} 1 & 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 8 & a-1 \end{array}\right) \qquad 4^\circ - 2\cdot 3^\circ$$

$$\left[\begin{array}{l} R_2 \\ R_1-2R_2 \\ R_3-2R_2 \end{array}\right. \qquad \left(\begin{array}{ccc|cc} 1 & 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & a-3 \end{array}\right) \qquad \begin{array}{ccc} -1 & 0 & z_1 \\ 3 & 1 & z_2 \\ 4 & 1 & z_3 \end{array}$$

$$R_4+R_2-R_1-2R_3+4R_2$$

$$z_2 - z_3 = z_1$$

Condizione di esistenza di f è $a-3=0$ perché $f\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

La funzione richiesta esiste solo se $a=3$, essendo su una base tale funzione è unica.

$$\left(\begin{array}{ccc|cc} 1 & 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 & 1 \end{array}\right) \qquad \begin{array}{l} 1^\circ + 3^\circ \\ 2^\circ - 2\cdot 3^\circ \end{array} \qquad \boxed{a=3}$$

$$\left(\begin{array}{ccc|cc} 1 & 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & -5 & -1 \\ 0 & 0 & 1 & 4 & 1 \end{array}\right) \quad 1^\circ - 2^\circ \quad \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 8 & 2 \\ 0 & 1 & 0 & -5 & -1 \\ 0 & 0 & 1 & 4 & 1 \end{array}\right)$$

$$\boxed{f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix} 8 & -5 & 4 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix} 8x-5y+4z \\ 2x-y+z \end{pmatrix}}$$

$$f\begin{pmatrix}2\\3\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \qquad x=2 \quad y=3 \quad z=0 \qquad f\begin{pmatrix}2\\3\\0\end{pmatrix} = \begin{pmatrix} 8\cdot 2 - 5\cdot 3 + 4\cdot 0 \\ 2\cdot 2 - 3 + 0 \end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

2) $B=\{v_1, v_2, v_3\}$ è base di $\mathbb{R}^3$

La riduzione a scala di

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \text{è} \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$ che ha rango 3

quindi v_1, v_2, v_3 generano $\mathbb{R}^3$ perciò sono anche lin. ind.

quindi sono una base di $\mathbb{R}^3$.

$B' = \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{w_1}, \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{w_2} \right\}$ è base di $\mathbb{R}^2$ perché sono due vettori lin. ind. - in quanto non multipli fra loro.

Calcoliamo $A = A_{B, B', f}$ $\qquad B=\{v_1, v_2, v_3\}$

$B' = \{w_1, w_2\}$

$f(v_1) = w_1 \qquad f(v_2) = w_2 \qquad f(v_3) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad a=3$

$f(v_1) = w_1 = x w_1 + y w_2 = 1 \cdot w_1 + 0 \cdot w_2 \qquad A = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & d \end{pmatrix}$

$f(v_2) = w_2 = 0 \cdot w_1 + 1 w_2$

$f(v_3) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = c w_1 + d w_2 = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} c-d \\ c \end{pmatrix}$

$\begin{cases} c-d=2 \\ c=1 \end{cases} \qquad \begin{cases} d=c-2=1-2=-1 \\ c=1 \end{cases} \qquad \boxed{A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}}$

3) Essendo $\dim \operatorname{Im} f = \operatorname{rg} A = 2 \qquad \operatorname{Im} f \subseteq \mathbb{R}^2$ con $\dim \operatorname{Im} f = 2$

$\Rightarrow \operatorname{Im} f = \mathbb{R}^2$ cioè f è suriettiva $\quad B_{\operatorname{Im} f} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\dim \mathbb{R}^3 = 3 = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$

$= \dim \operatorname{Ker} f + 2 \quad \Rightarrow \quad \boxed{\dim \operatorname{Ker} f = 1}$

