FOGLIO 1

ESERCIZIO 1

* a) $z = (1+i)^4$

Si ha $(1+i)^2 = (1+i)(1+i) = 1 + 2i + i^2 = 1 + 2i - 1 = 2i$

$\Rightarrow z = (1+i)^4 = ((1+i)^2)^2 = (2i)^2 = 2^2 \cdot i^2 = 4(-1) = -4$

b) $z = (1-i)(2+i) = 2 + i - 2i - i^2 = 3 - i$

utilizzo la distributività del prodotto e l'identità $i^2 = -1$

* c) $z = 3e^{\frac{5}{6}\pi i} = 3\left(\cos \underbrace{\frac{5}{6}\pi}_{150°} + i \sin \frac{5}{6}\pi\right) = 3\left(-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right)$

utilizzo la definizione di forma esponenziale e i coseni e i seni di angoli notevoli

* d) $z = \frac{\sqrt{2}i + \sqrt{3}}{\sqrt{2} - \sqrt{3}i} \cdot \frac{\sqrt{2} + \sqrt{3}i}{\sqrt{2} + \sqrt{3}i} = \frac{2i + \sqrt{6}i^2 + \sqrt{6} + 3i}{2 + 3}$

$= \frac{5i}{5} = i$

e) $z = \frac{(1+i)(3-2i)}{i} \cdot \frac{i}{i} = -i(3 - 2i + 3i + 2)$

$= -i(5 + i) = 1 - 5i$

f) $z = (i)^{2020} = (i^4)^{505}$ ma $i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$= (1)^{505} = 1$

ESERCIZIO 2

a) $z = \sqrt{3} - i$ $\qquad \|z\| = \sqrt{3+1} = 2$

$\theta = \operatorname{arctg} \frac{-1}{\sqrt{3}} = -\frac{\pi}{6}$

$= 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

oppure si noti $z = 2\left(\underset{\cos\left(-\frac{\pi}{6}\right)}{\underset{\uparrow}{\frac{\sqrt{3}}{2}}} - i \underset{\sin\left(-\frac{\pi}{6}\right)}{\underset{\uparrow}{\frac{1}{2}}}\right)$

* b) $z = (1-i)^5 = (1-i)^2(1-i)^2(1-i)$

$= (-2i)\cdot(-2i)\cdot(1-i) = -4(1-i)$

$= 4(-1+i)$

$= 4\cdot\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 4\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$

* c) $z = \left(\frac{i-1}{i+1}\right)^3 = \left(\frac{\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)}{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}\right)^3$

$z = \rho(\cos\theta + i\sin\theta)$

$z^m = \rho^m(\cos m\theta + i\sin m\theta)$ corollario $\rightarrow$

$= \left(\cos\left(\frac{3}{4}\pi - \frac{\pi}{4}\right) + i\sin\left(\frac{3}{4}\pi - \frac{\pi}{4}\right)\right)^3$

$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^3 = \cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi = -i$

$z_1 = \rho_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = \rho_2(\cos\theta_2 + i\sin\theta_2)$

$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}\cdot\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$

$= \frac{\rho_1}{\rho_2}\cdot\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\cdot\frac{\cos\theta_1 - i\sin\theta_1}{\cos\theta_2 - i\sin\theta_2}$

dalle formule trigonometriche

$= \frac{\rho_1}{\rho_2}\cdot\left(\underbrace{(\cos\theta_1\cos\theta_1 + \sin\theta_1\sin\theta_2)}_{\cos(\theta_1-\theta_2)} + i\underbrace{(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}_{\sin(\theta_1-\theta_2)}\right)$

d) $z = \frac{4i}{\sqrt{3}+i} \cdot \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{4i(\sqrt{3}-i)}{4}$

$= 1 + \sqrt{3}\,i = 2 \cdot \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) = 2 \cdot \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

e) $z = (1+i)^2(3+\sqrt{3}i)$

$= (1+2i-1)(3+\sqrt{3}i) = 2i(3+\sqrt{3}i) = 2(-\sqrt{3}+3i)$

$= 2 \cdot 2\sqrt{3}\left(-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)$ $\qquad \frac{3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{2 \cdot 3} = \frac{\sqrt{3}}{2}$

$= 4 \cdot \sqrt{3} \cdot \left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)$

* f) $z = \frac{(1+i)}{(1-i)(\sqrt{3}+i)} \quad \frac{(1+i)}{(1+i)} = \frac{2i}{2(\sqrt{3}+i)} \cdot \frac{(\sqrt{3}-i)}{(\sqrt{3}-i)}$

$= \frac{+1+\sqrt{3}i}{4} = +\frac{1}{4} + \frac{\sqrt{3}}{4}\,i = \frac{1}{4}(+1+\sqrt{3}\,i) =$

