Esercizio: dati i sottospazi

$$W_1 = \left\{ \begin{pmatrix} a & a+b \\ 2a & 0 \end{pmatrix} \middle| \ a,b \in \mathbb{R} \right\}$$

$$W_2 = \left\{ \begin{pmatrix} c-d & d \\ e & c+e \end{pmatrix} \middle| \ c,d,e \in \mathbb{R} \right\}$$

di $M_{2,2}(\mathbb{R})$

Determinare $\underline{B_{W_1}}$, $\underline{\dim W_1}$
$\underline{B_{W_2}}$, $\underline{\dim W_2}$
$\underline{B_{W_1 \cap W_2}}$, $\underline{\dim (W_1 \cap W_2)}$
$\underline{B_{W_1 + W_2}}$, $\underline{\dim (W_1 + W_2)}$

Determinare $T \leq M_{2,2}(\mathbb{R})$ tale che

$$(W_1 \cap W_2) + T = W_2 \qquad \dim T = 2$$

La somma è diretta?

Si scrive $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ come somma di un vettore di W_1 con un vettore di W_2.

Svolgimento:

$$W_1 = \left\{ \begin{pmatrix} a & a+b \\ 2a & 0 \end{pmatrix} \middle| \ a,b \in \mathbb{R} \right\}$$

$$W_1 = \left\langle \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle$$ essendo non multiple le due matrici.

$$\boxed{B_{W_1} = \left\{ \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}} \qquad \boxed{\dim W_1 = 2}$$

$$W_2 = \left\{ \begin{pmatrix} c-d & d \\ e & c+e \end{pmatrix} \middle| \ c,d,e \in \mathbb{R} \right\}$$

$$W_2 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle \qquad \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \quad (x_1\ x_2\ x_3\ x_4) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Mettiamo i vettori come righe di una matrice, le righe non nulle della matrice ridotta a scala sono una base.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad 2^\circ R + 1^\circ R \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\boxed{B_{W_2} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}} \qquad \boxed{\dim W_2 = 3}$$

Certamente non sono in somma diretta

$\dim(W_1 \cap W_2) \geq 1$ per le formule di Grassmann

essendo $\dim M_{2,2}(\mathbb{R}) = 4$.

Calcoliamo $W_1 \cap W_2$, cercando eq. cartesiane per W_2

eq. parametriche per W_1.

$$a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \\ a+b+c \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \begin{cases} \cancel{x_1 = a} \\ \cancel{x_2 = b} \\ \cancel{x_3 = c} \\ x_4 = a+b+c \end{cases} \qquad \boxed{x_4 = x_1 + x_2 + x_3}$$

$$W_2 = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \;\middle|\; x_1 + x_2 + x_3 - x_4 = 0 \right\}$$

$$B_{W_1} = \left\{ \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

$$\omega \in W_1 \qquad \omega \qquad a\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ a+b \\ 2a \\ 0 \end{pmatrix} \qquad \omega \in W_2$$

$$x_1 + x_2 + x_3 - x_4 = 0$$

$$a + a+b + 2a - 0 = 0 \qquad 4a + b = 0 \qquad b = -4a \qquad \begin{pmatrix} a \\ -3a \\ 2a \\ 0 \end{pmatrix} \; \omega$$

$$a+b = a - 4a = -3a$$

$$W_1 \cap W_2 = \left\{ \begin{pmatrix} a & -3a \\ 2a & 0 \end{pmatrix} \middle| \; a \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix} \right\rangle$$

$$\boxed{B_{W_1 \cap W_2} = \left\{ \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix} \right\}} \qquad \boxed{\dim W_1 \cap W_2 = 1}$$

Formule di Grassmann:

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) =$$

$$= 2 + 3 - 1 = 4 \qquad W_1 + W_2 = M_{2,2}(\mathbb{R})$$

$$\boxed{B_{W_1+W_2} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}} \qquad \boxed{\dim(W_1 + W_2) = 4}$$

Determinare $T \leq M_{2,2}(\mathbb{R})$ tale che

$$(W_1 \cap W_2) + T = W_2 \qquad \dim T = 2$$

$$1 \;+\; 2 = 3$$

$$W_2 = \langle A_1, A_2, A_3 \rangle$$

$$W_1 \cap W_2 = \langle B \rangle$$

Se $(W_1 \cap W_2) + T = W_2$ e $\dim T = 2$ allora

$$\dim W_2 = \dim\left((W_1 \cap W_2) + T\right) = \dim(W_1 \cap W_2) + \dim T - \dim((W_1 \cap W_2) \cap T)$$

$$3 = 3 = 1 + 2 - \dim((W_1 \cap W_2) \cap T)$$

$\Rightarrow \dim((W_1 \cap W_2) \cap T) = 0$ quindi la somma è diretta

$$B_{W_2} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \cancel{\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}} \right\} \qquad B_{W_1 \cap W_2} = \left\{ \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix} \right\}$$

$$\left\{ \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix}, \boxed{\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$$ base di W_2

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \notin \left\langle \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix} \right\rangle$$

$$a\begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a & -3a \\ 2a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b & b \end{pmatrix} =$$

$$= \begin{pmatrix} a & -3a \\ 2a+b & b \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \left\langle \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

$$\boxed{T = \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle}$$ verifica la richiesta.