$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}8 & -5 & 4\\2 & -1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\-\end{pmatrix} = \begin{pmatrix}8x-5y+4z\\2x-y+z\end{pmatrix}$$

$$\mathrm{Ker} f = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix}8x-5y+4z=0\\2x-y+z=0\end{matrix} \right\}$$

$$\left(\begin{array}{ccc|c}2 & -1 & 1 & 0\\8 & -5 & 4 & 0\end{array}\right) \quad 2^\circ - 4 \, 1^\circ \quad \left(\begin{array}{ccc|c}2 & -1 & 1 & 0\\0 & -1 & 0 & 0\end{array}\right) \quad \begin{cases}2x+z=0\\y=0\end{cases}$$

$$\begin{cases}z=-2x\\y=0\end{cases} \qquad \mathrm{Ker} f = \left\{ \begin{pmatrix}x\\0\\-2x\end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\} = \left\langle \begin{pmatrix}1\\0\\-2\end{pmatrix} \right\rangle$$

$$B_{\mathrm{Ker} f} = \left\{ \begin{pmatrix}1\\0\\-2\end{pmatrix} \right\}$$

$$f\begin{pmatrix}2\\3\\0\end{pmatrix} = \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{w_1} \qquad f\begin{pmatrix}1\\1\\-1\end{pmatrix} = \underbrace{\begin{pmatrix}-1\\0\end{pmatrix}}_{w_2} \qquad f\begin{pmatrix}2\\2\\-1\end{pmatrix} = \underbrace{\begin{pmatrix}2\\1\end{pmatrix}}_{w_3}$$

Scrivere una relazione di dipendenza fra w_1, w_2, w_3

$$a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}-1\\0\end{pmatrix} + c\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{cases}a-b+2c=0\\a+c=0\end{cases} \qquad \begin{cases}a-b-2a=0\\c=-a\end{cases} \qquad \begin{cases}b=-a\\c=-a\end{cases}$$

$a=1$
$b=-1$
$c=-1$

$$1\cdot w_1 - 1\cdot w_2 - 1 w_3 = \begin{pmatrix}0\\0\end{pmatrix}$$

$$f(v_1) - f(v_2) - f(v_3) = \begin{pmatrix}0\\0\end{pmatrix}$$

$$f(v_1-v_2-v_3) = \begin{pmatrix}0\\0\end{pmatrix} \iff v_1-v_2-v_3 \in \mathrm{Ker} f$$

$$\begin{pmatrix}2\\3\\0\end{pmatrix} - \begin{pmatrix}1\\1\\-1\end{pmatrix} - \begin{pmatrix}2\\2\\-1\end{pmatrix} = \begin{pmatrix}\cancel{2}-1-\cancel{2}\\\cancel{3}-1-\cancel{2}\\0+1+1\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix} \in \mathrm{Ker} f.$$

9) Determinare B_1 di $\mathbb{R}^3$ B_2 di $\mathbb{R}^2$ tali che

$$A_{B_1,B_2,f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$B_1 = \{v_1, v_2, z\}$

$B_2 = \{w_1, w_2\}$

$f(v_1) = w_1 = 1 \cdot w_1 + 0 \cdot w_2$

$f(z) = 0 \cdot w_1 + 0 \cdot w_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad z \in \operatorname{Ker} f = \left\langle \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right\rangle$

$B_1 = \left\{ v_1, v_2, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right\}$ Verificate che è base

$B_2 = \{w_1, w_2\}$

$f(v_1) = w_1 = 1 \cdot w_1 + 0 \cdot w_2$

$f(v_2) = w_2 = 0 \cdot w_1 + 1 \cdot w_2$

$f\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot w_1 + 0 \cdot w_2$

$$A_{B_1,B_2,f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

2×3

Proposizione: Sia data $f: V \longrightarrow W$ lineare con $\dim V = n$, $\dim W = m$ $\dim \operatorname{Im} f = r$, allora esistono sempre

$B_V = \{v_1, \dots, v_n\}$ base di V

$B_W = \{w_1, \dots, w_m\}$ base di W tali che

$$A_{B_V,B_W,f} = \begin{array}{c} \overbrace{\qquad\qquad}^{r\text{-colonne}} \quad r+1 \quad n \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots & \vdots & & \vdots \\ \vdots & 0 & \ddots & & & & \\ \vdots & \vdots & & 1 & & & \\ & & & 0 & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} \in M_{m,n}(\mathbb{R}).$$

Dim:

$f(v_1) = 1 \cdot w_1 + 0 \cdot w_2 + \cdots + 0 w_m = w_1$

$f(v_2) = 0 \cdot w_1 + 1 \cdot w_2 + 0 \cdot w_3 + \cdots + 0 w_m = w_2$

$\vdots$

$f(v_r) = w_r$

$f(v_i) = w_i$

$\forall i = 1, \dots, r$

$r = \dim \operatorname{Im} f \qquad \begin{cases} r \leq m \\ r \leq n \end{cases}$

