Spazi vettoriali fin. gen.

$K = \mathbb{R}$

$$V = \langle v_1, \dots, v_n \rangle = \left\{ \sum_{i=1}^{n} a_i v_i \;\middle|\; a_i \in \mathbb{R} \; \forall i = 1, \dots, n \right\}$$

Es:

1) $\mathbb{R}^n = \langle e_1, \dots, e_n \rangle$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i e_i$$

2) $M_{m,n}(\mathbb{R}) = \langle e_{ij} \;\; \begin{smallmatrix} i=1,\dots,m \\ j=1,\dots,n \end{smallmatrix} \rangle$ con

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & - & - & - \\ \vdots & & & \\ 0 & - & - & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & 0 & \dots \\ 0 & - & - & - \\ 0 & - & - & 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & a_{mn} \end{pmatrix}$$

$$= a_{11} \underbrace{\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & - & - & - \\ \vdots & & & \\ 0 & - & - & 0 \end{pmatrix}}_{e_{11}} + a_{12} e_{12} + \dots + a_{mn} e_{mn}$$

$$A = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} e_{ij}$$

3) $\mathbb{R}^{\leq d}[x] = \{ a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0 \mid a_i \in \mathbb{R} \; \forall i = 0, \dots, d \}$

$$= \langle \underbrace{x^d}_{v_d}, \underbrace{x^{d-1}}_{v_{d-1}}, \dots, \underbrace{x}_{v_1}, \underbrace{1}_{v_0} \rangle$$

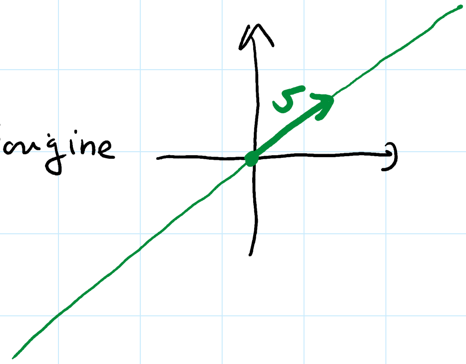
$$\sum_{i=0}^{d} a_i v_i = a_d v_d + \dots + a_0 v_0 = a_d x^d + \dots + a_0$$

Sottospazi di $\mathbb{R}^2$

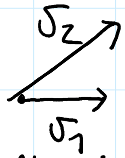
- $\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$

- Se $v \in \mathbb{R}^2$ $v \neq \begin{pmatrix}0\\0\end{pmatrix}$ $\langle v \rangle$ rette per l'origine

  $\langle v \rangle = \{ av \mid a \in \mathbb{R} \}$



- $\langle v_1, v_2 \rangle = \mathbb{R}^2$



  se i due vettori non sono multipli.

3 Modi per descrivere i sottospazi:

Modo ① con generatori

① $\mathbb{R}^3$ $W = \left\langle \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\rangle$

Prendo una combinazione lineare dei generatori

$a\begin{pmatrix}1\\2\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}a+b\\2a\\-b\end{pmatrix}$ e ottengo

Modo ② forma parametrica

② $W = \left\{ \begin{pmatrix}a+b\\2a\\-b\end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

da ② per tornare a ① pongo $\begin{cases} a=1 \\ b=0 \end{cases}$ $\begin{cases} a=0 \\ b=1 \end{cases}$

e riottengo $\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}$.

① $\Rightarrow$ ② $W = \langle v_1, \dots, v_n \rangle$ $W = \left\{ \sum_{i=1}^{n} a_i v_i \;\middle|\; a_i \in \mathbb{R} \;\; \forall i = 1, \dots, n \right\}$

② $\Rightarrow$ ① $W = \langle v_1, \dots, v_n \rangle$

v_i lo trovo ponendo $a_i = 1$ e $a_j = 0$ $\forall j \neq i$

Modo ③ con equazioni cartesiane

③ Equazioni cartesiane $\mathbb{R}^3$

② ⇒ ③

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+b \\ 2a \\ -b \end{pmatrix}$$

Eliminazione dei parametri:

$$\begin{cases} x = a+b \\ \cancel{y = 2a} \\ z = -b \end{cases} \qquad a = \frac{y}{2} \qquad \begin{cases} x = \frac{y}{2} + b \\ \cancel{z = -b} \end{cases} \qquad b = -z$$

$$x = \frac{y}{2} - z \qquad 2x - y + 2z = 0$$

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; \underbrace{2x - y + z = 0}_{} \right\}$$

