Proiezioni e Simmetrie

Endomorfismi di uno spazio vettoriale V

$$\boxed{V = U \oplus W}$$

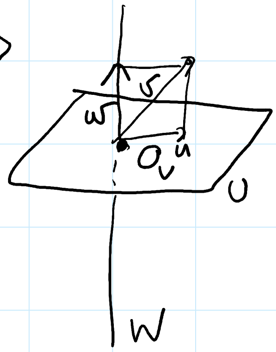
Def.: La proiezione su U di direzione W è

$p = p_U^W : V \longrightarrow V \qquad v = u + w \quad u \in U, w \in W$

$p(v) = u$

Esempio: $V = \mathbb{R}^3 \quad U = \langle e_1, e_2 \rangle \quad W = \langle e_3 \rangle$

$p\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$



Proprietà:

① Linearità:

$v = \underset{U}{\underset{\cap}{u}} + \underset{W}{\underset{\cap}{w}} \qquad av = \underset{U}{\underset{\cap}{au}} + \underset{W}{\underset{\cap}{aw}}$

$p(av) = au = a\,p(v)$ rispetta il pr. per scalari

$v_1 = u_1 + w_1 \qquad v_1 + v_2 = \underset{U}{(u_1 + u_2)} + \underset{W}{(w_1 + w_2)}$

$v_2 = u_2 + w_2$

$p(v_1 + v_2) = u_1 + u_2 = p(v_1) + p(v_2)$ rispetta la $+$

② $\operatorname{Ker} p = W$ direzione di proiezione

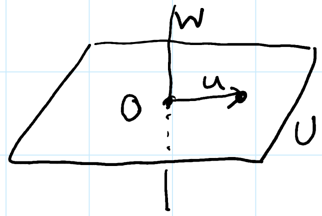
$v = u + w \qquad p(v) = u$

$\operatorname{Ker} p = \{ v \mid p(v) = 0_V \} = \{ 0_V + w \mid w \in W \} = W$

$\operatorname{Im} p = U$

$\operatorname{Im} p = \{ p(v) \mid v \in V \} \subseteq U$ perché $p(v) = u \in U$

se $u \in U$ $\quad u = \underset{U}{u} + \underset{W}{0_V}$

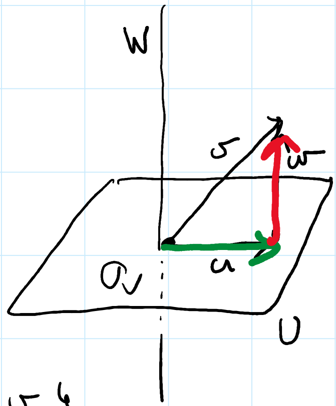


$p(u) = u \quad \forall u \in U$

quindi $\operatorname{Im} p = U$

③ $p^2 := p \circ p = p$ $\qquad v = u + w$

$p(p(v)) = p(u) = u = p(v)$



④ $v = u + w = P_U^W(v) + P_W^U(v)$

$P_U^W(v) = u \qquad P_W^U(v) = w$

$id_V(v) = v$

$$\boxed{id_V = P_U^W + P_W^U}$$

$B_V = \{v_1, \dots, v_n\}$

chiamiamo $A_{P_U^W}, A_{P_W^U}$ le matrici associate rispetto a B_V.

$$\boxed{I_n = A_{P_U^W} + A_{P_W^U}}$$

$P_{\langle e_1, e_2\rangle}^{\langle e_3\rangle} = p \qquad V = \mathbb{R}^3 \quad U = \langle e_1, e_2\rangle \quad W = \langle e_3 \rangle$

$p\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A$

$P_{\langle e_3\rangle}^{\langle e_1, e_2\rangle} = q \qquad q\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A + B = I_3 \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$

Simmetrie:

