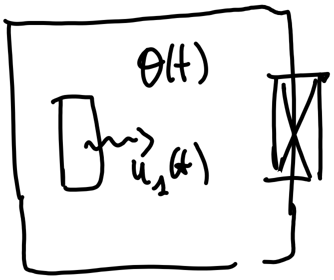
Esempio : sistema di raffreddamento



$\theta_a(t) = u_2(t)$

$$\dot{Q}(t) = u_1(t) - R\,(\theta(t) - u_2(t))$$

$R = u_3(t)$ Relazione tra $Q(t)$ $\theta(t)$

$Q(t) = M c_p\, \theta(t)$ $\theta(t) = x(t)$

$$\dot{x}(t) = \frac{1}{M c_p}\left[u_1(t) - u_3(t)\,(x(t) - u_2(t)) \right] = f(x, u_1 u_2 u_3)$$

Equilibrio $x(t) = \bar{x}$ $u_1(t) = \bar{u}_1$

$$0 = \bar{u}_1 - \bar{u}_3(\bar{x} - \bar{u}_2)$$

Parto da $\bar{x}$, $\bar{u}_1$, $\bar{u}_2$ $\longrightarrow$ $\bar{u}_3$

$$\boxed{\bar{u}_3 = \frac{u_1}{\bar{x} - \bar{u}_2}}$$

$\tilde{x}(t) \overset{def}{=} x(t) - \bar{x}$ $\tilde{u}_1(t) \overset{def}{=} u_1(t) - \bar{u}_1$

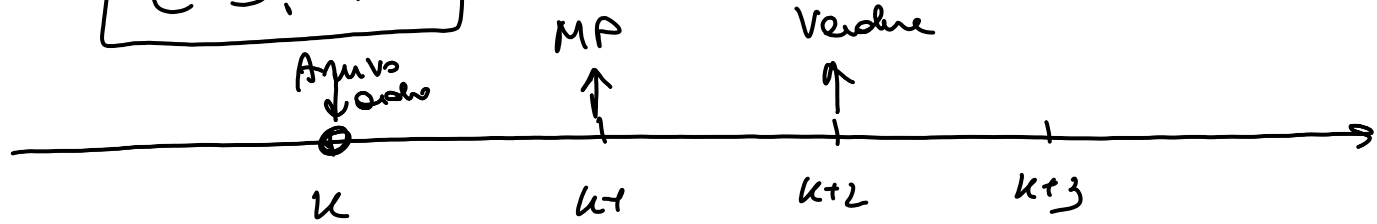
$$\dot{\tilde{x}}(t) = a_{11}\, \tilde{x}(t) + b_{11}\, \tilde{u}_1(t) + b_{12}\, \tilde{u}_2(t) + b_{13}\, \tilde{u}_3(t)$$

$$a_{11} = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \frac{1}{M c_p}(-\bar{u}_3) = -\frac{\bar{u}_3}{M c_p}$$

$$b_{11} = \frac{\partial f}{\partial u_1}(\bar{x}, \bar{u}) = \frac{1}{M c_p} \qquad b_{12} = \frac{\partial f}{\partial u_2}(\bar{x}, \bar{u}) = \frac{1}{M c_p}\bar{u}_3$$

$$b_{13} = \frac{\partial f}{\partial u_3}(\bar{x},\bar{u}) = \frac{1}{MC_p}(\bar{x}-\bar{u}_2)$$

ES. 19



ORD(k) = ordini ricevuti nel mese k

MP(k) = materie prime ricevute nel mese k — costo 100

$\rightarrow$ $MP(k) = ORD(k-1)$ $\longrightarrow$ $\underline{MP(k+1)} = ORD(k) = u(k)$

VEN(k) = merce venduta nel mese k

$\rightarrow VEN(k) = ord(k-2) = \underline{MP(k-1)}$

$costo(k) = 100 \cdot MP(k)$

$ricavo(k) = 120\ VEN(k)$

CAP(k) = capitale dell'azienda nel mese k

$$CAP(k+1) - CAP(k) = ricavo(k) - costo(k)$$

$$= 120\ VEN(k) - 100\ MP(k)$$

1) $u(k) = ORD(k)$ 2) $\underline{u(k) = MP(k)}$

2) $CAP(k) = x_2(k)$

$$x_2(k+1) = x_2(k) + 120\ \overbrace{VEN(k)}^{x_1(k)} - 100\ u(k) \leftarrow$$

$$x_1(k+1) = VEN(k+1) = MP(k) = u(k)$$

$$A = \begin{bmatrix} 0 & 0 \\ 120 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -100 \end{bmatrix}$$

