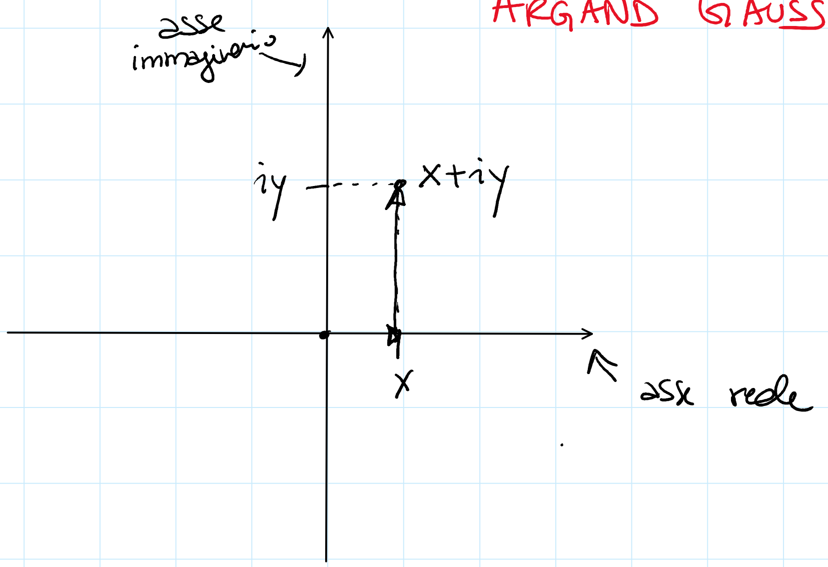
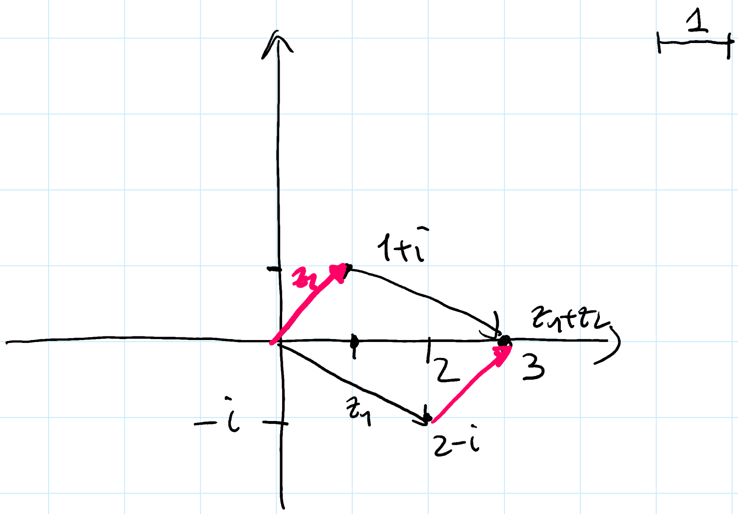
Lezione2