Provo ora ad eliminare prima b

$$\begin{cases} x = a+b \\ y = 2a \\ \cancel{z = -b} \end{cases} \qquad b = -z \qquad \begin{cases} \cancel{x = a - z} \\ y = 2a \end{cases} \qquad a = x + z$$

$$y = 2x + 2z \qquad 2x - y + 2z = 0$$

$$-2x + y - 2z = 0$$

③ ⇒ ② $\quad W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; 2x - y + 2z = 0 \right\} \qquad y = 2x + 2z$

② $W = \left\{ \begin{pmatrix} x \\ 2x+2z \\ z \end{pmatrix} \middle| \; x, z \in \mathbb{R} \right\}$ ① $W = \left\langle \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\rangle$

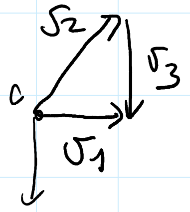
$x = 1$, $z = 0$ $\qquad$ $x = 0$, $z = 1$

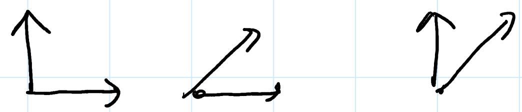
$$\underset{v_2}{\begin{pmatrix}1\\0\\-1\end{pmatrix}} = \underset{v_1}{\begin{pmatrix}1\\2\\0\end{pmatrix}} - \underset{v_3}{\begin{pmatrix}0\\2\\1\end{pmatrix}}$$

$v_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}$ $\quad v_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ $\quad v_2 = v_1 - v_3$

$v_3 = v_1 - v_2$

$v_2 + v_3 = v_1$





Proposizione:

1. ~~Dimostrare che~~ Dati U, W sottospazi di uno stesso k-spazio vettoriale V allora l'intersezione $U \cap W$ è un sottospazio di V.

Dimostrazione:

$U \leq V$

$W \leq V$

① $\begin{cases} 0_V \in U & \text{perché } U \leq V \\ 0_V \in W & \text{perché } W \leq V \end{cases}$ $\qquad 0_V \in U \cap W$

② Chiusura per la somma

Siano $z_1, z_2 \in U \cap W$ allora

$\begin{cases} z_1, z_2 \in U, & U \leq V \Rightarrow U \text{ chiuso per la somma} \Rightarrow z_1 + z_2 \in U \\ z_1, z_2 \in W, & W \leq V \Rightarrow W \text{ chiuso per la somma} \Rightarrow z_1 + z_2 \in W \end{cases}$

$\Rightarrow z_1 + z_2 \in U \cap W.$

③ Chiusura per prodotto per scalari:

sia $z \in U \cap W$ $\quad a \in \mathbb{R}$ $\quad$ allora $az \in U \cap W$ perché

$\begin{cases} z \in U & \text{ma } U \leq V \text{ chiuso per prodotto per scalari} \Rightarrow az \in U \\ z \in W & \text{ma } W \leq V \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \Rightarrow az \in W \end{cases}$

$\Rightarrow az \in U \cap W.$ $\qquad \square$

Esercizio:

Dato $V=\mathbb{R}^4$

$$U=\left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\in\mathbb{R}^4 \,\middle|\, \begin{matrix}x_1-x_2-3x_3=0\\x_2-x_4=0\end{matrix}\right\}$$

$$W=\left\langle\begin{pmatrix}1\\-1\\0\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\\-1\end{pmatrix}\right\rangle$$

determinare generatori di U e generatori di $U\cap W$.

Svolgimento:

U:
$$\begin{cases}x_1-x_2-3x_3=0\\x_2-x_4=0\end{cases} \Leftrightarrow \begin{cases}x_1=x_2+3x_3=x_4+3x_3\\x_2=x_4\end{cases}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \quad U=\left\{\begin{pmatrix}3x_3+x_4\\x_4\\x_3\\x_4\end{pmatrix} \,\middle|\, x_3,x_4\in\mathbb{R}\right\}=\left\{x_3\begin{pmatrix}3\\0\\1\\0\end{pmatrix}+x_4\begin{pmatrix}1\\1\\0\\1\end{pmatrix}\,\middle|\,x_3,x_4\in\mathbb{R}\right\}$$

$$U=\left\langle\begin{pmatrix}3\\0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right\rangle$$

