$$f(x) \simeq f(\bar{x}) + f'(\bar{x})(x-\bar{x})$$

LINEARIZZAZIONE

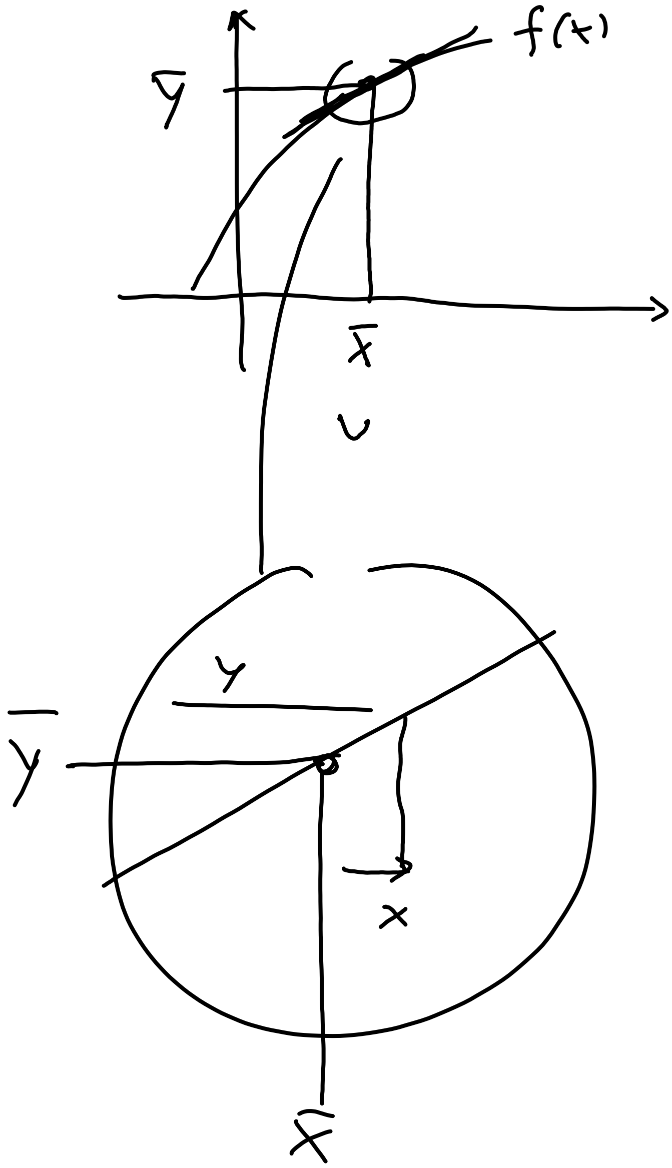
$\bar{y} = f(\bar{x})$

Se x è vicino $\bar{x}$

$x \simeq \bar{x}$ numero

$$\underbrace{y}_{= f(x)} - \underbrace{\bar{y}}_{= f(\bar{x})} \simeq f'(\bar{x})\,(x-\bar{x})$$

derivata di $f(x)$ valutata in $x = \bar{x}$



$\tilde{x} \overset{def}{=} x - \bar{x}$

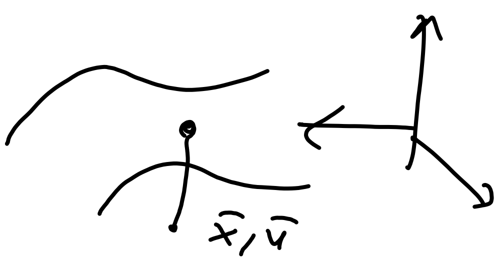
$\tilde{y} \overset{def}{=} y - \bar{y}$

$$\boxed{\tilde{y} \simeq f'(\bar{x})\,\tilde{x}}$$

Caso di funzioni in + variabili

$f(x,y)$ $\qquad (x,y) \in \mathbb{R}^2 \longrightarrow z \in \mathbb{R}$

Scelgo $\bar{x}, \bar{y} \rightarrow \bar{z} = f(\bar{x},\bar{y})$



$$(x,y) \simeq (\bar{x},\bar{y})$$

$$f(x,y) \simeq \underbrace{f(\bar{x},\bar{y})} + \frac{\partial f}{\partial x}(\bar{x},\bar{y})(x-\bar{x}) + \frac{\partial f}{\partial y}(\bar{x},\bar{y})(y-\bar{y})$$