Definiamo $\hat{u}(k) = 100\, u(k)$ $\qquad u = \frac{\hat{u}}{100}$

$\hat{x}_1(k) = 100\, \underline{x_1(k)}$ $\qquad x_1 = \frac{\hat{x}_1}{100}$

$\hat{x}_2(k) = x_2(k)$

$$\hat{x}_1(k+1) = 100\, x_1(k+1) = 100\, u(k) = \hat{u}(k)$$

$$\hat{x}_2(k+1) = \hat{x}_2(k) + 120\, x_1(k) - 100\, u(k)$$

$$= \hat{x}_2(k) + 120\, \frac{\hat{x}_1(k)}{100} - \cancel{100}\, \frac{\hat{u}(k)}{\cancel{100}}$$

1,2

$$A = \begin{bmatrix} 0 & 0 \\ 1{,}2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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1)

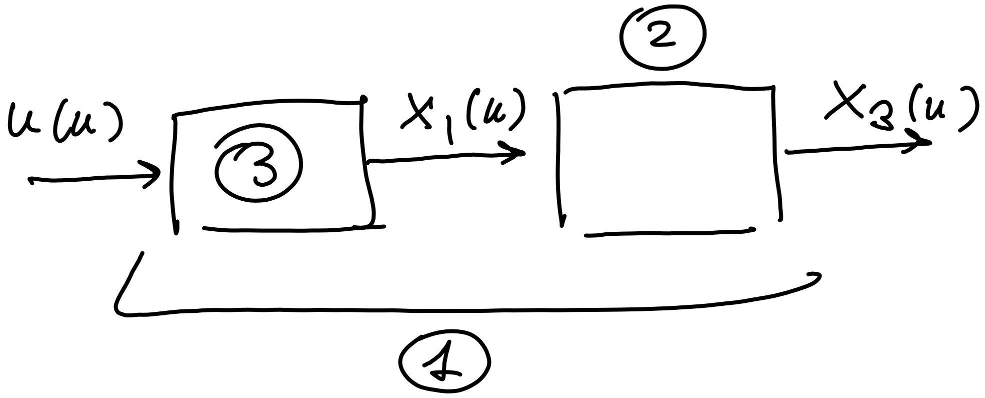
$ORD(k) = u(k)$

$MP(k) = x_1(k)$

$VEN(k) = x_2(k)$

$CAP(k) = x_3(k)$

$$\begin{cases} x_1(k+1) = u(k) & (3) \\ x_2(k+1) = x_1(k) \\ x_3(k+1) = x_3(k) + 120\, x_2(k) - 100\, x_1(k) \end{cases}$$



Esempi

$$\begin{cases} \dot{x}_1(t) = x_2(t) + x_1(t)\,u_1(t) \\ \dot{x}_2(t) = -x_1(t) + x_2^2(t) \end{cases}$$

$x_1(t),\ x_2(t)$

$u_1(t)$

1) Equilibri $\quad x_1(t) = \bar{x}_1 \quad x_2(t) = \bar{x}_2 \quad u(t) = \bar{u}$

$$\begin{cases} 0 = \bar{x}_2 + \bar{x}_1 \bar{u}_1 \\ 0 = -\bar{x}_1 + \bar{x}_2^2 \end{cases}$$

2 eq 3 inc.

- Fisso $\bar{u} = 1$ trovare gli equilibri se esistono

$$\begin{cases} 0 = \bar{x}_2 + \bar{x}_1 \\ 0 = -\bar{x}_1 + \bar{x}_2^2 \end{cases} \longrightarrow \bar{x}_2 = -\bar{x}_1 \qquad -\bar{x}_1 + \bar{x}_1^2 = 0$$

