Proposizione:

Dati V e W spazi vettoriali

$B_V = \{v_1, \dots, v_n\}$ e $w_1, \dots, w_n$ vettori di W

esiste un'unica applicazione lineare

$$\exists!\ f: V \longrightarrow W \qquad f(v_i) = w_i \quad \forall i = 1, \dots, n.$$

$$\boxed{f\left(\sum_{i=1}^{n} a_i v_i\right)} = \sum_{i=1}^{n} a_i f(v_i) = \boxed{\sum_{i=1}^{n} a_i w_i}$$

Esempi:

Determinare un'applicazione lineare

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ tale che

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \text{e} \quad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}.$$

Tale applicazione è unica?

Sv:

$v_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad \underline{w_1} = \boxed{\begin{pmatrix}1\\0\\0\end{pmatrix}}$ $\quad f(v_1) = w_1$

$v_2 = \begin{pmatrix}0\\1\end{pmatrix}$ $\quad \underline{w_2} = \boxed{\begin{pmatrix}1\\2\\1\end{pmatrix}}$ $\quad f(v_2) = w_2$

$B_{\mathbb{R}^2} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ per la proposizione una applicazione lineare che soddisfa la richiesta esiste ed è unica

$$f\begin{pmatrix}x\\y\end{pmatrix} = f\left(x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}\right) = x f\begin{pmatrix}1\\0\end{pmatrix} + y f\begin{pmatrix}0\\1\end{pmatrix} = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}1\\2\\1\end{pmatrix} =$$

$$\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix} \qquad = \begin{pmatrix}x+y\\2y\\y\end{pmatrix}$$

$$\underline{f\begin{pmatrix}x\\y\end{pmatrix}} = \begin{pmatrix}x+y\\2y\\y\end{pmatrix} = \underline{A\begin{pmatrix}x\\y\end{pmatrix}} \qquad A \in M_{3,2}(\mathbb{R})$$

$$A = \begin{pmatrix}\boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}1\\2\\1\end{matrix}}\end{pmatrix}$$

è unica perché è definita in modo unico su una base.

Esempio: Determinare tutte le applicazioni lineari $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ tali che

$$f\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}4\\6\\3\end{pmatrix} \qquad f\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\2\\1\end{pmatrix}.$$

Sv: $B = \{v_1', v_2'\}$ $\quad v_1' = \begin{pmatrix}1\\3\end{pmatrix}$ $\quad v_2' = \begin{pmatrix}-1\\1\end{pmatrix}$ $\quad w_1' = \begin{pmatrix}4\\6\\3\end{pmatrix}$ $\quad w_2' = \begin{pmatrix}0\\2\\1\end{pmatrix}$

$f(v_i') = w_i'$ per $i = 1, 2$.

B è base perché $\dim \mathbb{R}^2$ e v_1', v_2' sono lin. indip. in quanto non multipli fra loro.

$$\underline{\begin{pmatrix}x\\y\end{pmatrix} = a\begin{pmatrix}1\\3\end{pmatrix} + b\begin{pmatrix}-1\\1\end{pmatrix}}$$

determinare a e b in funzione di x, y.

$$\underline{f\begin{pmatrix}x\\y\end{pmatrix} = a\begin{pmatrix}4\\6\\3\end{pmatrix} + b\begin{pmatrix}0\\2\\1\end{pmatrix}}$$

Costruiamo una matrice mettendo in riga i vettori

$$\left(\begin{array}{c|c} v_1 & w_1 \\ v_2 & w_2 \end{array}\right) \qquad \left(\begin{array}{cc|ccc} 1 & 3 & 4 & 6 & 3 \\ -1 & 1 & 0 & 2 & 1 \end{array}\right)$$

