Numeri complessi: F. algebrica $a+ib$
F. cartesiana (a,b)
F. Trigonometrica $\rho(\cos\theta + i\sin\theta)$
F. Esponenziale $\rho e^{i\theta}$

Esercizio:

1) $z^4 = \frac{i(\bar{z})^3}{8}$ [R. 0, $\frac{1}{8}\left(\cos\left(\frac{\pi}{14}+\frac{2}{7}k\pi\right)+i\sin\left(\frac{\pi}{14}+\frac{2}{7}k\pi\right)\right)$.]

Determinare tutti i $z \in \mathbb{C}$ tali che $z^4 = \frac{i(\bar{z})^3}{8}$.

Scriviamo z in forma esponenziale: $z = \rho e^{i\theta}$ con $\rho \geq 0$, $\theta \in \mathbb{R}$

allora $\bar{z} = \rho e^{-i\theta}$

Il termine di sx $z^4 = \rho^4 e^{i4\theta}$

Il termine di dx $\frac{i(\bar{z})^3}{8} = \frac{e^{i\pi/2}(\rho e^{-i\theta})^3}{8} = \frac{\rho^3}{8} e^{i\pi/2} e^{-3i\theta}$

eguagliando

$$\rho^4 e^{i4\theta} = \frac{\rho^3}{8} e^{i\pi/2} e^{-3i\theta}$$ moltiplichiamo $e^{i3\theta}$

$$\boxed{\rho^4} e^{i7\theta} = \boxed{\frac{\rho^3}{8}} e^{i\pi/2} \iff \begin{cases} \rho^4 = \frac{\rho^3}{8} \\ 7\theta = \frac{\pi}{2} + 2k\pi \end{cases} \quad \begin{cases} \rho^4 - \frac{\rho^3}{8} = 0 \\ \theta = \frac{\pi}{14} + \frac{2k\pi}{7} \end{cases}$$

$$\begin{cases} \rho^3\left(\rho - \frac{1}{8}\right) = 0 \\ \theta = \frac{\pi}{14} + \frac{2k\pi}{7} \end{cases} \Rightarrow \rho = 0 \Rightarrow z = 0 \quad \text{oppure} \quad \begin{cases} \rho = 1/8 \\ \theta = \frac{\pi}{14} + \frac{2k\pi}{7} \end{cases}$$

$$z_k = \frac{1}{8} e^{i\left(\frac{\pi}{14} + \frac{2k\pi}{7}\right)} \quad \text{con} \quad k = 0, 1, \dots, 6.$$

Teorema fondamentale dell'algebra

Ogni polinomio

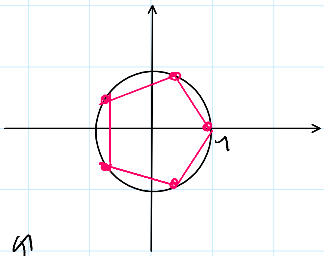
$$a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0 = \sum_{i=0}^{d} a_i x^i \quad d \geq 1$$

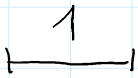
$a_d \neq 0 \quad a_i \in \mathbb{C} \quad \forall\, i = 0, \dots, d$

ha almeno una radice in $\mathbb{C}$.

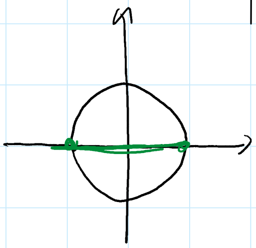
Es:

$x^5 = 1$





$x^2 = 1$



Corollario: se $p(x) = \sum_{i=0}^{d} a_i x^i \quad a_i \in \mathbb{R}$ ha grado $d \geq 1$

e supponiamo che $z \in \mathbb{C}$ sia una radice di $p(x)$ allora anche $\bar{z}$ è radice di $p(x)$.