Per intersecare U e W

$$U=\left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\in\mathbb{R}^4 \,\middle|\, \begin{matrix}x_1-x_2-3x_3=0\\x_2-x_4=0\end{matrix}\right\}$$

$$W=\left\langle\begin{pmatrix}1\\-1\\0\\2\end{pmatrix},\begin{pmatrix}0\\2\\0\\-1\end{pmatrix}\right\rangle$$

$z\in U\cap W$ $\qquad$ $z\in W$ se e solo se

$$z=a\begin{pmatrix}1\\-1\\0\\2\end{pmatrix}+b\begin{pmatrix}0\\2\\0\\-1\end{pmatrix}=\boxed{\begin{pmatrix}a\\-a+2b\\0\\2a-b\end{pmatrix}}\in W$$

$$z \in U \quad \begin{cases} x_1 - x_2 - 3x_3 = 0 \\ x_2 - x_4 = 0 \end{cases} \qquad \begin{cases} a - (-a+2b) - 3 \cdot 0 = 0 \\ -a+2b-2a+b=0 \end{cases}$$

$$\begin{cases} 2a - 2b = 0 \\ -3a + 3b = 0 \end{cases} \qquad \begin{cases} a = b \\ a = b \end{cases}$$

$$U \cap W = \left\{ \begin{pmatrix} a \\ -a+2a \\ 0 \\ 2a-a \end{pmatrix} \middle| \, a \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ a \\ 0 \\ a \end{pmatrix} \middle| \, a \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

Generatore di $U \cap W$ $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Calcolo di intersezioni:

U con equazioni cartesiane

W con generatori $W = \langle w_1, \dots, w_k \rangle$

$U \cap W$ si impone $z \in W$ cioè $z = a_1 w_1 + \dots + a_k w_k$

chiediamo che questo $z \in U$ cioè soddisfi le sue equazioni cartesiane.

Oss: $0_V \in U \cap W$ sempre quindi $U \cap W$ non è MAI l'insieme vuoto

Esempio:

$V = \mathbb{R}^3$

$$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \begin{matrix} x - 2y - z = 0 \\ -x + 2y + 3z = 0 \end{matrix} \right\}$$

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, 2x - 4y - 4z = 0 \right\}$$

$$U: \begin{cases} x-2y-z=0 \\ -x+2y+3z=0 \end{cases} \qquad W: 2x-4y-4z=0$$

$$U \cap W: \begin{cases} x-2y-z=0 \\ -x+2y+3z=0 \\ 2x-4y-4z=0 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -1 & 2 & 3 & 0 \\ 2 & -4 & -4 & 0 \end{array}\right)$$

$$\begin{array}{r} \\ II^\circ + I^\circ \\ III^\circ - 2I^\circ \end{array} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} x=2y+\cancel{z} \\ z=0 \end{cases}$$

$$U \cap W = \left\{ \begin{pmatrix} 2y \\ y \\ 0 \end{pmatrix} \middle| \, y \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\rangle \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Esempio:

$M_{2,2}(\mathbb{R}) = V$

$$U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} a+b=0 \\ c+d=0 \end{array} \right\}$$

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a+b-c+d=0 \right\}$$

$$U \cap W \quad \begin{cases} a+b=0 \\ c+d=0 \\ a+b-c+d=0 \end{cases} \qquad \text{incognite} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 \end{array}\right) \xrightarrow{3^\circ - 1^\circ} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right) \quad 3^\circ + 2^\circ$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right) \qquad \begin{cases} a+b=0 \\ c+d=0 \\ 2d=0 \end{cases} \qquad \begin{cases} a=-b \\ c=0 \\ d=0 \end{cases} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$U \cap W = \left\{ \begin{pmatrix} -b & b \\ 0 & 0 \end{pmatrix} \middle| \, b \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle$$

$b=1$

Fatto: dati U e W sottospazi di un $\mathbb{R}$-spazio vettoriale V

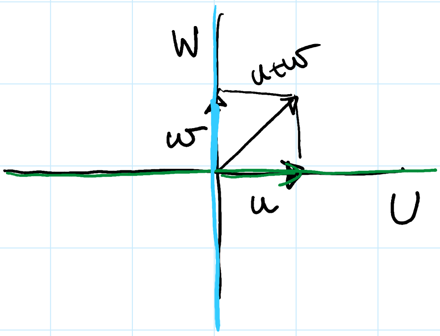
$U \cup W$ è sottospazio se e solo se $\iff$ $U \subseteq W$ oppure $W \subseteq U$