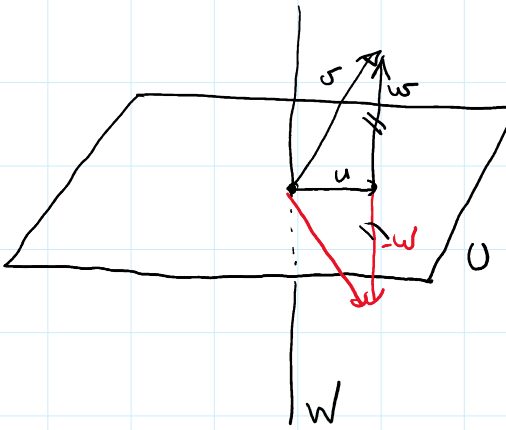
$V = U \oplus W \quad s = s_U^W$ $\quad U$ asse di simmetria

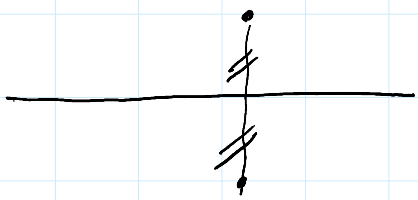
W direzione di simmetria

$s : V \longrightarrow V \qquad v = u + w$

$s(v) = u - w$

$U = \langle e_1, e_2 \rangle \qquad W = \langle e_3 \rangle \qquad s\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$





Proprietà:

1) Linearità: $\qquad v = u + w$

$$s(v) = u - w = 2u - u - w = 2\underset{\substack{\| \\ p(v)}}{u} - \underbrace{(u+w)}_{v} = 2p(v) - v$$

$\boxed{s = 2p - id_V}$ $\qquad$ vale per matrici associate

$A_{B_V, B_V, -} \qquad B_V = \{v_1, \dots, v_n\}$

$$2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ -z \end{pmatrix} = s\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

2) Dimostriamo che $s^2 = id_V$ $\quad v = u + w$

$s(s(v)) = s(u-w) = u + w = v$

perciò s è biiettiva $\quad \operatorname{Ker} s = \{0_V\}$ è iniettiva

$\operatorname{Im} s = V$ è suriettiva

3) $s(v) = u - w \quad v = u + w$

$s(u) = s(u + 0_V) = u - 0_V = u \quad$ cioè $s(u) = u \quad \forall u \in U$

$s(w) = s(0_V + w) = 0_V - w = -w \quad$ cioè $s(w) = -w \quad \forall w \in W.$

Esercizio: $\mathbb{R}^3$

$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x - y + 2z = 0 \right\} \quad W = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

1. Verificare che $U \oplus W = \mathbb{R}^3$ e determinare una base di U B_U.
2. Fissata $\mathcal{E} = \{e_1, e_2, e_3\}$ base canonica di $\mathbb{R}^3$ determinare la matrice associata alla $p_U^W = p$ rispetto a $\mathcal{E}$ (usata come base sia nel dominio che nel codominio).
3. Determinare la matrice di $s_U^W = s$ rispetto ad $\mathcal{E}$.
4. Prendere $B := B_U \cup \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}}_{B_W}$ e determinare la matrice associata a p rispetto a B sia nel dominio che nel codominio.

Svolgimento:

$$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x - y + 2z = 0 \right\} \qquad W = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

① $\dim U = 2$ $\qquad U \cap W \qquad w = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \qquad w \in W$

$\dim W = 1$

$w \in U \qquad a - 0 + 2 \cdot 0 = 0$

$a = 0$

$$U \cap W = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow \text{somma diretta}$$

Per la F. di Grassmann $\dim(U+W) = 2 + 1 - 0 = 3$

$\Rightarrow U + W = \mathbb{R}^3$

$B_U \qquad y = x + 2z \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{pmatrix} x \\ x+2z \\ z \end{pmatrix}$

$$B_U = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

2) $P\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x - y + 2z = 0 \right\} \qquad W = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \quad w = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = u + w = u + \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x-a \\ y \\ z \end{pmatrix} \qquad u \in U \quad x - a - y + 2z = 0$$

$$a = x - y + 2z$$

$$w = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x-y+2z \\ 0 \\ 0 \end{pmatrix} \qquad u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x-y+2z \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cancel{x} - \cancel{x} + y - 2z \\ y \\ z \end{pmatrix}$$

$$u = \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix} \qquad P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A \cdot A = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3) $S = 2p - id$ $\quad v = u + w \quad p(v) = u$

$S(v) = u - w = 2u - (u + w) = 2p(v) - id(v)$

$$S = 2A - \mathbb{I}_3 = 2\begin{pmatrix} 0 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 2 & -4 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Verificate $S \cdot S = \mathbb{I}_3$.