Dim:

$a_i \in \mathbb{R} \iff a_i = \overline{a_i}$

z è radice di $p(x) \iff p(z) = 0$

$$0 = p(z) = \sum_{i=0}^{d} a_i z^i$$

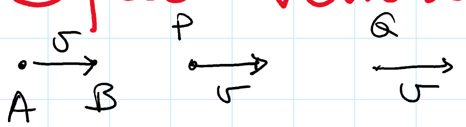
$$0 = \bar{0} = \overline{p(z)} = \overline{\sum_{i=0}^{d} a_i z^i} = \sum_{i=0}^{d} \overline{a_i z^i} = \sum_{i=0}^{d} \overline{a_i}\, \bar{z}^i = \sum_{i=0}^{d} a_i \bar{z}^i = p(\bar{z})$$

$\square$

Cor: se $p(x)$ ha coefficienti reali e ha grado dispari allora esiste sempre una radice reale.

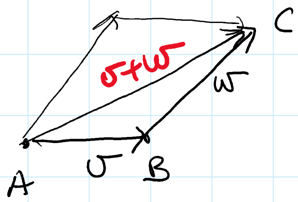
Spazi vettoriali

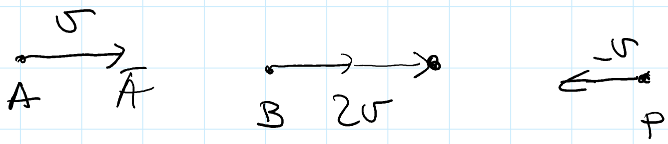
Vettore



$v = 0_V$

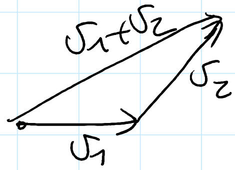
Somma di vettori





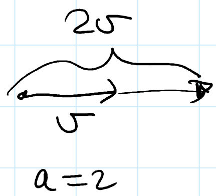
Definizione: dato K un campo uno **spazio vettoriale** su K è un insieme non vuoto dotato di due operazioni

$$+ : V \times V \longrightarrow V$$
$$(v_1, v_2) \longmapsto v_1 + v_2$$



somma di vettori

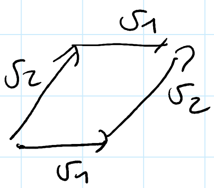
$$\cdot : K \times V \longrightarrow V$$
$$(a, v) \longmapsto av$$



prodotto per scalari

gli elementi di K vengono detti scalari.

a) $+$ è commutativa



associativa

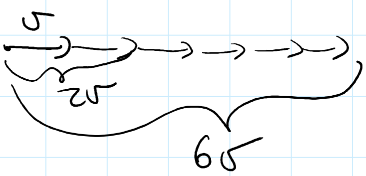
$(v_1+v_2)+v_3 = v_1+(v_2+v_3)$

Esiste un vettore nullo 0_V elemento neutro per $+$

$v + 0_V = v$

Ogni vettore $v \in V$ ha un opposto $-v \in V$

b) $a(bv) = (ab)v$ $\quad \forall a,b \in K$ $\quad \forall v \in V$



$(a+b)v = av + bv$

$a(v_1+v_2) = av_1 + av_2$

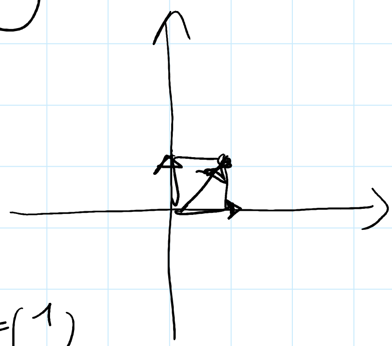
$1 \cdot v = v$

Esempi:

$K = \mathbb{R}$ piano cartesiano

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

$$v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$



$$v_1 + v_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$0_{\mathbb{R}^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

vettore colonne

$$\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$(1,5) + (2,3) - (1,6)$$

Esempio $\mathbb{R}^3$

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \qquad 3 \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 21 \\ 6 \\ -9 \end{pmatrix}$$

$$a = 3 \qquad v = \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} \qquad a\, v \qquad a \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ az \end{pmatrix}$$