$\bar{x}_2 = 0 \quad \bar{x}_2 = -1$

$\bar{x}_1 = 0,\ \bar{x}_1 = 1$

$\quad\ \bar{x}_1 \quad \bar{x}_2 \quad \bar{u}$

Eq 1 : $(0, 0, 1)$

Eq 2 : $(1, -1, 1)$

- Fisso $\bar{x}_1 = 4$ trovare gli equilibri se esistono

$$\begin{cases} 0 = \bar{x}_2 + 4\bar{u}_1 \\ 0 = -4 + \bar{x}_2^2 \end{cases} \qquad \bar{u} = -\frac{\bar{x}_2}{4} \begin{cases} -1/2 \\ +1/2 \end{cases}$$

$$\longrightarrow \bar{x}_2 = \pm 2$$

Eq 3 : $(4, 2, -1/2)$

Eq 4 : $(4, -2, 1/2)$

- $\bar{x}_1 = -4$ trovare gli equilibri se esistono

$$\begin{cases} 0 = \bar{x}_2 - 4\bar{u}_1 \\ 0 = 4 + x_2^2 \leftarrow \text{MAI} \end{cases}$$

Linearizzo nell'intorno di (Eq 2)

$$\begin{cases} \dot{x}_1 = x_2 + x_1 u_1 = f_1(x_1, x_2, u) \leftarrow \\ \dot{x}_2 = -x_1 + x_2^2 = f_2(x_1, x_2, u) \end{cases}$$

$$\tilde{x}_1(t) \overset{def}{=} x_1(t) - \bar{x}_1 \quad \dots \qquad (\bar{x}, \bar{u}) = \overset{\bar{x}_1 \; \bar{x}_2 \; \bar{u}_1}{(1, -1, 1)}$$

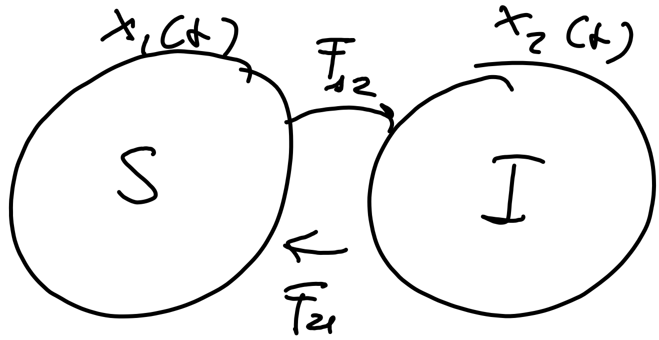
$$\frac{\partial f_1}{\partial x_1} = u_1 \xrightarrow{\bar{x}, \bar{u}} \underline{1} \qquad \frac{\partial f_1}{\partial x_2} = \underline{1} \qquad \frac{\partial f_1}{\partial u_1} = x_1 \longrightarrow \underline{1}$$

$$\frac{\partial f_2}{\partial x_1} = \underline{-1} \qquad \frac{\partial f_2}{\partial x_2} = 2x_2 \longrightarrow \underline{-2} \qquad \frac{\partial f_2}{\partial u_1} = 0$$

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u} \qquad A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

SIS

Popolazione totale = N
fissa



$\dot{x}_1(t) = -F_{12} + F_{21}$

$\dot{x}_2(t) = F_{12} - F_{21}$

$P(t) = x_1(t) + x_2(t)$ $\qquad \dot{P}(t) = \dot{x}_1 + \dot{x}_2 = 0$

$P(t) = N$

$F_{12} = \alpha_{12} x$

$F_{21}(t) = b x_2$

$F_{12}(t) = a x_1 x_2$

$$\begin{cases} \dot{x}_1 = -a x_1 x_2 + b x_2 = f_1 \\ \dot{x}_2 = a x_1 x_2 - b x_2 = f_2 \end{cases}$$

Equilibrio

$$\begin{cases} 0 = -a \bar{x}_1 \bar{x}_2 + b \bar{x}_2 \\ 0 = a \bar{x}_1 x_2 - b \bar{x}_2 \end{cases}$$

2 eq 2 inc

$\downarrow$

1 eq

$\bar{x}_1 + \bar{x}_2 = N$

$a \bar{x}_1 \bar{x}_2 - b \bar{x}_2 = 0$

$\rightarrow (a \bar{x}_1 - b) \bar{x}_2 = 0$

1 $\quad \bar{x}_2 = 0 \qquad \bar{x}_1 = N$

② $\bar{x}_2 \neq 0 \Rightarrow a\bar{x}_1 - b = 0 \quad \bar{x}_1 = \frac{b}{a}$