$= \frac{1}{4} \cdot 2\left(+\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) = \frac{1}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

ESERCIZIO 3

* a) $z^5 = \dfrac{\sqrt{3}-i}{\sqrt{3}+i} = \dfrac{(\sqrt{3}-i)^2}{4} = \dfrac{2-2\sqrt{3}i}{4} = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2} i$

$= \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$

per De Moivre si ha: $z = \cos\left(-\frac{\pi}{15} + \frac{2k\pi}{5}\right) + i\sin\left(-\frac{\pi}{15} + \frac{2k\pi}{5}\right)$

le radici si ottengono per $k = 0, 1, 2, 3, 4$

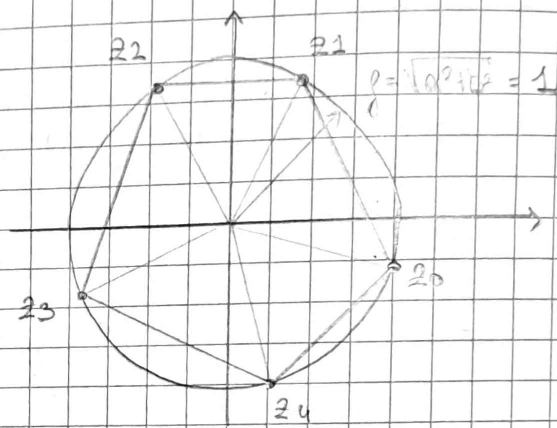
$z_0 = \cos\left(-\frac{\pi}{15}\right) + i\sin\left(-\frac{\pi}{15}\right)$

$z_1 = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

$z_2 = \cos\left(\frac{11}{15}\pi\right) + i\sin\left(\frac{11}{15}\pi\right)$

$z_3 = \cos\left(\frac{17}{15}\pi\right) + i\sin\left(\frac{17}{15}\pi\right)$

$z_4 = \cos\left(\frac{23}{15}\pi\right) + i\sin\left(\frac{23}{15}\pi\right)$



b) $z^3 = \dfrac{1-i}{i+1} = \dfrac{-2i}{2} = -i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$

per De Moivre: $z = \cos\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i\sin\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)$

le radici si ottengono per $k = 0, 1, 2$

c) $z^4 = 1 = \cos(0) + i\sin(0) \Rightarrow z = \cos\left(0 + \frac{2k\pi}{4}\right) + i\sin\left(0 + \frac{2k\pi}{4}\right)$

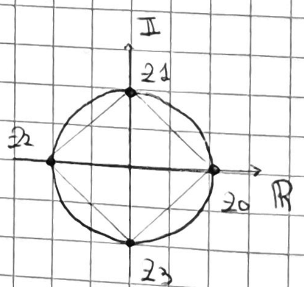
con $k = 0, 1, 2, 3$

$z_0 = \cos 0 + i\sin 0 = 1$

$z_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$z_2 = \cos\pi + i\sin\pi = -1$

$z_3 = \cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi = -i$



Scomporre $P(z) = z^4 - 1 = (z-1)(z-i)(z+1)(z+i)$ in fattori irriducibili reali

$\Rightarrow$ associo i prodotti $(z-\alpha)(z-\bar{\alpha})$ $\forall$ radice complessa non reale α

$\Rightarrow P(z) = (z-1)(z+1)(z^2+1)$

d) * $z^3 = \dfrac{(i-1)^4}{(i+1)^2} = \dfrac{\left(\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)\right)^4}{\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^2}$

$= 2\left(\cos\left(3\pi - \frac{\pi}{2}\right) + i\sin\left(3\pi - \frac{\pi}{2}\right)\right) = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$\Rightarrow z = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$ con $k = 0, 1, 2$

* e) $z^2 = -\dfrac{2i}{i-1} = \dfrac{2\left(\cos\left(\frac{3}{2}\pi\right) + i\sin\left(\frac{3}{2}\pi\right)\right)}{\sqrt{2}\left(\cos\left(\frac{3}{4}\pi\right) + i\sin\left(\frac{3}{4}\pi\right)\right)} = \sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$

$\Rightarrow z = \sqrt[4]{2}\cdot\left(\cos\left(\frac{3}{8}\pi + \frac{2k\pi}{2}\right) + i\sin\left(\frac{3}{8}\pi + \frac{2k\pi}{2}\right)\right)$ con $k = 0, 1$

* f) $(\bar{z})^4 = \dfrac{1+i}{i} = -(i-1) = 1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$\bar{z} = \sqrt[8]{2}\left(\cos\left(-\frac{\pi}{16} + \frac{2k\pi}{4}\right) + i\sin\left(-\frac{\pi}{16} + \frac{2k\pi}{4}\right)\right) = a - ib$

