Es:

Risolvere il sistema lineare

$$\begin{cases} -x+y=1 \\ x+2y-2z=-1 \\ y+z=5 \end{cases}$$

3 equazioni in 3 incognite x, y, z con coefficienti reali:

$(A|b)$ $\left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 5 \end{array}\right)$ matrice completa associata al sistema lineare

$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ matrice incompleta associata al sist. lin.

$b = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ colonna dei termini noti

$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ colonna delle incognite

$\Sigma: \quad A\underline{x} = b$

$$\begin{cases} -x+y=1 \\ x+2y-2z=-1 \\ y+z=5 \end{cases} \qquad \left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 5 \end{array}\right) \quad A|b$$

Elim. di Gauss sulla matrice completa

Elim. di Gauss sulla matrice completa

$$\begin{array}{r} -1^\circ R \\ 2^\circ R + 1^\circ R \\ 3^\circ R \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 3 & -2 & 0 \\ 0 & 1 & 1 & 5 \end{array}\right)$$

scambio 2° con $3^\circ R$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & 3 & -2 & 0 \end{array}\right)$$

$3^\circ R - 3 \cdot 2^\circ R$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -5 & -15 \end{array}\right) \qquad -5z = -15$$

$-\frac{1}{5} 3^\circ R$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\begin{cases} x - y = -1 \\ y + z = 5 \\ z = 3 \end{cases} \qquad \uparrow \text{ x sostituzione} \qquad \begin{cases} x = -1 + y = -1 + 2 = 1 \\ y = 5 - z = 5 - 3 = 2 \\ z = 3 \end{cases}$$

Ha un'unica soluzione $Sol_{A|b} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

Prop: se $A\underline{x} = b$ è un sistema lineare e
$A'\underline{x} = b'$ è un sistema lineare tale che

la matrice completa $(A'|b')$ è ottenuta da $(A|b)$ tramite operazioni elementari allora

$$Sol_{A|b} = Sol_{A'|b'}$$

operazioni elementari allora

$$Sol_{A|b} = Sol_{A'|b'}.$$

Esempio: risolvere il sistema lineare dipendente da $k \in \mathbb{R}$

$$\begin{cases} kx - ky + z = 0 \\ x + y + z = 1 \\ k^2x + k^2y + z = 0 \end{cases}$$

nelle incognite x, y, z.

Sol:

Passo 1 determinare la matrice completa associata al sistema lineare

$$(A_k|b) = \left(\begin{array}{ccc|c} k & -k & 1 & 0 \\ 1 & 1 & 1 & 1 \\ k^2 & k^2 & 1 & 0 \end{array}\right) \qquad M_{3,4}$$

Passo 2: ridurre con Gauss

Sc. 1° con 2°

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ k & -k & 1 & 0 \\ k^2 & k^2 & 1 & 0 \end{array}\right)$$

$$\boxed{\begin{array}{ll} 1 & \\ k & 2° - k\,1° \\ k^2 & 3° - k^2\,1° \end{array}}$$

$$\begin{array}{l} \\ 2° - k\,1° \\ 3° - k^2\,1° \end{array} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2k & 1-k & -k \\ 0 & 0 & 1-k^2 & -k^2 \end{array}\right)$$

$$\begin{array}{ll} 1 & \\ 2 & 2° - 2\,1° \\ -5 & 3° + 5\,1° \\ 4 & 4° - 4\,1° \end{array}$$

$2^\circ - k\,1^\circ$

$3^\circ - k^2\,1^\circ$

$$\left(\begin{array}{ccc|c} 0 & -2k & 1-k & -k \\ 0 & 0 & 1-k^2 & -k^2 \end{array}\right)$$

$2 \quad 2^\circ - 2\,1^\circ$

$-5 \quad 3^\circ + 5\,1^\circ$

$4 \quad 4^\circ - 4\,1^\circ$

Passo 3: segnare i "candidati pivot" e discutere al variare di $k \in \mathbb{R}$ quando sono pivot oppure no

$$\begin{cases} -2k \neq 0 \longrightarrow k \neq 0 \\ 1-k^2 \neq 0 \longrightarrow k \neq 1 \ \text{e} \ k \neq -1 \end{cases}$$

$\forall\, k \in \mathbb{R} \setminus \{0, 1, -1\}$ $\quad rg\,A = 3 = rg(A|b)$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2k & 1-k & -k \\ 0 & 0 & 1-k^2 & -k^2 \end{array}\right) \qquad \begin{cases} x+y+z = 1 \\ (-2k)y + (1-k)z = -k \\ (1-k^2)z = -k^2 \end{cases}$$

$$z = \frac{-k^2}{1-k^2}$$

$$(-2k)y = -k - (1-k)z = -k - \frac{(\cancel{1-k})(-k^2)}{(\cancel{1-k})(1+k)} = -k + \frac{k^2}{1+k} =$$

$$= \frac{-k - \cancel{k^2} + \cancel{k^2}}{1+k} = -\frac{k}{1+k}$$

$$\boxed{(+2k)y = +\frac{k}{1+k}} \qquad y = \frac{-k}{(-2k)(1+k)} \qquad \alpha y = \beta \quad \alpha \neq 0 \qquad y = \frac{\beta}{\alpha}$$