Si scrive $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ come somma di un vettore di W_1

con un vettore di W_2.

$$W_2 = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \middle| \ x_1 + x_2 + x_3 - x_4 = 0 \right\}$$

$$W_1 = \left\langle \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle \qquad w_1 = a\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ a+b \\ 2a \\ 0 \end{pmatrix}$$

$$A \begin{pmatrix}1\\3\\2\\1\end{pmatrix} = w_1 + w_2 \quad \text{con} \quad \begin{matrix} w_1 \in W_1 \\ w_2 \in W_2 \end{matrix}$$

$$\begin{pmatrix}1\\3\\2\\1\end{pmatrix} = \begin{pmatrix}a\\a+b\\2a\\0\end{pmatrix} R + \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \quad \text{con} \quad x_1+x_2+x_3-x_4=0$$

$$\begin{pmatrix}1\\3\\2\\1\end{pmatrix} - \begin{pmatrix}a\\a+b\\2a\\0\end{pmatrix} = \begin{pmatrix}1-a\\3-a-b\\2-2a\\1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \quad \begin{cases} x_1 = 1-a \\ x_2 = 3-a-b \\ x_3 = 2-2a \\ x_4 = 1 \end{cases}$$

$$\cancel{1} - a + 3 - a - b + 2 - 2a - \cancel{1} = 0$$

$$\boxed{5 - 4a - b = 0} \qquad b = 5 - 4a \qquad \begin{matrix} a=0 \\ b=5 \end{matrix} \qquad \begin{matrix} a=1 \\ b=1 \end{matrix}$$

$$\begin{matrix} a=1 \\ b=1 \end{matrix} \qquad \begin{pmatrix}1\\3\\2\\1\end{pmatrix} = \underset{w_1}{\begin{pmatrix}1\\2\\2\\0\end{pmatrix}} + \underset{w_2}{\begin{pmatrix}0\\1\\0\\1\end{pmatrix}} \qquad \begin{matrix} x_1+x_2+x_3-x_4=0 \\ 0+1+0-1=0 \end{matrix}$$

$$\begin{matrix} a=0 \\ b=5 \end{matrix} \qquad \begin{pmatrix}1\\3\\2\\1\end{pmatrix} = \begin{pmatrix}0\\5\\0\\0\end{pmatrix} + \begin{pmatrix}1\\-2\\2\\1\end{pmatrix} \qquad \begin{matrix} x_1+x_2+x_3-x_4=0 \\ \cancel{1} - \cancel{2} + \cancel{2} - \cancel{1} = 0 \end{matrix}$$

$$w_1' + w_2'$$

W_1 e W_2 non erano in somma diretta

$$\boxed{\begin{pmatrix}1 & 3\\2 & 1\end{pmatrix} = \underset{w_1}{\begin{pmatrix}1 & 2\\2 & 0\end{pmatrix}} + \underset{w_2}{\begin{pmatrix}0 & 1\\0 & 1\end{pmatrix}}} \qquad \begin{matrix} w_1 \in W_1 \\ w_2 \in W_2 \\ u \in W_1 \cap W_2 \\ u \neq 0_V \end{matrix}$$

$$A = \underset{W_1}{w_1} + \underset{W_2}{w_2} = \underset{W_1}{(w_1+u)} + \underset{W_2}{(w_2-u)}$$

Esercizio:

Verificare che $T = \left\langle \begin{pmatrix}1\\2\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\rangle$ è sottospazio di $U = \left\langle \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix} \right\rangle$ e determinare un sottospazio $S \leq \mathbb{R}^4$ di dimensione 2 tale che $S+T=U$.

Svolg:

$\dim T = 2 \qquad \dim U = 3 \qquad U: x_3 = 0$

1° metodo $\quad T \leq U \quad T+U = U \quad$ dim

$T \leq U$ se e solo se $\dim(T+U)=3$

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ t_1 \\ t_2 \end{array} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Riduciamo

$$\begin{array}{c} u_2 \\ u_3-u_2 \\ t_1-u_2 \\ t_2 \\ u_1 \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{c} u_2 \\ \frac{u_3-u_2}{-2} \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[4^\circ-2^\circ]{3^\circ-2^\circ} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[5^\circ R - \frac{1}{2} 3^\circ]{\frac{1}{2} 3^\circ R} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

abbiamo dim. che $\dim(T+U)=3 \qquad \underset{3}{U} \leq \underset{3}{T+U} \Rightarrow T+U=U$

$\Rightarrow T \leq U$

$U: x_3=0$ verifichiamo che i generatori di T verifichino l'eq. cartesiana di U

$T = \left\langle \begin{pmatrix}1\\2\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\rangle \qquad T \leq U$

$S \leq \mathbb{R}^4$ di dimensione 2 tale che

$S+T=U$.