martedì 1 ottobre 2024 12:09

PIANO DI ARGAND GAUSS

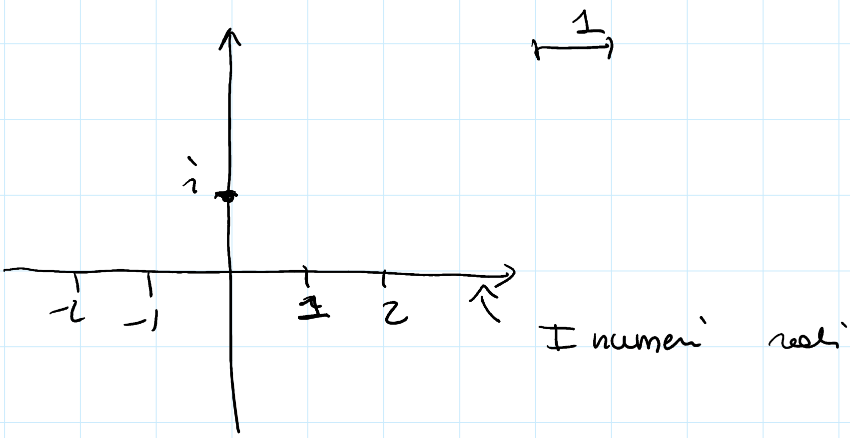


$$\mathbb{C} = \{ a+ib \mid a,b \in \mathbb{R} \} \qquad z = x+iy$$



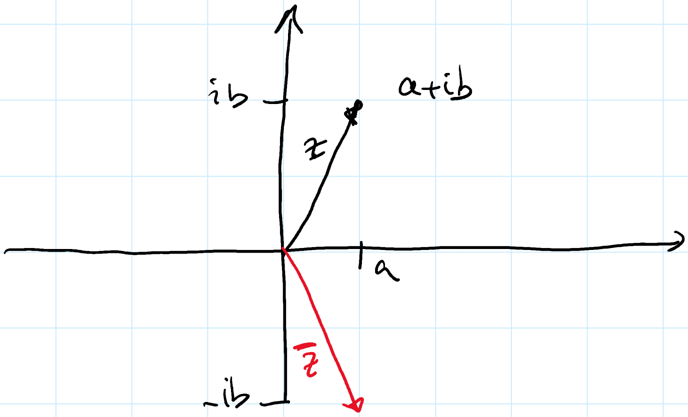
$$z_1 = 2-i$$

$$z_1+z_2 = 2-i+1+i = 3 \qquad z_2 = 1+i$$



Coniugato di un numero complesso

Def: dato $z = a + ib \in \mathbb{C}$ si dice **coniugato di z**
$\bar{z} = a - ib$



1) $\bar{\bar{z}} = z$ $\quad z = a+ib \quad \bar{z} = a - ib$

$\bar{\bar{z}} = a + ib$

2) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ $\quad a_1 + a_2 + i(b_1 + b_2)$

$\overline{z_1+z_2} = \boxed{a_1 + a_2 - i(b_1+b_2)} =$

$(a_1 - ib_1) + (a_2 - ib_2)$

3) $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$ $\quad$ verifica per casa

$z_1 = a + ib$

$z_2 = a' + ib'$

$\overline{(a+ib)(a'+ib')} = (a-ib)(a'-ib')$ $\quad i^2 = -1$

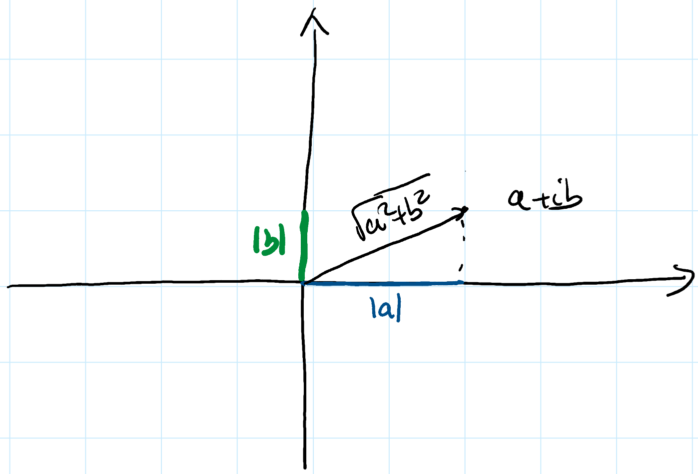
4) $z = a + ib$

$z \cdot \bar{z} = (a+ib)(a-ib) = a^2 - \cancel{iab} + \cancel{iab} - i^2 b^2 =$

$= a^2 + b^2 \in \mathbb{R}$

$a^2 + b^2 \geq 0 \quad \forall z \in \mathbb{C}$

$z \cdot \bar{z} = 0 \iff \begin{matrix} a = 0 \\ b = 0 \end{matrix}$ cioè $z = 0$

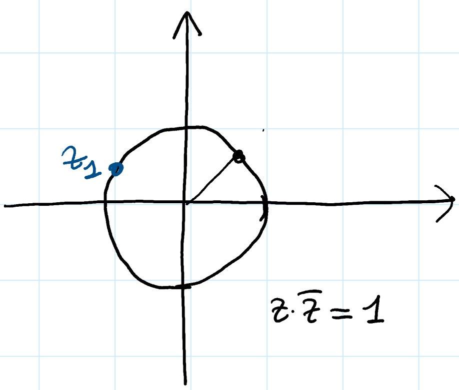


Def. dato $z=a+ib \in \mathbb{C}$ si dice **modulo di z** il numero reale $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2+b^2}$

Es:

$z = 1-3i \qquad |z| = \sqrt{1+9} = \sqrt{10}$

$z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \qquad |z| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$



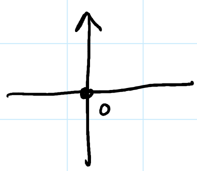
$z \cdot \bar{z} = 1$

L'insieme dei numeri z tali che $|z|=1$ è la circonf. di centro 0 e raggio 1 nel piano di AG