Applicazione ai sistemi non lineari

<u>Tempo continuo</u>

$$\dot{x}_1(t) = f_1[x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)]$$

Prendo $\bar{x}_1 \dots \bar{x}_n$ $\bar{u}_1 \dots \bar{u}_m$ equilibri

Se $x_1(t), \dots x_n(t)$ $\simeq$ $\bar{x}_1 \dots \bar{x}_n$

$u_1(t) \dots u_m(t)$ $\simeq$ $\bar{u}_1 \dots \bar{u}_m$

$$f(\bar{x},\bar{u}) = 0$$

$$\dot{x}_1(t) = f_1[x_1(t), \dots x_n(t), u_1(t), \dots u_m(t)] \simeq \overbrace{\cancel{f_1[\bar{x}_1 \dots \bar{x}_n, \bar{u}_1 \dots \bar{u}_m]}}^{=0} +$$

$$+ \overbrace{\frac{\partial f_1}{\partial x_1}(\bar{x},\bar{u})}^{a_{11}} \overbrace{(x_1(t)-\bar{x}_1)}^{\tilde{x}_2(t)} + \overbrace{\frac{\partial f_1}{\partial x_2}(\bar{x},\bar{u})}^{a_{12}} \overbrace{(x_2(t)-\bar{x}_2)}^{\text{def } \tilde{x}_2(t)} + \dots$$

$$+ \overbrace{\frac{\partial f_1}{\partial u_1}(\bar{x},\bar{u})}^{b_{11}} \overbrace{(u_1(t)-\bar{u}_1)}^{\tilde{u}_1(t)} + \overbrace{\dots}^{b_{12}}$$

$$a_{ij} \overset{\text{def}}{=} \frac{\partial f_i}{\partial x_j}(\bar{x},\bar{u}) \qquad b_{ij} \overset{\text{def}}{=} \frac{\partial f_i}{\partial u_j}(\bar{x},\bar{u})$$

$$\tilde{x}_i(t) \overset{\text{def}}{=} x_i(t) - \bar{x}_i \qquad \tilde{u}_i(t) \overset{\text{def}}{=} u_i(t) - \bar{u}_i$$

$$\dot{\tilde{x}}_1(t) = \dot{x}_1(t) \simeq a_{11}\tilde{x}_1(t) + \dots + a_{1n}\tilde{x}_n(t) + b_{11}\tilde{u}_1(t) + \dots + b_{1m}\tilde{u}_m(t)$$

$$\underbrace{\frac{d}{dt}\tilde{x}_1(t)}_{\dot{\tilde{x}}_1(t)} = \frac{d}{dt}\left(x_1(t) - \bar{x}_1\right) = \underbrace{\frac{d}{dt}x_1(t)}_{\dot{x}_1(t)} - \cancel{\frac{d}{dt}\bar{x}_1}$$

$$\boxed{\dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t)}$$

IPOTESI

$\tilde{x}(t) \simeq 0$

$\tilde{u}(t) \simeq 0$

$$\begin{cases} x(t) \simeq \bar{x} \\ u(t) \simeq \bar{u} \end{cases}$$

Tempo discreto

$$x_1(k+1) = f_1[x_1(k) \dots \quad u_1(k) \dots]$$

se $x_1(k) \simeq \bar{x}_1$ equilibrio

$u_1(k) \simeq \bar{u}_1$ equilibrio

$$x_1(k+1) \simeq \overbrace{f_1[\bar{x}_1 \dots \bar{u}_1 \dots]}^{\bar{x}_1} + \overbrace{\frac{\partial f}{\partial x_1}(\bar{x}, \bar{u})}^{a_{11}} \overbrace{(x_1(k) - \bar{x}_1)}^{\tilde{x}_1(k)} + \dots$$

$$+ \overbrace{\frac{\partial f}{\partial u_1}(\bar{x}, \bar{u})}^{b_{11}} (u_1(k) - \bar{u}_1) + \dots$$