Spazi vettoriali standard di dimensione $n \geq 1$

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; x_i \in \mathbb{R} \quad \forall\, i = 1, \dots, n \right\}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix} \qquad 0_{\mathbb{R}^n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Esempio:
Polinomi $\displaystyle\sum_{i=0}^{d} a_i x^i$

$$\mathbb{R}[x] = \left\{ \sum_{i=0}^{d} a_i x^i \;\middle|\; d \in \mathbb{N} \quad a_i \in \mathbb{R} \quad \forall i = 0, .., d \right\}$$

$$3x^3 - x + 1 = 3x^3 + 0x^2 + (-1)x + 1$$

$\mathbb{R}[x]$ ha dimensione infinita

$$\mathbb{R}_{\leq d}[x] = \left\{ \sum_{i=0}^{d} a_i x^i \;\middle|\; a_i \in \mathbb{R} \quad \forall i = 0, .., d \right\}$$

$$\mathbb{R}_{\leq 2}[x] = \left\{ ax^2 + bx + c \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

$a_d \quad a_{d-1} \ldots \quad a_1 \quad a_0 \qquad d+1$

MATRICI

Definizione: una **matrice $m \times n$** con $m, n \in \mathbb{N}$ $m, n \geq 1$ ad entrate in un campo K è il dato di mn elementi di K

$$a_{ij} \qquad \begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix} \qquad \text{(sono } mn \text{ numeri)}$$

disposti in una tabella con m righe e n colonne

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}.$$

Indichiamo con

$$M_{m,n}(\mathbb{R}) = \left\{ A \,\middle|\, \begin{array}{l} A \text{ matrice } m \text{ righe} \\ n \text{ colonne a coeff. in } \mathbb{R} \end{array} \right\}$$

Es:

$M_{3,2}(\mathbb{R})$
$$\begin{array}{c} \\ \to \\ \to \\ \to \end{array} \begin{array}{c} \begin{array}{cc} \downarrow & \downarrow \end{array} \\ \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix} \end{array} \in M_{3,2}(\mathbb{R})$$

$$\underset{A}{\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix}} + \underset{B}{\begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 3 & 4 \end{pmatrix}} = \underset{C}{\begin{pmatrix} 3 & -1 \\ 1 & -1 \\ 6 & 1 \end{pmatrix}}$$

$A = (a_{ij})_{\substack{1\le i\le 3 \\ 1\le j\le 2}}$ $\quad a_{31} = 3 \quad a_{22} = -2$

$B = (b_{ij})_{\substack{1\le i\le 3 \\ 1\le j\le 2}}$ $\quad b_{12} = 0 \quad b_{32} = 4$

$A+B = C = (c_{ij}) \qquad c_{ij} = a_{ij} + b_{ij}$

$$0_{M_{3,2}(\mathbb{R})} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad -\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}$$

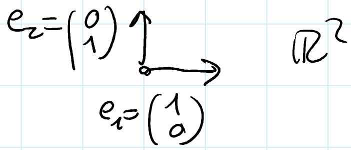
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Def. una matrice quadrata di ordine n è $M_{n,n}(\mathbb{R}) =: M_n(\mathbb{R})$

2×2

$$\begin{array}{c} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{matrix} e_1 & e_2 \end{matrix} \end{array} = \mathbb{I}_2$$

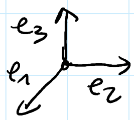
matrice identità



3×3

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \end{array} = \mathbb{I}_3$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



$$\mathbb{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$a_{11} = 1$

$a_{22} = 1$

$a_{33} = 1$

$a_{ij} = 0$ con $i \neq j$

$$\mathbb{I}_n = (\delta_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}} \qquad \delta_{ij} = \begin{cases} 1 & \text{se } i = j \\ 0 & \text{se } i \neq j \end{cases}$$

Delta di Kronecker

$\mathbb{R}^1 = \mathbb{R} = M_{1,1}(\mathbb{R})$ $\qquad 3 \in \mathbb{R}$

$(3) \quad 3 \quad (3)$