$$Y = \frac{\beta}{\alpha}$$

$$y = \frac{1}{2+2k}$$

$$x = 1 - y - z = 1 - \underbrace{\frac{1}{2+2k}}_{2(1+k)} + \underbrace{\frac{k^2}{1-k^2}}_{(1-k)(1+k)}$$

$$= \frac{2 - \cancel{2k^2} - 1 + k + \cancel{2k^2}}{2(1-k)(1+k)} = \frac{\cancel{k+1}}{2(1-k)\cancel{(1+k)}} = \frac{1}{2-2k}$$

$$Sol_{A_k|b} = \left\{ \begin{pmatrix} \frac{1}{2-2k} \\ \frac{1}{2+2k} \\ -\frac{k^2}{1-k^2} \end{pmatrix} \right\} \qquad \forall\, k \in \mathbb{R} \setminus \{0, 1, -1\}$$

Passo 4: studiare separatamente i casi rimasti

$k=0$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2k & 1-k & -k \\ 0 & 0 & 1-k^2 & -k^2 \end{array}\right)$$

$$A_0|b = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{3°-2°} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$rg\, A_0 = 2 = rg(A_0|b)$$

$$\begin{cases} x + y + z = 1 \\ z = 0 \\ 0 = 0 \end{cases} \qquad \begin{cases} x = 1 - y \\ z = 0 \end{cases} \qquad y = 1 - x$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} 1-y \\ y \\ 0 \end{pmatrix}$$

$$\boxed{Sol_{A_0|b} = \left\{ \begin{pmatrix} 1-y \\ y \\ 0 \end{pmatrix} \;\middle|\; y \in \mathbb{R} \right\}}$$

$$\begin{pmatrix} y \\ z \end{pmatrix} \quad \begin{pmatrix} \cdot \\ y \\ 0 \end{pmatrix} \quad A_0|b \quad \left\{ \begin{pmatrix} y \\ 0 \end{pmatrix} \right\}$$

$$y=0 \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underbrace{y}_{\text{coeff. di } y}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-y \\ y \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$y=0$

$$k=1 \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2k & 1-k & -k \\ 0 & 0 & 1-k^2 & -k^2 \end{array}\right)$$

$$A_1|b \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right) \qquad 0=-1 \quad \text{impossibile}$$

$$rg A_1 = 2 \neq rg(A_1|b) = 3$$

$$Sol_{A_1|b} = \emptyset$$

$$k=-1 \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2k & 1-k & -k \\ 0 & 0 & 1-k^2 & -k^2 \end{array}\right)$$

$$(A_{-1}|b) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right) \qquad \begin{array}{l} rg A_{-1} = 2 \\ rg(A_{-1}|b) = 3 \end{array}$$

$$\left(\begin{array}{ccc|c} 0 & 2 & 2 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right) \qquad rg(A_{-1}|b) = 3$$

$0 = -1$ impossibile $\quad Sol_{A_{-1}|b} = \emptyset.$

Per $k=1$ e $k=-1$ $\quad Sol_{A_k|b} = \emptyset$

Per $k=0$ $\quad Sol_{A_0|b} = \left\{ \begin{pmatrix} 1-y \\ y \\ 0 \end{pmatrix} \middle| \, y \in \mathbb{R} \right\}$

Per $k \in \mathbb{R} \setminus \{0,1,-1\}$ $\quad Sol_{A_k|b} = \left\{ \begin{pmatrix} \frac{-1}{2k-2} \\ \frac{1}{2k+2} \\ \frac{-k^2}{1-k^2} \end{pmatrix} \right\}$

Enunciato del Teorema di Rouché-Capelli

Un sistema lineare

$$A\underline{x} = b \quad \text{con} \quad A \in M_{m,n}(\mathbb{R}) \quad b \in \mathbb{R}^m$$

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

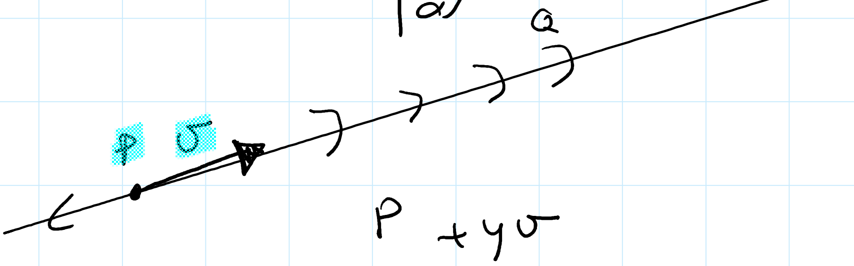
ha soluzione se e solo se $rgA = rg(A|b) = r$

Le soluzioni dipendono da $n-r$.