$$\begin{cases} \bar{x} = f(\bar{x}, \bar{u}) \\ \tilde{x}_1(k+1) \simeq a_{11}\tilde{x}_1(k) + \dots + b_{11}\tilde{u}_1(k) + \dots \\ \overbrace{x_1(k+1) - \bar{x}_1}^{\tilde{x}_1(k+1)} = \frac{\partial f_1}{\partial x_1}(\bar{x}, \bar{u})\tilde{x}_1(k) + \dots \end{cases}$$

$$\tilde{x}(k+1) = A\,\tilde{x}(k) + B\,\tilde{u}(k)$$

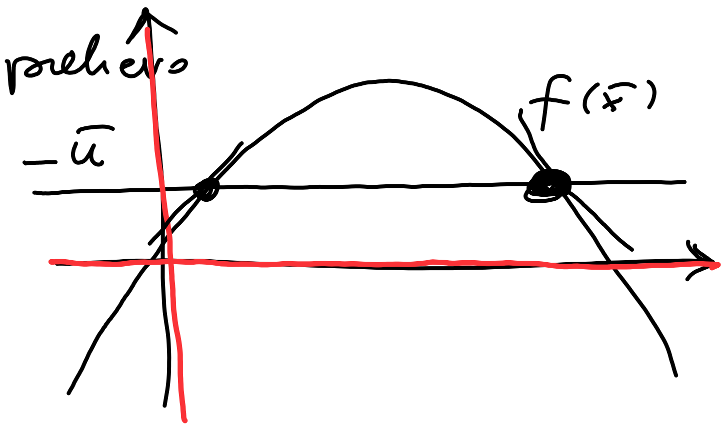
Esempio: Popolazione

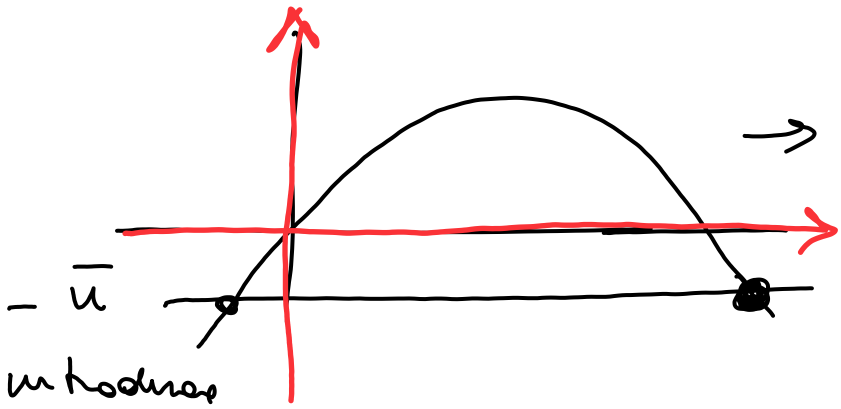
$$\dot{x}(t) = a\left(1 - \frac{x(t)}{K}\right)x(t) + u(t)$$

Equilibrio

$$0 = \underbrace{a\left(1 - \frac{\bar{x}}{K}\right)}_{f(\bar{x})}\bar{x} + \bar{u}$$

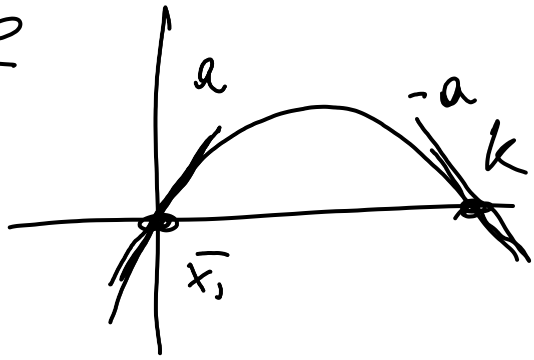
$$f(\bar{x}) = -\bar{u}$$





Consideriamo il caso $\bar{u} = 0$

$$\dot{x} = \overbrace{a\left(1 - \frac{x}{K}\right)x + u}^{f(x,u)}$$



1) $(\bar{x}_1, \bar{u}_1) = (0,0) \leftarrow$

2) $(\bar{x}_2, \bar{u}_2) = (K,0)$

(I equilibrio)