Propr. del modulo: $\forall z \in \mathbb{C}$

$|z| \geq 0 \qquad \forall z \in \mathbb{C}$

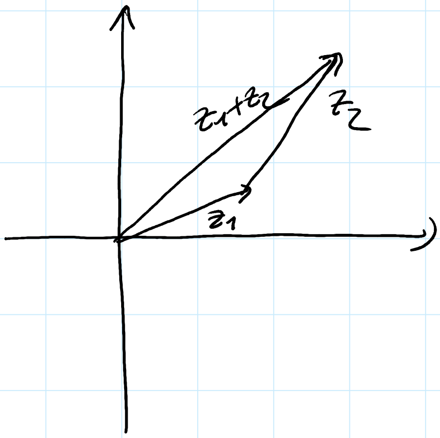
$|z| = 0 \iff z = 0$

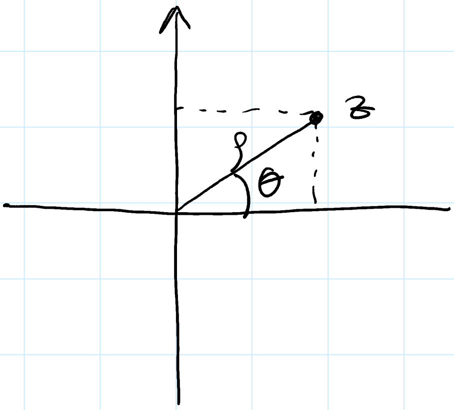


$|z_1 \cdot z_2| = |z_1|\,|z_2|$

$|z_1+z_2| \le |z_1|+|z_2|$

dis. triangolare





$z = \rho\cos\theta + i\rho\,\mathrm{sen}\,\theta$

Forma trigonometrica di un numero complesso

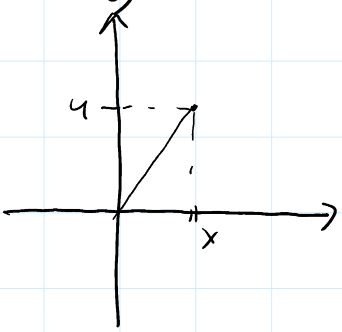
$z = \rho(\cos\theta + i\,\mathrm{sen}\,\theta)$

$\rho = |z|$

θ argomento radianti

$\theta + 2k\pi \quad k \in \mathbb{Z}$

$P = (x,y) \qquad z = x+iy$



$\rho = \sqrt{x^2+y^2}$

$(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta) = (x,y)$

$$\boxed{\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{cases}}$$

$x^2+y^2 = \rho^2\cos^2\theta + \rho^2\mathrm{sen}^2\theta = \rho^2 \qquad \rho = \sqrt{x^2+y^2}$

$z = 1+i \qquad \rho = \sqrt{1^2+1^2} = \sqrt{2}$

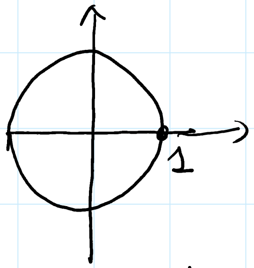
Se $z=0$ la forma trigon. è 0

Se $z\neq 0$ $\rho=|z|\neq 0$ $\begin{cases}\cos\theta = \frac{x}{\rho}\\ \operatorname{sen}\theta = \frac{y}{\rho}\end{cases}$

Es:

$1 = 1(\cos 0 + i \operatorname{sen} 0)$

$1+i\cdot 0$



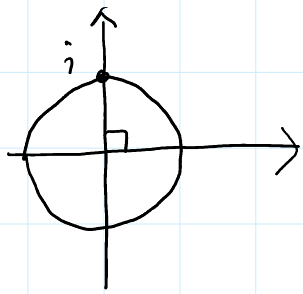
$z=1$ $\rho=|z|=\sqrt{1^2+0^2}=1$

$\begin{cases}\cos\theta = \frac{x}{\rho} = \frac{1}{1} = 1\\ \operatorname{sen}\theta = 0\end{cases}$