$\underline{\bar{x}_2 = N - \frac{b}{a}} \geq 0$ lo sono solo se $\bar{x}_2 \geq 0$

Questo equilibrio esiste solo se $N \geq b/a$

<u>Linearizzazione</u>

$$\frac{\partial f_1}{\partial x_1} = -a x_2 \qquad \frac{\partial f_1}{\partial x_2} = -a x_1 + b$$

$$\frac{\partial f_2}{\partial x_1} = a x_2 \qquad \frac{\partial f_2}{\partial x_2} = a x_1 - b$$

$$A = \begin{bmatrix} 0 & -aN+b \\ 0 & aN-b \end{bmatrix} \qquad \text{eq } ①$$

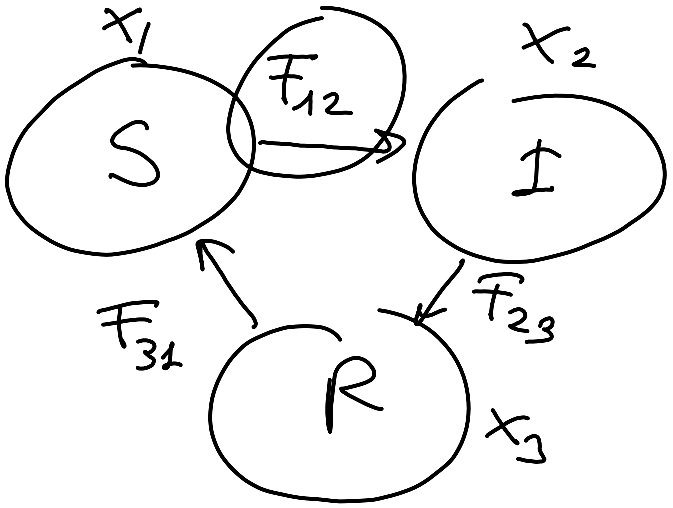
$$A = \begin{bmatrix} -a\left(N-\frac{b}{a}\right) & 0 \\ a\left(N-\frac{b}{a}\right) & 0 \end{bmatrix} \qquad \text{eq } ②$$

$$-a x_1 + b = -\cancel{a}\frac{b}{\cancel{a}} + b = -b + b = 0$$

$$= \begin{bmatrix} -aN+b & 0 \\ aN-b & 0 \end{bmatrix}$$

SIRS

$$\dot{x}_1 = -F_{12} + F_{31}$$
$$\dot{x}_2 = F_{12} - F_{23}$$
$$\dot{x}_3 = F_{23} - F_{31}$$



$F_{12} = a x_1 x_2$ $\quad F_{23} = b x_2$ $\quad F_{31} = c x_3$

$$\begin{cases} \dot{x}_1 = -a x_1 x_2 + c x_3 \\ \dot{x}_2 = a x_1 x_2 - b x_2 \\ \dot{x}_3 = b x_2 - c x_3 \end{cases}$$

$\underline{Pop(t) = N}$

$Pop = \underline{x_1 + x_2 + x_3} = N$

$\dot{Pop} = 0$

Equilibri

$$\begin{cases} 0 = -a \bar{x}_1 \bar{x}_2 + c \bar{x}_3 \\ 0 = a \bar{x}_1 \bar{x}_2 - b \bar{x}_2 \\ 0 = b \bar{x}_2 - c \bar{x}_3 \end{cases}$$

2 vincoli

3 incognite

1) Se $\bar{x}_2 = 0$ terzo $\rightarrow \bar{x}_3 = 0 \rightarrow \bar{x}_1 = N$

2) Se $\bar{x}_2 \neq 0$ secondo $\cancel{\bar{x}_2}(a\bar{x}_1 - b) = 0$

$$\underline{\underline{\bar{x}_1 = b/a}}$$

terzo $\bar{x}_3 = \dfrac{b}{c} \bar{x}_2$

$$\overline{x_1} + \overline{x_2} + \overline{x_3} = N$$

$$\frac{b}{a} + \overline{x_2} + \frac{b}{c}\overline{x_2} = N$$

$$\overline{x_2}\left(1 + \frac{b}{c}\right) = N - b/a$$

$$\overline{x_2} = \frac{N - b/a}{1 + b/c}$$

C'è soluzione se

$$N - \frac{b}{a} > 0$$