$u(t) \simeq 0$

$x(t) \simeq \bar{x}_1 = 0$

$$\frac{\partial f}{\partial x} = a\left(-\frac{1}{K}\right)x + a\left(1 - \frac{x}{K}\right) = a\left(1 - \frac{2x}{K}\right)$$

$$\frac{\partial f}{\partial x}(0,0) = a \qquad \frac{\partial f}{\partial u} = 1$$

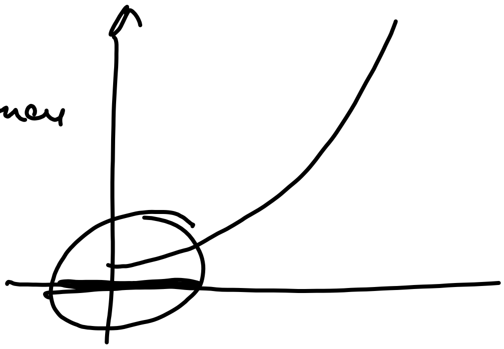
$$\tilde{x}(t) \triangleq x(t) - \bar{x} = x(t) \qquad \hat{u}(t) = u(t)$$

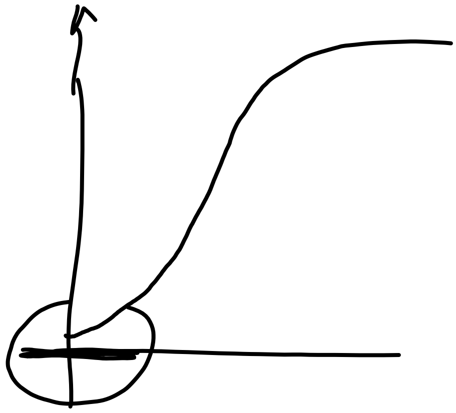
$$\boxed{\dot{\tilde{x}}(t) = a\,\tilde{x}(t) + \tilde{u}(t)}$$

$$\dot{x}(t) = a\,x(t) + u(t)$$

$a > 0 \qquad \tilde{x}(t) = x(t)$

linearizzazione





II equilibrio

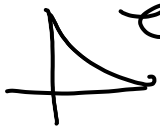
$$\bar{x}_2 = k \qquad \bar{u}_2 = 0$$

k

$$\frac{\partial f}{\partial x} = a\left(1 - \frac{2x}{k}\right)\Big|_{x=k} = a(1-2) = -a$$