Riduciamo con Gauss fino col avere

$$\left(\begin{array}{c|c} v_1 & f(v_1) \\ v_2 & f(v_2) \end{array}\right) \qquad \left(\begin{array}{c|c} v_2 & f(v_2) \\ v_1 & f(v_1) \end{array}\right)$$

$$\left(\begin{array}{c|c} v_1 & f(v_1) \\ v_1+v_2 & \underbrace{f(v_1)+f(v_2)}_{= f(v_1+v_2)} \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & 0 & f(e_1) \\ 0 & 1 & f(e_2) \end{array}\right)$$

$$\left(\begin{array}{cc|ccc} 1 & 3 & 4 & 6 & 3 \\ -1 & 1 & 0 & 2 & 1 \end{array}\right) 2°+1° \qquad \left(\begin{array}{cc|ccc} 1 & 3 & 4 & 6 & 3 \\ 0 & 4 & 4 & 8 & 4 \end{array}\right) \tfrac{1}{4} 2°R$$

$$\left(\begin{array}{cc|ccc} 1 & 3 & 4 & 6 & 3 \\ 0 & 1 & 1 & 2 & 1 \end{array}\right) 1°R-3\cdot 2°R \qquad \left(\begin{array}{cc|ccc} \boxed{1 \quad 0} & & \boxed{1 \quad 0 \quad 0} & & \\ \boxed{0 \quad 1} & & \boxed{1 \quad 2 \quad 1} & & \end{array}\right)$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}$$

Verifichiamo $\quad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2y \\ y \end{pmatrix}$

$$f\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix} \qquad f\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+3 \\ 2\cdot 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}$$

$$f\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}0\\2\\1\end{pmatrix} \qquad f\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}-1+1\\2\cdot 1\\1\end{pmatrix}=\begin{pmatrix}0\\2\\1\end{pmatrix} \qquad \#$$

Esercizio: determinare tutte le applicazioni lineari $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ tali che

$$f\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}2\\1\\5\end{pmatrix} \quad e \quad f\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}3\\1\\8\end{pmatrix}.$$

Sv:

$$v_1=\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad w_1=\boxed{\begin{pmatrix}2\\1\\5\end{pmatrix}} \qquad f(v_1)=w_1$$

$$v_2=\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad w_2=\boxed{\begin{pmatrix}3\\1\\8\end{pmatrix}} \qquad f(v_2)=w_2$$

$$v_3=\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad w_3=\boxed{\begin{pmatrix}a\\b\\c\end{pmatrix}} \qquad a,b,c\in\mathbb{R} \qquad f(v_3)=w_3$$

$$f\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2&3&a\\1&1&b\\5&8&c\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} \qquad \text{con } a,b,c\in\mathbb{R}.$$

esistono infinite appl. lineari che verificano la richiesta.

Esempio: determinare, se esistono, tutte le applicazioni lineari tali che: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f\Big(\overset{v_1}{\begin{pmatrix}1\\0\end{pmatrix}}\Big)=\overset{w_1}{\begin{pmatrix}1\\1\end{pmatrix}}$$

$$f\Big(\overset{v_2}{\begin{pmatrix}2\\0\end{pmatrix}}\Big)=\overset{w_2}{\begin{pmatrix}1\\1\end{pmatrix}} \qquad \begin{pmatrix}2\\0\end{pmatrix}=2\begin{pmatrix}1\\0\end{pmatrix} \qquad f\begin{pmatrix}2\\0\end{pmatrix}=2f\begin{pmatrix}1\\0\end{pmatrix}=2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}$$

$$f\Big(\overset{v_3}{\begin{pmatrix}1\\1\end{pmatrix}}\Big)=\overset{w_3}{\begin{pmatrix}3\\3\end{pmatrix}}.$$

Sv:

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 3 & 3 \end{array}\right) \begin{array}{l} 2^\circ R - 2\cdot 1^\circ R \\ 3^\circ R - 1^\circ R \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 0 & f(e_1) \\ 0 & 1 & f(e_2) \\ 0 & 0 & f(0_V) \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

$$\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & -1 \end{array} = f\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

impossibile

l'applicazione lineare non esiste.