Inter. geom:

$k=0$ $\quad \left\{ \begin{pmatrix} 1-y \\ y \\ 0 \end{pmatrix} \middle| \, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$

$y=0 \qquad P=\begin{pmatrix}1\\0\\a\end{pmatrix}$



$$\begin{pmatrix}1-y\\y\\0\end{pmatrix}=\underbrace{\begin{pmatrix}1\\0\\a\end{pmatrix}}_{}\!\!\!\!\!\!\!\!\!^{P}+y\begin{pmatrix}-1\\1\\a\end{pmatrix} \qquad P+yv$$

$$\begin{pmatrix}1\\0\\a\end{pmatrix}+\left\langle\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\rangle \quad v$$

$$\begin{pmatrix}0\\0\\a\end{pmatrix}+\left\langle\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\rangle$$

$x+y+z=1 \qquad x=1-y-z$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \qquad \left\{\begin{pmatrix}1-y-z\\y\\z\end{pmatrix} \,\middle|\, y,z\in\mathbb{R}\right\}$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}+\left\langle\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\rangle$$

Osservazione:

Un sistema lineare a coeff. reali può avere solo 3 tipi di soluzioni:

(A) $\emptyset$ non ha soluzioni

(B) un' unica soluzione

(C) infinite soluzioni

Prodotto di matrici

Prodotto riga per colonne

$$(a_1\ a_2\ \dots\ a_n)\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix} = a_1x_1+a_2x_2+\dots+a_nx_n=\sum_{i=1}^{n} a_ix_i$$

$M_{1,\cancel{n}}$ $\quad M_{\cancel{n},1}$ $\quad M_{1,1}$

$$(1\ \ 3)\begin{pmatrix}-1\\2\end{pmatrix}=-1+6=5$$

$1\times\cancel{2}$ $\quad \cancel{2}\times 1$ $\quad 1\times 1$ $\quad M_{1,\cancel{n}}$ $\quad M_{\cancel{n},1}$

Se $A\in M_{m,\underline{n}}(\mathbb{R})$ $\quad B\in M_{\underline{n},p}(\mathbb{R})$

$AB\in M_{m,p}(\mathbb{R})$

$$(AB)_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$$

$$c_{11}=(1\ 0\ 2)\begin{pmatrix}2\\0\\1\end{pmatrix}=4$$

Esempi:

1) $$\begin{pmatrix}1&0&2\\-1&5&1\end{pmatrix}\begin{pmatrix}2&0\\0&1\\1&0\end{pmatrix}=\begin{pmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{pmatrix}=\begin{pmatrix}4&0\\-1&5\end{pmatrix}$$

$2\times\cancel{3}$ $\quad \cancel{3}\times 2$ $\quad 2\times 2$

$$c_{12}=(1\ 0\ 2)\begin{pmatrix}0\\1\\0\end{pmatrix}=1\cdot 0+0\cdot 1+2\cdot 0=0$$

$$c_{21} = (-1\ 5\ 1)\begin{pmatrix}2\\0\\1\end{pmatrix} = (-1)\cdot 2 + 5\cdot 0 + 1\cdot 1 = -2+1 = -1$$

$$c_{22} = (-1\ 5\ 1)\begin{pmatrix}0\\1\\0\end{pmatrix} = (-1)\cdot 0 + 5\cdot 1 + 1\cdot 0 = 5$$

$$\begin{pmatrix}2 & 0\\0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 2\\-1 & 5 & 1\end{pmatrix} = \begin{pmatrix}d_{11} & d_{12} & d_{13}\\d_{21} & d_{22} & d_{23}\\d_{31} & d_{32} & d_{33}\end{pmatrix} = \begin{pmatrix}2 & 0 & 4\\-1 & 5 & 1\\1 & 0 & 2\end{pmatrix}$$

$3 \times \cancel{2}$ $\quad$ $\cancel{2} \times 3$ $\quad\quad$ $M_{3,3}(\mathbb{R})$

$$d_{11} = (2\ 0)\begin{pmatrix}1\\-1\end{pmatrix} = 2+0 = 2$$

2×3 $\quad$ 2×2

A $\quad$ B $\quad\quad$ $\cancel{AB}$

$BA \in M_{2,3}(\mathbb{R})$

B $\quad$ A

$2 \times \cancel{2}$ $\quad$ $\cancel{2} \times 3$

Se $A, B \in M_{2,2}(\mathbb{R})$ $\quad$ AB

BA

$$(AB)C = A(BC)$$

$A \in M_{2,n}(\mathbb{R})$

$$\mathbb{I}_2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad \mathbb{I}_2 A = A$$

$$B\,\mathbb{I}_2 = B \qquad \forall B \in M_{m,2}(\mathbb{R})$$

$m \times \cancel{2}\ \cancel{2} \times 2$

Def. Se $A \in M\ (\mathbb{R})$ ...

$m \times \cancel{x}$ $x \times x$

Def: se $A \in M_{n,n}(\mathbb{R})$ quadrata si dice invertibile se $\exists$ $B \in M_{n,n}(\mathbb{R})$ tale che

$$AB = \mathbb{I}_n = BA \qquad B = A^{-1}$$

Osservazione:

Se $S, C \in M_{n,n}(\mathbb{R})$ sono invertibili allora

$$C^{-1}S^{-1}SC = C^{-1}C = \mathbb{I}_n \rightarrow (SC)^{-1} = C^{-1}S^{-1}.$$