$$\tilde{x}(t) = x(t) - k \qquad \tilde{u}(t) = u(t) - \cancel{\bar{u}} = u(t)$$

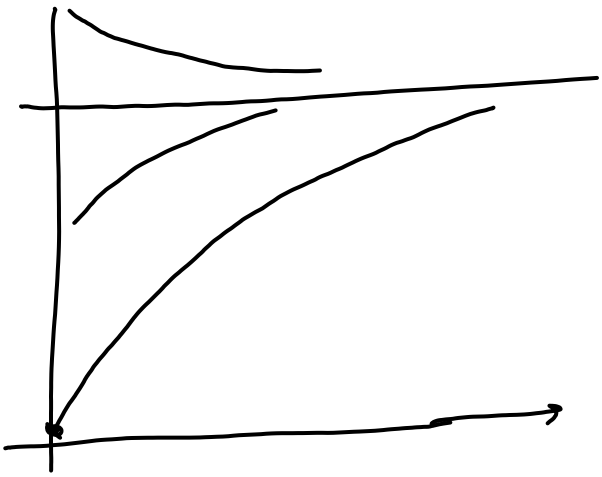
e^{-at}

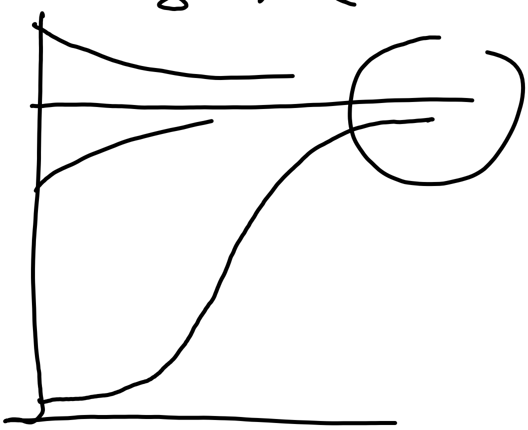


$$\boxed{\dot{\tilde{x}}(t) = -a\,\tilde{x}(t) + u(t)}$$

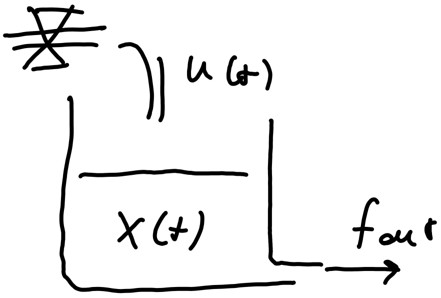
stabile

k

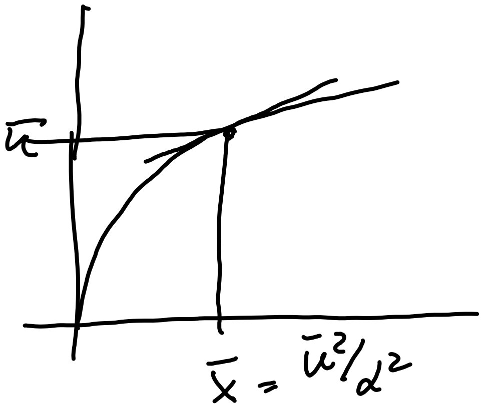




Esempio Serbatoio



$$f_{out} = \alpha\sqrt{x}$$



$$\dot{x}(t) = u(t) - \alpha\sqrt{x(t)}$$

Fissato $\bar{u} > 0$

$$0 = \bar{u} - \alpha\sqrt{\bar{x}} \Longrightarrow \alpha\sqrt{\bar{x}} = \bar{u} \Rightarrow \boxed{\bar{x} = \frac{\bar{u}^2}{\alpha^2}}$$

$$f(x, u) = u - \alpha\sqrt{x}$$

$$\frac{\partial f}{\partial x} = -\alpha\frac{1}{2\sqrt{x}} \xrightarrow{x = \bar{x} = \bar{u}^2/\alpha^2} -\frac{\alpha}{2}\frac{\alpha}{\bar{u}} = -\frac{\alpha^2}{2\bar{u}}$$

$$\frac{\partial f}{\partial u} = 1 \qquad \tilde{x}(t) \triangleq x(t) - \bar{x}$$

$$\dot{\tilde{x}}(t) = \boxed{-\frac{\alpha^2}{2\bar{u}}}\,\tilde{x}(t) + \tilde{u}(t)$$

$\bar{u} < 0$ la linearizzazione non esiste

Esempio pendolo

$$-ml^2\ddot{\theta}(t) - b\dot{\theta}(t) - mgl\sin(\theta(t)) + u(t) = 0$$

$x_1(t) = \theta(t)$

$x_2(t) = \dot{\theta}(t)$

① $\boxed{\dot{x}_1(t) = x_2(t)}$

$$\dot{x}_2(t) = \ddot{\theta}(t) = \frac{1}{ml^2}\left[-mgl\sin(\theta(t)) - b\dot{\theta}(t) + u(t)\right]$$

② $$\boxed{\dot{x}_2(t) = -\frac{g}{l}\sin(x_1(t)) - \frac{b}{ml^2}x_2(t) + \frac{1}{ml^2}u(t)}$$