Es: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad f\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 4 & 4 \end{array}\right) \begin{array}{l} 2^\circ R - 1^\circ R \\ 3^\circ R - 2\cdot 1^\circ R \end{array} \quad \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right) \quad 3^\circ R - 2^\circ R$$

$$\left(\begin{array}{cc|cc} \underline{1} & \underline{0} & 1 & 1 \\ \underline{0} & \underline{1} & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ x+2y \end{pmatrix}$$

Esiste <u>unica la funzione</u> richiesta.

Esempio: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad f\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Svolg:

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) \quad 2^\circ R - 2\, 1^\circ R \quad \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & a & b \end{array}\right) \quad I^{\circ}R - II^{\circ}R \quad \left(\begin{array}{cc|cc} 1 & 0 & \boxed{1-a} & \boxed{-b} \\ 0 & 1 & \boxed{a} & \boxed{b} \end{array}\right)$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-a & a \\ -b & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Come associare una matrice ad un'applicazione lineare

Ingredienti:

1. $f: V \to W$ applicazione lineare
2. $B_V = \{v_1, \dots, v_n\}$ base di V
3. $B_W = \{w_1, \dots, w_m\}$ base di W,

Ricetta

Prendi v_1 calcola $f(v_1) \in W$

$$f(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m = \sum_{i=1}^{m} a_{i1} w_i$$

$$A = \begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix} \qquad \begin{array}{l} m = \dim W \\ n = \dim V \end{array}$$

$$f(v_2) = a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m$$

$$\vdots$$

$$f(v_n) = a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m$$

Matrice associata ad f rispetto alle basi B_V e B_W.

$$A_{B_V, B_W, f} \qquad A_{B_V, B_W}(f).$$

Esempio:

Determinare $A_{B_V, B_W, f}$

① $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\quad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{\begin{pmatrix} x+2y+3z \\ 4x+5y+6z \end{pmatrix}}$

② $B_{\mathbb{R}^3} = \{e_1, e_2, e_3\} = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

③ $B_{\mathbb{R}^2} = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ $\quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Ricetta

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+2\cdot 0+3\cdot 0 \\ 4\cdot 1+5\cdot 0+6\cdot 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0+2\cdot 1+3\cdot 0 \\ 4\cdot 0+5\cdot 1+6\cdot 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+2\cdot 0+3\cdot 1 \\ 4\cdot 0+5\cdot 0+6\cdot 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Calcoliamo $A_{B_1, B, f}$ con

$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ 4x+5y+6z \end{pmatrix}$ $\quad B_1 = \left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

$B = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

$$f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad A_{B_1, B, f} = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

mentre avevamo $A_{B_{\mathbb{R}^3}, B_{\mathbb{R}^2}, f} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$A_{B, B_2, f}$ $\qquad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ 4x+5y+6z \end{pmatrix}$

$B = \{e_1, e_2, e_3\}$

$B_2 = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ $\qquad A_{B, B_2, f} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$

$f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 6\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Osservazione:

$B_V = \{v_1, \dots, v_n\}$ permette di definire

$\phi_V : V \longrightarrow \mathbb{R}^n$ biiettiva

$$\phi_V\left(\sum_{i=1}^{n} x_i v_i\right) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{array}{ccc} V & \longleftrightarrow & \mathbb{R}^n \\ \sum_{i=1}^{n} x_i v_i & & \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ f \downarrow & & \downarrow A_{B_V, B_W, f} \\ W & \longleftrightarrow & \mathbb{R}^m \\ \sum_{i=1}^{m} y_i w_i & & \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \end{array}$$

$B_V = \{v_1, \dots, v_n\}$

$B_W = \{w_1, \dots, w_m\}$

Def: $A \in M_{m,n}(\mathbb{R})$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

cioè $\quad A = A_{\mathcal{E}_n, \mathcal{E}_m, f}$

$$\operatorname{Ker} f = \operatorname{Ker} A = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

$$\operatorname{Im} f = \operatorname{Im} A = \langle f(e_1), \dots, f(e_n) \rangle = \left\langle \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\rangle$$

cioè

$\operatorname{Im} A$ è generata dalle colonne di A.