Dim:

$\Leftarrow$ Se $U \subseteq W$ $\quad U \cup W = W \leq V$

Se $W \subseteq U$ $\quad U \cup W = U \leq V$

$\Rightarrow$ Se $U \not\subseteq W$ e $W \not\subseteq U$ allora siano



$u \in U \quad u \notin W$

$w \in W \quad w \notin U$

$\Rightarrow u+w \notin U$ se

$u+w \notin W$

se $u+w$ appartenesse a U $\quad (u+w)-u = w \in U$ perché U è chiuso per comb. lineari ma avevamo supposto $w \notin U$

$\Rightarrow$ $U \cup W$ non è chiuso per la somma

Definizione: dati U e W sottospazi di V $\mathbb{R}$-spazio vettoriale si def.

$$U + W = \{ u + w \mid u \in U, w \in W \}$$ somma dei due sottospazi

Proposizione:

$\underline{U+W \leq V}$ ed è il più piccolo sottospazio che contiene $U \cup W$.

Dim:

Dimostriamo che $U+W \leq V$

① $0_V = 0_V + 0_V \in U+W$
$0_V \in U \quad 0_V \in W$

② Ch. per la somma $u_1+w_1 \quad u_2+w_2$

$(u_1+w_1)+(u_2+w_2) = (u_1+u_2)+(w_1+w_2) \in U+W$
$U \quad + \quad W$

③ Ch. per prod. per scalari $u+w \quad a \in \mathbb{R}$

$a(u+w) = au + aw \in U+W.$
$U + W$

$U \subseteq U+W \qquad u = u + 0_V$
$U+W$

$W \subseteq U+W \qquad w = 0_V + w$
$U+W$

Se $Z \leq V$ che contiene e U e W essendo chiuso per la somma $\{u+w \mid u \in U \ w \in W\} \subseteq Z$.

Esempi:

① $V = \mathbb{R}^4$

$U = \left\langle \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ a \end{pmatrix} \right\rangle \qquad W = \left\langle \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

Determinare generatori di $U+W$ e di $U\cap W$.

$$\left\langle \begin{pmatrix}1\\2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\\2\end{pmatrix}, \begin{pmatrix}2\\0\\0\\1\end{pmatrix} \right\rangle = U+W$$

$U=\langle u_1,\dots u_h\rangle \qquad W=\langle w_1,\dots,w_k\rangle$

$U+W=\langle u_1,\dots,u_h, w_1,\dots,w_k\rangle$

Cerchiamo eq. cartesiane per U

$$U=\left\langle \begin{pmatrix}1\\2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\rangle = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \;\middle|\; \begin{matrix}x_1=x_4\\x_3=0\end{matrix} \right\}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = a\begin{pmatrix}1\\2\\0\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}a\\2a+b\\0\\a\end{pmatrix} \qquad \begin{cases}x_1=a\\x_2=2a+b\\x_3=0\\\cancel{x_4=a}\end{cases} \qquad a=x_4$$

$$\begin{cases}x_1=x_4\\\cancel{x_2=2x_4+b}\\x_3=0\end{cases} \qquad b=x_2-2x_4$$

$$W=\left\langle \begin{pmatrix}2\\0\\1\\2\end{pmatrix}, \begin{pmatrix}2\\0\\0\\1\end{pmatrix} \right\rangle \qquad U: \begin{cases}x_1=x_4\\x_3=0\end{cases}$$

$$z= a\begin{pmatrix}2\\0\\1\\2\end{pmatrix} + b\begin{pmatrix}2\\0\\0\\1\end{pmatrix} = \begin{pmatrix}2a+2b\\0\\a\\2a+b\end{pmatrix} \in W$$

$$z\in U \qquad \begin{cases}\cancel{2a}+2b=\cancel{2a}+b\\a=0\end{cases} \qquad \begin{cases}b=0\\a=0\end{cases}$$

$$z=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \qquad \text{quindi} \qquad U\cap W=\left\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right\}$$