$z = \sqrt[8]{2}\left(\cos\left(-\frac{\pi}{16} + \frac{2k\pi}{4}\right) - i\sin\left(-\frac{\pi}{16} + \frac{2k\pi}{4}\right)\right) = a + ib$

$\cos(x) = \cos(-x)$ →
$-\sin(x) = \sin(-x)$

$= \sqrt[8]{2}\left(\cos\left(\frac{\pi}{16} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{\pi}{16} + \frac{2k\pi}{4}\right)\right)$

h) $z^4 = 2(\bar{z})^2$ dalla formula esponenziale: $z = \rho e^{\theta i}$, $\bar{z} = \rho e^{-\theta i}$

$\rho^4 e^{4\theta i} = 2\cdot\rho^2 e^{-2\theta i} \rightarrow \rho^4 e^{6\theta i} = 2\rho^2$

$\rho^2\cdot(\cos(6\theta) + i\sin(6\theta)) = 2\rho^2(\cos(0) + i\sin(0))$

$\begin{cases} \rho^4 = 2\rho^2 \\ 6\theta = 0 + 2k\pi \end{cases}$ $\begin{cases} \rho = \sqrt{2} \vee \rho = 0 \\ \theta = \frac{0}{6} + \frac{2k\pi}{6} = \frac{k\pi}{3} \end{cases}$

$z = \sqrt{2}\cdot e^{\left(\frac{k\pi}{3}\right)i} = \sqrt{2}\cdot\left(\cos\left(\frac{k\pi}{3}\right) + i\sin\left(\frac{k\pi}{3}\right)\right)$ (*)

per $k = 0, 1, 2, 3, 4, 5$ ho 6 radici in tutto ho 7 radici

$z = 0$ e $z = $ (*)

i) $z^2 = (\sqrt{6} - \sqrt{2}i)\,|z|^3$

$z = \rho(\cos\theta + i\sin\theta) = a + ib$ $\qquad \rho = \sqrt{a^2+b^2} = |z|$

$\rho^2(\cos 2\theta + i\sin 2\theta) = (\sqrt{6} - \sqrt{2}i)\cdot \rho^3 = \rho^3 \cdot 2\sqrt{2}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$\rho^2(\cos 2\theta + i\sin 2\theta) = \rho^3 \cdot 2\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$\begin{cases} \rho^2 = 2\sqrt{2}\,\rho^3 \\ 2\theta = -\frac{\pi}{6} + 2k\pi \end{cases} \rightarrow \begin{cases} \rho = 0 \ \vee\ \rho = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \\ \theta = -\frac{\pi}{12} + k\pi \end{cases}$

a meno dei multipli di 2π

$z = 0$

$z = \rho(\cos\theta + i\sin\theta) = \frac{\sqrt{2}}{4}\left(\cos\left(-\frac{\pi}{12} + k\pi\right) + i\sin\left(-\frac{\pi}{12} + k\pi\right)\right)$ $\quad k=0,1$

k) $z^4 = \frac{i}{(2\bar{z})^3} \rightarrow z^4 \cdot \bar{z}^3 = \frac{i}{8}$ (*)

$|z| = \sqrt{z\cdot\bar{z}} \Rightarrow |z|^2 = z\cdot\bar{z}$ ★

(*) $z^3\cdot\bar{z}^3\cdot z = (z\cdot\bar{z})^3\cdot z \underset{★}{=} (|z|^2)^3\cdot z = |z|^6\cdot z = \frac{i}{8}$ (**)

(*) faccio il modulo a dx e sx : $|z^4\cdot\bar{z}^3| = \left|\frac{i}{8}\right| \Rightarrow |z|^7 = \frac{1}{8}$

$\Rightarrow |z| = \frac{1}{\sqrt[7]{8}}$

(**) $\left(\frac{1}{\sqrt[7]{8}}\right)^6 \cdot z = \frac{i}{8} \Rightarrow z = \frac{i}{8}\cdot 8^{\frac{6}{7}} = \frac{i}{\sqrt[7]{8}}$

$= \frac{1}{\sqrt[7]{8}}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

oppure $z^4 \cdot \bar{z}^3 = \dfrac{i}{8}$

$$\rho^4 \cdot e^{4\theta i} \cdot \rho^3 \cdot e^{-3\theta i} = \frac{1}{8} \cdot e^{\frac{\pi}{2} i}$$

$$\underbrace{\rho^7} \cdot e^{\theta i} = \underbrace{\frac{1}{8}} e^{\frac{\pi}{2} i}$$

$$\begin{cases} \rho^7 = \dfrac{1}{8} \\ \theta = \dfrac{\pi}{2} + 2k\pi \end{cases} \qquad \begin{cases} \rho = \dfrac{1}{\sqrt[7]{8}} \\ 1\theta = \dfrac{\pi}{2} + 2k\pi \end{cases}$$

$$z = \frac{1}{\sqrt[7]{8}} \left(\cos\left(\frac{\pi}{2} + \cancel{2k\pi} \right) + i \sin\left(\frac{\pi}{2} + \cancel{2k\pi} \right) \right) \qquad k = 0$$