$|z|=1$

$\theta = 2k\pi \quad k\in\mathbb{Z}$

$i = 1\left(\cos\frac{\pi}{2} + i\operatorname{sen}\frac{\pi}{2}\right)$

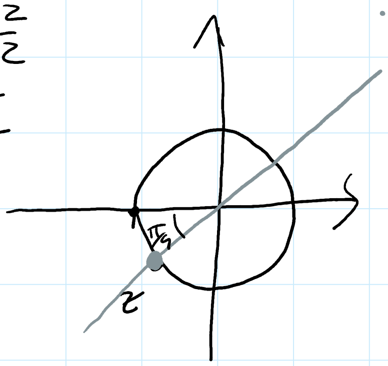


$i = 0 + i\cdot 1$ $\begin{cases}y=1\\ x=0\end{cases}$

$\rho = |i| = \sqrt{x^2+y^2} = \sqrt{0^2+1^2} = 1$

$\begin{cases}\cos\theta = \frac{x}{\rho} = 0\\ \operatorname{sen}\theta = \frac{y}{\rho} = 1\end{cases}$

$z = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ $x=-\frac{\sqrt{2}}{2}$ $y=-\frac{\sqrt{2}}{2}$



$|z| = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$

$\theta = \frac{5}{4}\pi$

$\begin{cases}\cos\theta = \frac{x}{\rho} = -\frac{\sqrt{2}}{2}\\ \operatorname{sen}\theta = \frac{y}{\rho} = -\frac{\sqrt{2}}{2}\end{cases}$

Proposizione:

Dati due complessi non nulli in forma trigonometrica

$z_1 = \rho_1 (\cos\theta_1 + i \operatorname{sen}\theta_1)$ e

$z_2 = \rho_2 (\cos\theta_2 + i \operatorname{sen}\theta_2)$

$$\boxed{z_1 z_2 = \rho_1\rho_2 [\cos(\theta_1+\theta_2) + i \operatorname{sen}(\theta_1+\theta_2)]}$$

cioè il modulo del prodotto è il prodotto dei moduli

l'argomento del prodotto è la somma degli argomenti.

Dim:

$$z_1 z_2 = \rho_1 (\cos\theta_1 + i \operatorname{sen}\theta_1)\, \rho_2 (\cos\theta_2 + i \operatorname{sen}\theta_2) =$$

$$= \rho_1 \rho_2 [\underbrace{\cos\theta_1 \cos\theta_2 - \operatorname{sen}\theta_1 \operatorname{sen}\theta_2} + \underbrace{i \operatorname{sen}\theta_1 \cos\theta_2 + i \cos\theta_1 \operatorname{sen}\theta_2}_{i(\cos\theta_1 \operatorname{sen}\theta_2 + \operatorname{sen}\theta_1 \cos\theta_2)}]$$

$$= \rho_1 \rho_2 [\cos(\theta_1+\theta_2) + i \operatorname{sen}(\theta_1+\theta_2)]$$

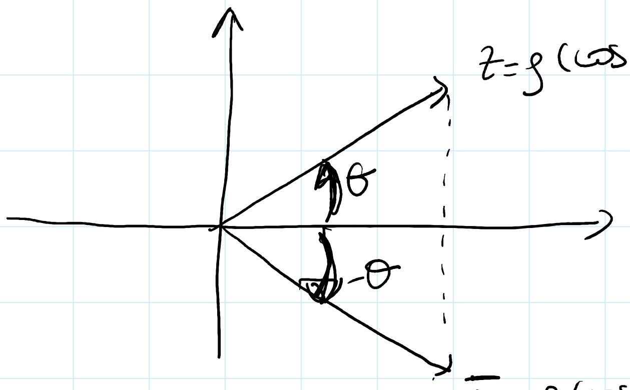
Corollario: se $z \neq 0$ $\quad z = \rho(\cos\theta + i \operatorname{sen}\theta)$

allora $\boxed{z^{-1} = \frac{1}{z} = \frac{1}{\rho} [\cos(-\theta) + i \operatorname{sen}(-\theta)]}$

Dim:

$$z \cdot \frac{1}{z} = 1 = 1 (\cos 0 + i \operatorname{sen} 0)$$

$$\rho(\cos\theta + i \operatorname{sen}\theta)\, \frac{1}{\rho}\big(\cos(-\theta) + i \operatorname{sen}(-\theta)\big) = 1$$



$z=\rho(\cos\theta+i\,\text{sen}\,\theta)$

$\bar{z}=\rho(\cos\theta-i\,\text{sen}\,\theta)$

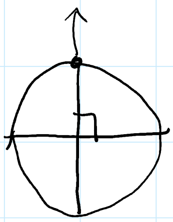
$|z|=\sqrt{z\cdot\bar{z}}$

$$\boxed{\frac{1}{z}=\frac{1}{z}\,\frac{\bar{z}}{\bar{z}}=\frac{\bar{z}}{z\cdot\bar{z}}=\frac{\bar{z}}{|z|^2}}$$