Equilibri. $u(t) = \bar{u}$ $x_1(t) = \bar{x}_1$ $x_2(t) = \bar{x}_2$

① $0 = \bar{x}_2$

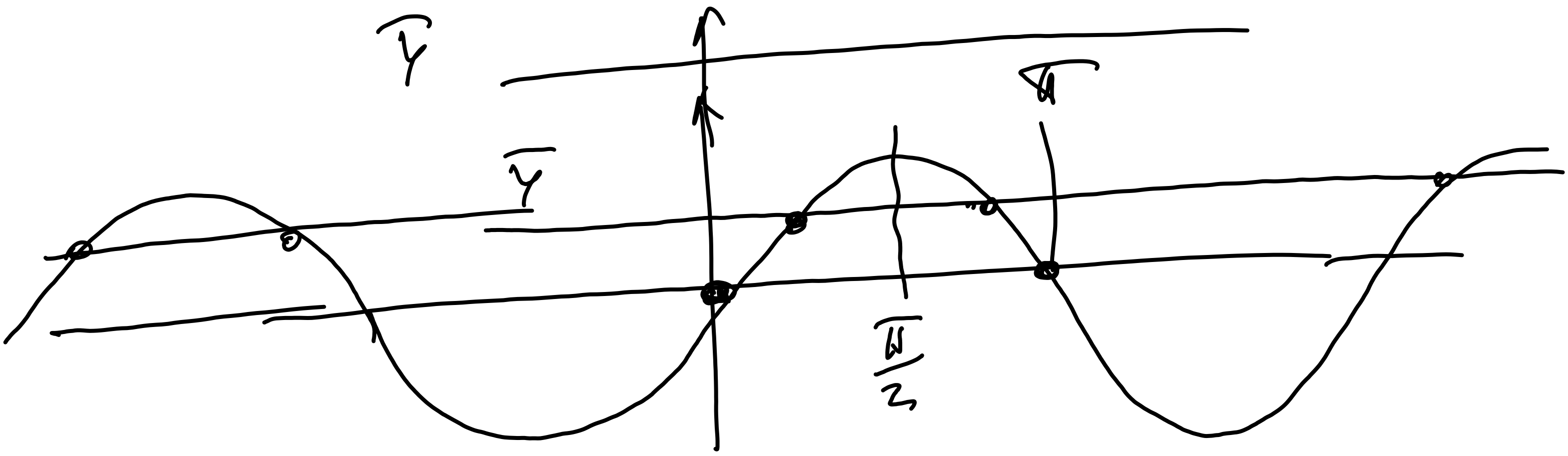
② $0 = -\frac{g}{l}\sin(\bar{x}_1) + \frac{1}{ml^2}\bar{u}$

$$\sin(\bar{x}_1) = \frac{l}{g}\frac{1}{ml^2}\bar{u} = \frac{1}{gml}\bar{u}$$

$\bar{u} = gml\sin(\bar{x}_1)$ dato $\bar{x}_1$ trovare $\bar{u}$

dato $\bar{u}$ trovare $\bar{x}_1$

risolvere equazione $\sin(\bar{x}_1) = \boxed{\frac{1}{mgl}\bar{u}} - \bar{y}$



Caso $\bar{u} = 0$

Equilibrio 1 $\quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0$

Equilibrio 2 $\quad \bar{x}_1 = \pi, \quad \bar{x}_2 = 0$

$$\begin{cases} \dot{x}_1 = \overbrace{x_2}^{f_1(x_1, x_2, u)} \\ \dot{x}_2 = \underbrace{-\frac{g}{l}\sin(x_1) - \frac{b}{ml^2}x_2 + \frac{1}{ml^2}u}_{f_2(x_1, x_2, u)} \end{cases}$$

$$\frac{\partial f_1}{\partial x_1} = 0 \qquad \frac{\partial f_1}{\partial x_2} = 1 \qquad \frac{\partial f_1}{\partial u} = 0$$

$$\frac{\partial f_2}{\partial x_1} = -\frac{g}{l}\cos(x_1) \quad \begin{cases} \text{Eq1: } -g/l \\ \text{Eq2: } g/l \end{cases} \qquad \frac{\partial f_2}{\partial x_2} = -\frac{b}{ml^2}$$

$$\frac{\partial f_2}{\partial u} = \frac{1}{ml^2}$$

$$\tilde{x}_1(t) = x_1(t) - \overset{0}{\overset{\shortparallel}{\bar{x}_1}} \qquad \tilde{x}_2(t) = x_2(t) - \overset{0}{\overset{\shortparallel}{\bar{x}_2}}$$

$$\tilde{u}(t) = u(t) - \overset{0}{\overset{\shortparallel}{\bar{u}}}$$

$\boxed{\text{Equilibri 1}}$

$$\begin{cases} \dot{\tilde{x}}_1(t) = 0\,\tilde{x}_1(t) + 1\,\tilde{x}_2(t) + 0\,\tilde{u}(t) = \tilde{x}_2(t) \\ \dot{\tilde{x}}_2(t) = -g/\ell\,\tilde{x}_1(t) - \dfrac{b}{m\ell^2}\,\tilde{x}_2(t) + \dfrac{1}{m\ell^2}\,\tilde{u}(t) \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 \\ -g/\ell & -\dfrac{b}{m\ell^2} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \dfrac{1}{m\ell^2} \end{bmatrix}$$