Attenzione: operazioni elementari sulle righe cambiano $\operatorname{Im} A$

Es: $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad 2°R - 1°R \qquad \operatorname{Im} A = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$

$\neq$

$A' = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \qquad \operatorname{Im} A' = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$

$\operatorname{Ker} A = \operatorname{Ker} A' \qquad \begin{cases} x + 2y = 0 \\ x + 2y = 0 \end{cases} \qquad \operatorname{Ker} A = \left\{ \begin{pmatrix} -2y \\ y \end{pmatrix} \middle| y \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\rangle \quad x = -2y$

$\begin{cases} x + 2y = 0 \\ 0 = 0 \end{cases} \qquad \operatorname{Ker} A' = \left\langle \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\rangle$

Conseguenza: cambiando basi cambia la matrice

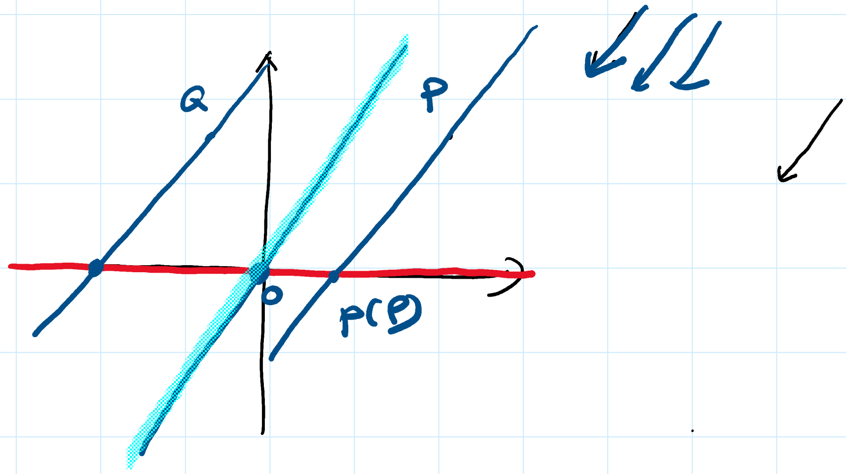
$\dim \operatorname{Im} f = \dim \langle A_1, \dots, A_n \rangle$ sottospazio generato dalle colonne di A

$\dim \operatorname{Ker} f = \dim \operatorname{Ker} A$

per ogni $f: V \longrightarrow W$ $\quad B_V, B_W \quad A = A_{B_V, B_W, f}$

Proiezioni e simmetrie:

$p: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



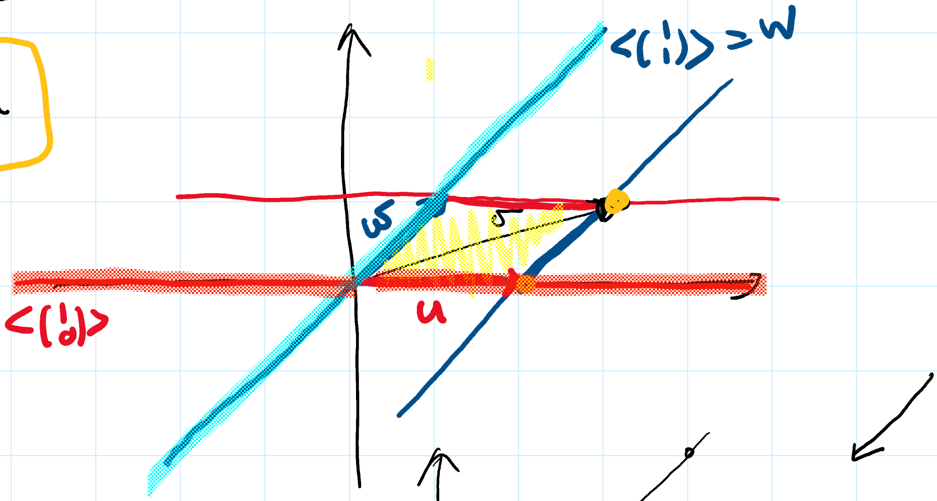
Schermo: $U = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$