1) Trovare i numeri complessi che soddisfano l'equazione

$$z^3 = (1+\sqrt{3}i)^2 + \frac{6-2\sqrt{3}i}{1+\sqrt{3}i}$$

$$= -2+2\sqrt{3}i + \frac{-2\sqrt{3}i\,(1+\sqrt{3}i)}{1+\sqrt{3}i}$$

$$= -2 = 2(\cos\pi + i\sin\pi)$$

per De Moivre: $z = \sqrt[3]{2}\cdot\left(\cos\left(\frac{\pi}{3}+\frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{3}+\frac{2k\pi}{3}\right)\right)$ $k=0,1,2$

$z_0 = \sqrt[3]{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z_1 = \sqrt[3]{2}\left(\cos\pi + i\sin\pi\right)$

$z_3 = \sqrt[3]{2}\left(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi\right)$

2) Trovare le radici complesse del polinomio

$$\left(\frac{5}{2}+\frac{1}{2}i\right)X^4 + (6+9i)X$$

$$= \left[\left(\frac{1}{2}(5+i)\right)X^3 + 3(2+3i)\right]X$$

$\Rightarrow$ le radici sono le soluzioni dell'eq. $\frac{1}{2}(5+i)X^3 + 3(2+3i) = 0$ e $X=0$

$$X^3 = -6\,\frac{2+3i}{5+i}\cdot\frac{5-i}{5-i} = -6\cdot\frac{13+13i}{26}$$

$$X^3 = -3(1+i) = 3\sqrt{2}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right)$$

per De Moivre: $X_k = \sqrt[3]{3}\,\sqrt[6]{2}\left(\cos\left(\frac{5}{12}\pi+\frac{2k\pi}{3}\right) + i\sin\left(\frac{5}{12}\pi+\frac{2k\pi}{3}\right)\right)$

$k=0,1,2$

3) Determinare i numeri complessi z t.c. $2\dfrac{z}{1-i} + (1+2i)\bar{z} = 8+9i$

Sia $z = a+ib$ $\quad a,b \in \mathbb{R}$

$$2 \cdot \frac{a+ib}{1-i} + (1+2i)(a-ib) = 8+9i$$

$$\frac{2}{(1-i)(1+i)} \cdot (a+ib)(1+i) + (1+2i)(a-ib) = 8+9i$$

$$(a-b) + i(a+b) + (a+2b) + i(2a-b) = 8+9i$$

$$2a+b + i(3a) = 8+9i$$

$$\Rightarrow \begin{cases} 2a+b = 8 \\ 3a = 9 \end{cases} \Rightarrow z = 3+2i$$

g) $\bar{z}^3 = -8i$

$\rho^3 e^{-3\theta i} = 8e^{-\frac{1}{2}\pi}$

$\begin{cases} \rho^3 = 8 \\ -3\theta = -\frac{1}{2}\pi + 2k\pi \end{cases}$ $\begin{cases} \rho = 2 \\ \theta = \frac{\pi}{6} + \frac{2}{3}k\pi \end{cases}$

3 soluzioni

j) $z^3 = \frac{4+4i}{|z|^2}$

$\rho^3 e^{3\theta i} \cdot \rho^2 = 4+4i \rightarrow \rho^5 e^{3\theta i} = 4 \cdot \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= 4\sqrt{2} \cdot e^{\frac{\pi}{4}i}$

$\begin{cases} \rho^5 = 4\sqrt{2} \\ 3\theta = \frac{\pi}{4} + 2k\pi \end{cases}$ $\begin{cases} \rho = \sqrt{2} \\ \theta = \frac{\pi}{12} + \frac{2k\pi}{3} \end{cases}$ 3 soluzioni

e) $z^4 = \frac{i\bar{z}^3}{8}$

$\rho^4 e^{4\theta i} = \frac{1}{8} \cdot \rho^3 e^{-3\theta i} \cdot e^{\frac{\pi}{2}i} \rightarrow \rho^4 e^{7\theta i} = \frac{1}{8}\rho^3 e^{\frac{\pi}{2}i}$

$\begin{cases} \rho^4 = \frac{1}{8}\rho^3 \\ 7\theta = \frac{\pi}{2} + 2k\pi \end{cases} \rightarrow \begin{cases} \rho = 0 \lor \rho = \frac{1}{8} \\ \theta = \frac{\pi}{14} + \frac{2k\pi}{7} \end{cases}$ 7 + 1 soluzioni