Esempio:

$z_1=\rho(\cos\theta+i\,\text{sen}\,\theta)$

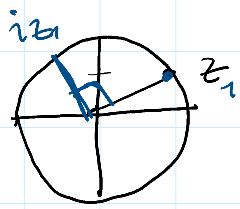
$z_2=i=1\left(\cos\frac{\pi}{2}+i\,\text{sen}\,\frac{\pi}{2}\right)$



$z_1z_2=\rho\left[\cos\left(\frac{\pi}{2}+\theta\right)+i\,\text{sen}\left(\frac{\pi}{2}+\theta\right)\right]$

$z_1=\frac{\sqrt{3}}{2}+\frac{i}{2}$ $\qquad |z_1|=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$

$$\begin{cases}\cos\theta=\frac{\sqrt{3}}{2}\\ \text{sen}\,\theta=\frac{1}{2}\end{cases}$$



$\theta=\frac{\pi}{6}$

$$iz_1 = i\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right) = \frac{i\sqrt{3}}{2}-\frac{1}{2} = -\frac{1}{2}+\frac{i\sqrt{3}}{2}$$

Definizione: dato $m \in \mathbb{Z}$ $z \in \mathbb{C}$ si dice **radice m-esima di z** ogni numero complesso $\omega \in \mathbb{C}$ tale che

$$\omega^m = z \qquad\qquad x^2 = 4$$

Se $z=0$ allora $\omega = 0$

Ci interessa il caso $z \neq 0$ $m \in \mathbb{N}$

Se $\omega^{-m} = (\omega^{-1})^m = \left(\frac{1}{\omega}\right)^m$

Formule di De Moivre: dato $z \in \mathbb{C}$ $z \neq 0$ e $m \in \mathbb{N}\setminus\{0\}$

z ha esattamente m-radici m-esime distinte

$\omega_0, \dots, \omega_{m-1}$ che nel piano di Argand Gauss si dispongono come vertici di un poligono regolare con m-lati iscritto in una circonferenza di centro O e raggio $\sqrt[m]{|z|}$

Se $z = \rho(\cos\theta + i\sin\theta)$ le sue radici m-esime sono

$$\boxed{\omega_k = \sqrt[m]{\rho}\left[\cos\left(\frac{\theta+2k\pi}{m}\right)+i\sin\left(\frac{\theta+2k\pi}{m}\right)\right]} \qquad k=0,\dots,m-1$$

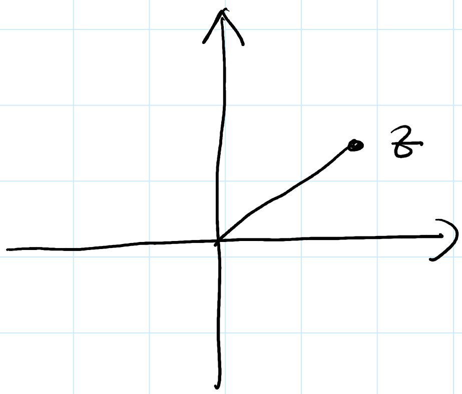
Dim:

$$\omega^m = z = \rho(\cos\theta + i\sin\theta)$$

$$\omega = r(\cos\alpha + i \sin\alpha)$$

$$\omega^m = \boxed{r^m}(\cos \boxed{(m\alpha)} + i \sin(m\alpha))$$

$$= \boxed{\rho}(\cos \boxed{\theta} + i \sin\theta)$$



$$\Longleftrightarrow \begin{cases} r^m = \rho & r = \sqrt[m]{\rho} \quad \text{perché reali} > 0 \\ m\alpha = \theta + 2k\pi & \alpha_k = \dfrac{\theta + 2k\pi}{m} \quad k \in \mathbb{Z} \end{cases}$$

$$\boxed{\alpha_0 = \frac{\theta}{m}} \qquad \alpha_1 = \frac{\theta + 2\pi}{m} \quad \dots \quad \alpha_{m-1} = \frac{\theta + 2(m-1)\pi}{m}$$

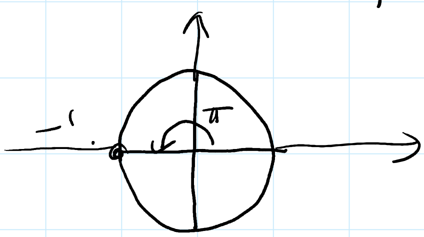
$$\boxed{\alpha_m} = \frac{\theta + 2m\pi}{m} = \boxed{\frac{\theta}{m} + 2\pi}$$