stabile →

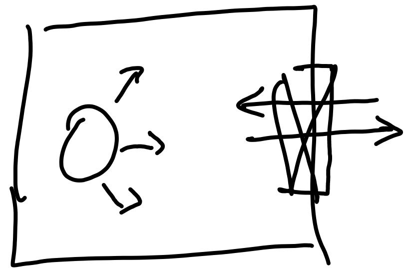
Equilibrio 2

$$A = \begin{bmatrix} 0 & 1 \\ +g/\ell & -b/m\ell^2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/m\ell^2 \end{bmatrix}$$

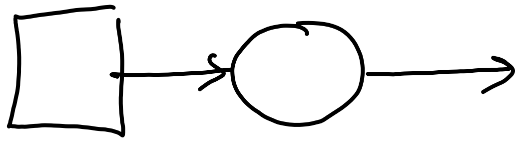
instabile →

Esempio: Ventola di raffreddamento

$Q(t)$ quantità di calore



$$\dot{Q}(t) = f_{in}(t) - f_{out}(t)$$



$$f_{in}(t) = u_1(t)$$

$-$ temp. interna

$$f_{out}(t) = R\,(\theta(t) - \theta_a(t))$$

resistenza termica $\qquad$ temp. ambiente

$$\theta_a(t) = u_2(t)$$

$$Q(t) = \overbrace{\underbrace{M}_{\text{massa}}\ \underbrace{C_p}_{\text{calore specifico}}}^{\text{capacità termica}}\ \theta(t)$$

$$\theta(t) = x(t)$$

$$M C_p \dot{\theta}(t) = u_1(t) - R\,(\theta(t) - \theta_a(t))$$

$$\boxed{\dot{x}(t) = -\frac{R}{M C_p} x(t) + \frac{1}{M C_p} u_1(t) + \frac{R}{M C_p} u_2(t)}$$

lineare

$$A = \left[-R/M C_p\right] \qquad B = \left[\frac{1}{M C_p} \;\middle|\; \frac{R}{M C_p}\right]$$

$$\dot{x}(t) = A\,x(t) + B \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

Supponiamo che $R = R(t) = u_3(t)$

$$\dot{x}(t) = -\frac{1}{MC_p} x(t) u_3(t) + \frac{1}{MC_p} u_1(t) + \frac{1}{MC_p} u_2(t) u_3(t)$$

Equilibri $x(t) = \bar{x}$ $u_1(t) = \bar{u}_1$

$$0 = -\frac{1}{MC_p} \bar{x}\,\bar{u}_3 + \frac{1}{MC_p} \bar{u}_1 + \frac{1}{MC_p} \bar{u}_2 \bar{u}_3$$

$$\boxed{-\bar{x}\,\bar{u}_3 + \bar{u}_1 + \bar{u}_2 \bar{u}_3 = 0}$$

1 eq

4 incognite

Fisso $\bar{x}, \bar{u}_1, \bar{u}_2 \longrightarrow \bar{u}_3$

$$\bar{u}_3 (\bar{x} - \bar{u}_2) = \bar{u}_1$$

$$\bar{u}_3 = \frac{\bar{u}_1}{\bar{x} - \bar{u}_2}$$

$$f(x, u_1, u_2, u_3) = -\frac{1}{MC_p} x u_3 + \frac{1}{MC_p} u_1 + \frac{1}{MC_p} u_2 u_3$$

$$\frac{\partial f}{\partial x} = -\frac{1}{MC_p} u_3 \qquad \frac{\partial f}{\partial u_1} = \frac{1}{MC_p} \qquad \frac{\partial f}{\partial u_2} = \frac{1}{MC_p} u_3$$

$$\frac{\partial f}{\partial u_3} = -\frac{1}{MC_p} x + \frac{1}{MC_p} u_2$$

Sostituiamo gli equilibri