$f(v_{r+1}) = 0 \cdot w_1 + \dots + 0 \cdot w_m = 0_W$

$\vdots$

$f(v_n) = 0 \cdot w_1 + \dots + 0 w_m = 0_W$

$B_V = \{v_1, \dots, v_n\}$ $\qquad \boxed{f(v_i) = w_i \quad \forall i = 1, \dots, r}$

$B_W = \{w_1, \dots, w_m\}$ $\qquad \boxed{f(v_j) = 0_W \quad \forall j = r+1, \dots, n.}$

$v_j \in \operatorname{Ker} f \quad \forall j = \underline{r+1, \dots, n} \qquad n-r$ vettori

$\dim V = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$

$\underline{n = \dim \operatorname{Ker} f + r} \qquad \underline{\dim \operatorname{Ker} f = n - r}$

$\{v_{r+1}, \dots, v_n\}$ sia una base di $\operatorname{Ker} f$ e completiamo a base di V

$\{v_1, \dots, v_r, \boxed{v_{r+1}, \dots, v_n}\} = B_V$

Definiamo

$\boxed{w_i = f(v_i)} \qquad \underline{i = 1, \dots, r}$ per il teorema delle dimensioni sono lin. indip. Completiamo a base di W

$B_W = \{\boxed{w_1, \dots, w_r}, w_{r+1}, \dots, w_m\}$

Essendo $f(v_j) = 0_W \quad \forall j = r+1, \dots, n$ le ultime $n-r$ colonne di $A = A_{B_V, B_W, f}$ sono nulle

Essendo $f(v_i) = w_i \quad \forall i = 1, \dots, r$ la colonna A_i con

$i = 1, \dots, r \qquad A_i = e_i \in \mathbb{R}^m \qquad A = \begin{pmatrix} 1 & 0 & & 0 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots & \vdots & & \vdots \\ \vdots & 0 & \ddots & 0 & & & \\ & \vdots & & 1 & & & \\ 0 & 0 & & 0 & 0 & & 0 \end{pmatrix} \; m \times n.$

Esercizio: sia $f_a : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ tale che la matrice associata a f_a rispetto alle basi canoniche sia

$$A_a = \begin{pmatrix} 0 & 1 & a-1 \\ 1-a & -1 & 0 \\ 2-2a & 2a & 0 \end{pmatrix}$$

1) Per ogni $a \in \mathbb{R}$ calcolare $f_a^{-1}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$.

2) Per ogni $a \in \mathbb{R}$ determinare $B_{\mathrm{Ker} f_a}$, $B_{\mathrm{Im} f_a}$.

3) Esistono dei valori di $a \in \mathbb{R}$ tali che f_a è biiettiva cioè isomorfismo?

4) Esistono dei valori di $a \in \mathbb{R}$ tali che

$$\mathrm{Ker}\, f_a \oplus \mathrm{Im}\, f_a = \mathbb{R}^3 \quad ?$$

5) Posto $a=1$ determinare B_1 e B_2 basi di $\mathbb{R}^3$ tali che $A_{B_1,B_2,f_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Svolgimento:

1) $$f_a^{-1}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\} = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; A_a \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \right\}$$

Il sistema lineare da risolvere ha matrice completa

$$\left(A_a \,\middle|\, \begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) =$$

$$= \left(\begin{array}{ccc|c} 0 & 1 & a-1 & 1 \\ 1-a & -1 & 0 & 0 \\ 2-2a & 2a & 0 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

Riduciamo con Gauss

$$\begin{array}{l} R_2 \\ R_1 \\ R_3-2R_2 \end{array}\left(\begin{array}{ccc|c} 1-a & -1 & 0 & 0 \\ 0 & 1 & a-1 & 1 \\ 0 & 2a+2 & 0 & 0 \end{array}\right) \qquad 3^\circ - (2a+2)\,2^\circ$$

$$\left(\begin{array}{ccc|c} \boxed{1-a} & -1 & 0 & 0 \\ 0 & 1 & a-1 & 1 \\ 0 & 0 & \boxed{-(a-1)(2a+2)} & -(2a+2) \end{array}\right)$$

$$\begin{cases} 1-a \neq 0 \\ -(a-1)(2a+2) \neq 0 \end{cases} \qquad \begin{array}{l} a \neq 1 \\ a \neq 1 \quad a \neq -1 \end{array}$$