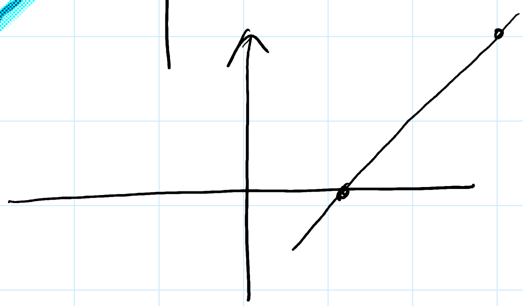
Direzione di proiezione $\quad W = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$

$U \oplus W = \mathbb{R}^2$ $\qquad$ Proiezione su U con direzione W

$u + w = v$

$\boxed{p_U^W(v) = u}$





$\mathrm{Ker}\, p_U^W = W$

$\mathrm{Im}\, p_U^W = U$

$W = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle \qquad U = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \quad \boxed{y=0}$

$\begin{pmatrix} x \\ y \end{pmatrix} = u + w$ con $u \in \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$ $\quad w \in \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle \quad w = \begin{pmatrix} a \\ a \end{pmatrix}$

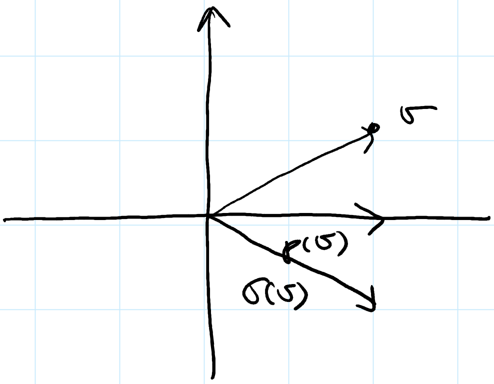
$\begin{pmatrix} x \\ y \end{pmatrix} = u + \begin{pmatrix} a \\ a \end{pmatrix} \qquad u = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a \\ a \end{pmatrix} = \begin{pmatrix} x-a \\ \boxed{y-a} \end{pmatrix} \in U \iff \underline{\underline{y-a=0}}$

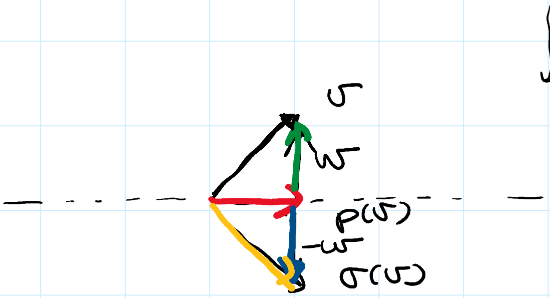
$a = y \qquad \begin{pmatrix} x \\ y \end{pmatrix} = u + \begin{pmatrix} y \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} x-y \\ 0 \end{pmatrix}}_{u} + \underbrace{\begin{pmatrix} y \\ y \end{pmatrix}}_{w}$

$p_U^W \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ 0 \end{pmatrix} \qquad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$\mathrm{Ker} f = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle = W$

$\mathrm{Im} f = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \rangle = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle = U$





$v = u + w \qquad u = p(v)$

$\boxed{\sigma(v) = u - w}$ Simmetria di asse U e direzione W

$p(v) = u \qquad p^2 = p$

$p(p(u+w)) = p(u) = u$

$\sigma(\sigma(v)) = \sigma(u - w) = u - (-w) = u + w = v$

$\sigma^2 = id$