$2+2 \neq 3$ $\quad \dim U = 3 \quad \dim T = 2 \quad \dim S = 2$

Non sono in somma diretta S e T

$B_U' = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$ completiamo $\left\{ \begin{pmatrix}1\\2\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\}$ a base di U aggiungendo $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$$\underbrace{\left\langle \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \right\rangle}_{S} + \underbrace{\left\langle \begin{pmatrix}1\\2\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\rangle}_{T} = U$$

$S \cap T$

Sono una base di U

$a\begin{pmatrix}1\\2\\0\\2\end{pmatrix} + b\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$\begin{pmatrix}a\\2a+b\\0\\2a\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\quad \begin{cases} a=1 \\ 2a=0 \end{cases}$ imposs.

$$S = \left\langle \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \right\rangle$$

Richiami su funzioni

Notazioni:

$$f: X \longrightarrow Y \qquad x \longmapsto f(x)$$

X dominio

Y codominio

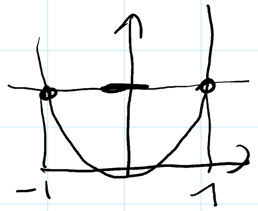
Esempi:

$f: \mathbb{R} \longrightarrow \mathbb{R}$ $\quad f(x) = x^2$ $\quad$ Non è suriettiva

$x \longmapsto x^2$

$1 \longmapsto 1$ $\quad f(1) = 1$ $\quad$ ma $1 \neq -1$ Non è iniettiva

$-1 \longmapsto 1$ $\quad f(-1) = 1$



$g: \mathbb{R} \longrightarrow [0, +\infty)$ $\qquad g(x) = x^2$ **È suriettiva**

$x \longmapsto x^2$

$1 \longrightarrow 1$

$-1 \longrightarrow 1$

Non è iniettiva

$g(1) = 1$ ma $1 \neq -1$

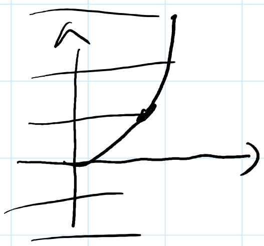
$g(-1) = 1$ $\quad x_1 = 1$

$x_2 = -1$

$h: [0, +\infty) \longrightarrow \mathbb{R}$ $\qquad h(x) = x^2$

$x \longmapsto x^2$ $\qquad$ è iniettiva

Non è suriettiva

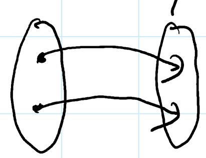


Una funzione si dice **iniettiva** se

$x_1 \neq x_2 \quad x_1, x_2 \in X$ allora $f(x_1) \neq f(x_2)$

se $f(x_1) = f(x_2)$ allora $x_1 = x_2$

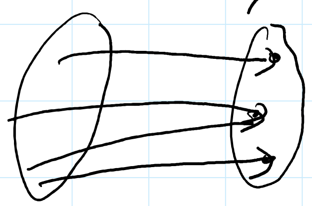
$X \xrightarrow{f} Y$



Una funzione si dice **suriettiva** se

$\forall y \in Y \ \exists x \in X : f(x) = y.$

$X \xrightarrow{f} Y$



Una funzione si dice **biiettiva** se è sia iniettiva che suriettiva.

Dato X un insieme $id_X : X \to X$ $\quad id_X(x) = x$

Una funzione $f : X \to Y$ si dice invertibile se esiste una funzione $f^{-1} : Y \to X$ tale che

$x \in [0, +\infty)$

$$\boxed{f^{-1}(f(x)) = x \quad \forall x \in X} \qquad x \quad x^2 \quad \sqrt{x^2} = x$$

$$\boxed{f(f^{-1}(y)) = y \quad \forall y \in Y} \qquad y \quad \sqrt{y} \quad (\sqrt{y})^2 = y$$

$$f^{-1} \circ f = id_X$$

$$f \circ f^{-1} = id_Y$$

$$\begin{tikzcd} X \arrow[rr, "id_X"] \arrow[dr, "f"'] & & X \\ & Y \arrow[ur, "f^{-1}"'] & \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[rr, "id_Y"] \arrow[dr, "f^{-1}"'] & & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

$$\operatorname{Im} f = \{ f(x) \mid x \in X \}$$
$$= \{ y \in Y \mid \exists x \in X \; f(x) = y \}$$

Suriettiva $\iff \operatorname{Im} f = Y$

Def: dato $B \subseteq Y$ si dice **controimmagine di B**

$$f^{-1}(B) = \{ x \in X \mid f(x) \in B \}$$

Esempio: $\quad f : \mathbb{R} \longrightarrow \mathbb{R} \quad f(x) = x^2$

$B = [4, +\infty) \qquad x^2 \geq 4$

$$f^{-1}(B) = \{ x \in \mathbb{R} \mid x^2 \in [4, +\infty) \} =$$

$$= (-\infty, 2] \cup [2, +\infty)$$