Esempio

Determinare le radici quarte di -1

$m = 4$

$$\omega^4 = -1$$

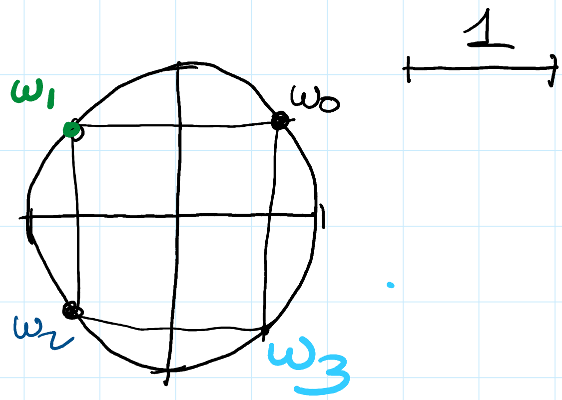


$$-1 = 1(\cos\pi + i \sin\pi)$$

$\rho = 1 \qquad \theta = \pi$

$$\omega_k = \sqrt[4]{\rho}\left(\cos\left(\frac{\theta + 2k\pi}{4}\right) + i \sin\left(\frac{\theta + 2k\pi}{4}\right)\right)$$

$$\omega_k = 1\left(\cos\left(\frac{\pi+2k\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{\pi+2k\pi}{4}\right)\right) \qquad k=0,\,1,\,2,\,3$$



$$\omega_0 = \cos\frac{\pi}{4} + i\,\mathrm{sen}\frac{\pi}{4}$$

$$\omega_1 = \cos\left(\frac{\pi+2\pi}{4}\right) + i\,\mathrm{sen}\,\frac{3\pi}{4}$$

$$\omega_2 = \cos\left(\frac{\pi+4\pi}{4}\right) + i\,\mathrm{sen}\,\frac{5\pi}{4}$$

$$\omega_3 = \cos\left(\frac{\pi+6\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{7\pi}{4}\right)$$

Ex:

$$x^4 - \frac{5+i}{3-2i} = 0$$

$$x^4 = \frac{5+i}{3-2i}$$

Scriviamo $\frac{5+i}{3-2i}$ in forma algebrica

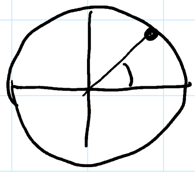
$$\frac{5+i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{15+10i+3i+\overbrace{2i^2}^{-2}}{9+4} =$$

$$= \frac{13+13i}{13} = 1+i$$

$$x^4 = 1+i$$

$$z = 1+i \qquad |z| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\begin{cases}\cos\theta = \frac{1}{\sqrt{2}}\\ \operatorname{sen}\theta = \frac{1}{\sqrt{2}}\end{cases} \qquad \theta$$



$\pi/4$

$$\theta = \frac{\pi}{4}$$

$$\sqrt{2} = 2^{1/2}$$

$$x^4 = \sqrt{2}\left(\cos\frac{\pi}{4} + i \operatorname{sen}\frac{\pi}{4}\right)$$

$$\sqrt[4]{\sqrt{2}} = \left(2^{1/2}\right)^{1/4} =$$

$$x_k = \sqrt[8]{2}\left(\cos\frac{\frac{\pi}{4}+2k\pi}{4} + i \operatorname{sen}\left(\frac{\pi+8k\pi}{16}\right)\right) \qquad k=0,1,2,3$$

$$\frac{1}{4}\left(\frac{\pi}{4}+2k\pi\right) = \frac{1}{4}\left(\frac{\pi+8k\pi}{4}\right) = \frac{\pi+8k\pi}{16}$$

Formula esponenziale dei numeri complessi

Formule di Eulero

$$\boxed{e^{i\theta} = \cos\theta + i \operatorname{sen}\theta}$$

$$z = \rho(\cos\theta + i \operatorname{sen}\theta) = \rho e^{i\theta} \qquad \text{Forma espon.}$$

$$\rho_1 e^{i\theta_1}\, \rho_2 e^{i\theta_2} = \rho_1\rho_2 e^{i(\theta_1+\theta_2)}$$

Identità di Eulero

$$e^{i\pi} = -1$$

Identità di Eulero

$$e^{i\pi} = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$