4) $B_U = \{u_1, u_2\}$ $\quad B_W = \{w\}$ $\quad B = B_U \cup B_W = \{u_1, u_2, w\}$

$A_{B,B,p}$ $\quad W = \operatorname{Ker} p$

$p(u_1) = u_1 = 1 \cdot u_1 + 0 \cdot u_2 + 0 \cdot w$

$p(u_2) = u_2 = 0 \cdot u_1 + 1 \cdot u_2 + 0 \cdot w$

$p(w) = 0_V = 0 \cdot u_1 + 0 \cdot u_2 + 0 w$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A_{B,B,p}$$

Esercizio per casa: determinare la matrice della proiezione $p = p_U^W$

$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x + y + z = 0 \right\}$ $\quad W = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$.

Osservazioni:

Sia $A \in M_{m,n}(\mathbb{R})$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ cioè

$A = A_{\mathcal{E}_n, \mathcal{E}_m, f}$

$\operatorname{Im} f = \langle f(e_1), \dots, f(e_n) \rangle = \left\langle \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\rangle = \langle A_1, \dots, A_n \rangle$

colonne di A

$rg_c A$ = dimensione dello sp. vettoriale generato dalle colonne di A
= massimo n° di colonne linearm. indipendenti.

$rg_r A$ = dimensione dello sp. vett. generato dalle righe di A
= massimo n° di righe linearm. indipendenti.
= n° di righe non nulle in una matrice ridotta scala (con op. elementari sulle righe).

$$\boxed{rg_c A = rg_r A = n° \text{ pivot}}$$

$$rg\begin{pmatrix} 2 & 1 & 3 \\ 8 & 1 & 5 \end{pmatrix} = 2 \qquad rg\begin{pmatrix} 2 & 1 \\ 8 & 0 \\ 10 & 1 \\ 20 & 0 \end{pmatrix} = 2$$

Teorema di Rouché-Capelli:

Si consideri il sistema lineare $Ax = b$ con $A \in M_{m,n}(\mathbb{R})$ $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ definiamo

$$S_{A|b} = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; Ax = b \right\}$$ insieme delle soluzioni

① $S_{A|b} \neq \emptyset \iff rg A = rg(A|b)$

② Se $rg A = rg(A|b) = r$ allora
$S_{A|b} = v + \operatorname{Ker} A$ con $\operatorname{Ker} A$ soluzione del sist. om. $Ax = \underline{0}$
$\dim \operatorname{Ker} A = n - r$

③ $S_{A|b}$ è sottospazio di $\mathbb{R}^n$ se e solo se $b = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ cioè il sistema è omogeneo.

Dim:

① Consideriamo l'applicazione lineare

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \begin{array}{l} Ax=b \\ f(x)=b \end{array}$

$$S_{A|b} = f^{-1}(\{b\}) \begin{cases} \emptyset & \text{se } b \notin \operatorname{Im} f \\ v + \operatorname{Ker} f = v + \operatorname{Ker} A & \text{se } b \in \operatorname{Im} f. \end{cases}$$

Dimostriamo che $S_{A|b} \neq \emptyset \iff b \in \operatorname{Im} f \iff rg(A) = rg(A|b)$

$\operatorname{Im} f = \underbrace{\langle A_1, \dots, A_n \rangle}_{\text{colonne di } A} \subseteq \langle A_1, \dots, A_n, b \rangle$

$b \in \operatorname{Im} f \iff \langle A_1, \dots, A_n \rangle = \langle A_1, \dots, A_n, b \rangle \iff$

$\dim \langle A_1, \dots, A_n \rangle = \dim \langle A_1, \dots, A_n, b \rangle \iff rg_c A = rg_c (A|b)$.