$\forall\, a \in \mathbb{R} \setminus \{1,-1\}$ $\quad rg\, A_a = 3 \qquad a \neq -1$

$+(a-1)\cancel{(2a+2)}\,z = +\cancel{(2a+2)} \qquad (a-1)\,z = 1 \qquad z = \frac{1}{a-1}$

$y + (a-1)z = 1 \qquad y = -\cancel{(a-1)}\frac{1}{\cancel{a-1}} + 1 = 0$

$(1-a)\,x - y = 0 \qquad x = \frac{y}{1-a} = 0$

$f_a^{-1}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}0\\0\\\frac{1}{a-1}\end{pmatrix}\right\}$ per $a \in \mathbb{R} \setminus \{-1,1\}$.

Per $a=1$

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right) \qquad f_1^{-1}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\} = \emptyset$$

$0 = -4$ impossibile

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad rg\, A_1 = 1 = \dim \operatorname{Im} f_1$$

Per $a=-1$ $\quad A_{-1}$ $\left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\quad \text{rg} A = 2 = \text{rg}(A|b)$

$$\begin{cases} y-2z=1 \\ 2x-y=0 \end{cases} \qquad \begin{cases} y=2z+1 \\ x=\frac{y}{2}=\frac{2z+1}{2}=z+\frac{1}{2} \end{cases} \qquad \begin{pmatrix} z+1/2 \\ 2z+1 \\ z \end{pmatrix}$$

$z=0$ $\qquad \text{Ker} f_{-1}$

$$\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$$

coeff. di z

$$f_a^{-1}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} = \begin{cases} \left\{\begin{pmatrix} 0 \\ 0 \\ \frac{1}{a-1} \end{pmatrix}\right\} & \text{se } a \in \mathbb{R} \setminus \{1,-1\} \\ \emptyset & \text{se } a=1 \\ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle & \text{se } a=-1 \end{cases}$$

2) $B_{\text{Ker} f_a}$ $\quad B_{\text{Im} f_a}$

$B_{\text{Ker} f_a}$ $\quad$ Se $a \in \mathbb{R} \setminus \{-1,1\}$ $\text{rg} A_a = 3$ quindi

$B_{\text{Ker} f_a} = \emptyset$ $\quad B_{\text{Im} f_a} = \{e_1, e_2, e_3\}$ è biiettiva
è isomorfismo.

Se $a=1$ $\quad A_1$ è ridotta a scala

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{Ker} f_1 \qquad y=0 \qquad \text{Ker} f_1 = \left\{\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \middle| x,z \in \mathbb{R}\right\}$$

$$= \langle e_1, e_3 \rangle$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Im} f_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \qquad B_{\text{Im} f_1} = \left\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right\}$$

$$\begin{pmatrix} 0 & 1 & a-1 \\ 1-a & -1 & 0 \\ 2-2a & 2a & 0 \end{pmatrix}$$

Se $a=-1$ $\quad \operatorname{Im} f_{-1} = \begin{pmatrix} 0 & 1 & -2 \\ 2 & -1 & 0 \\ 4 & -2 & 0 \end{pmatrix}$

Essendo $\operatorname{rg} A_{-1} = 2$ $\quad \dim \operatorname{Im} f_{-1} = 2$ $\quad \operatorname{Im} f_{-1} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

$\operatorname{Ker} f_{-1} = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$

3) $\forall a \in \mathbb{R}$ $\quad a \neq 1$ e $a \neq -1$

4) $\operatorname{Ker} f_a \oplus \operatorname{Im} f_a = \mathbb{R}^3$

$a \in \mathbb{R} \setminus \{1,-1\}$ $\quad \{0_{\mathbb{R}^3}\} \oplus \mathbb{R}^3 = \mathbb{R}^3$ $\quad$ sì

Se $a = 1$

$\langle e_1, e_3 \rangle \oplus \left\langle \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\rangle = \mathbb{R}^3$ $\qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

?

$a = -1$ $\quad \operatorname{Ker} f_{-1} = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$ $\quad \operatorname{Im} f_{-1} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{pmatrix}$$

$\operatorname{Ker} f_{-1} \oplus \operatorname{Im} f_{-1} = \mathbb{R}^3.$