2) $f^{-1}(\{b\}) = v + \operatorname{Ker} A \qquad$ se $b \in \operatorname{Im} f \quad rg(A) = rg(A|b)$

Teorema di Struttura delle controimmagine

$r = rg A = rg(A|b) \qquad$ dato che $\operatorname{Im} f = \langle A_1, \dots, A_n \rangle$

$r = rg_c A = \dim \operatorname{Im} f$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad n = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$

$\qquad\qquad\qquad\qquad\quad \overset{!}{=} \dim \operatorname{Ker} A + r$

$\dim \operatorname{Ker} A = n - r$

③ $S_{A|b} \leq \mathbb{R}^n \quad$ se $\overset{\Longleftrightarrow}{\text{e solo}}$ se $\quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

"$\Longrightarrow$" Supponiamo $S_{A|b} \leq \mathbb{R}^n$ allora contiene $0_{\mathbb{R}^n} \Rightarrow$

$S_{A|b} = \{ x \in \mathbb{R}^n \mid Ax = b \} \Rightarrow A 0_{\mathbb{R}^n} = b \quad \Rightarrow b = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

"$\Leftarrow$" Se $b=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$ $\quad Ax=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$ $\quad S_{A|0}=\operatorname{Ker} A \leq \mathbb{R}^n$. $\square$

Esempio: dato il sistema lineare $Ax=b$ con

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 5 & k \\ 0 & k & 1 & 8 & k \\ 0 & 0 & 3 & 0 & 0 \end{array}\right) = A|b \qquad S_{A|b} \leq \mathbb{R}^4 \iff k=0$$

perché $\quad b=\begin{pmatrix}k\\ k\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \iff k=0$

Oss: $\quad v+\operatorname{Ker} A$

Esercizio: si consideri il sottoinsieme di $\mathbb{R}^4$

$$S=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+\left\langle \underset{u_1}{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}, \underset{u_2}{\begin{pmatrix}0\\1\\1\\0\end{pmatrix}}, \underset{u_3}{\begin{pmatrix}0\\0\\1\\1\end{pmatrix}} \right\rangle$$

$v+U$

determinare un sistema lineare che abbia S come soluzioni.

Svolg: Verificare che siano lin. ind. u_1, u_2, u_3

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+a\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\1\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\\1\end{pmatrix}=\begin{pmatrix}1+a\\1+a+b\\b+c\\c\end{pmatrix}$$

$$\begin{cases} x_1=1+a \\ x_2=1+a+b \\ x_3=b+c \\ x_4=c \end{cases} \qquad \underline{a=x_1-1} \quad \begin{cases} x_2=\cancel{1}+x_1-\cancel{1}+b=x_1+b \\ x_3=b+c \\ x_4=c \end{cases} \qquad \begin{cases} x_2=x_1+b \\ x_3=b+x_1 \\ \underline{c=x_4} \end{cases}$$

$b=x_2-x_1$

$x_3 = x_2 - x_1 + x_4 \qquad x_1 - x_2 + x_3 - x_4 = 0 \qquad$ è $\qquad$ sottospazio

$$\overset{v}{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}} + \left\langle \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\rangle \qquad v = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = u_1$$

$$v + \boxed{\langle u_1, u_2, u_3 \rangle} = 0_{\mathbb{R}^4} + \boxed{\langle u_1, u_2, u_3 \rangle} \iff$$

$$u_1 = v - 0_{\mathbb{R}^4} \in \langle u_1, u_2, u_3 \rangle$$

$$\underset{\substack{\shortparallel \\ u_1+u_2}}{\begin{pmatrix}1\\2\\1\\0\end{pmatrix}} + \left\langle \underbrace{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}}_{u_1 + u_2}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\rangle$$