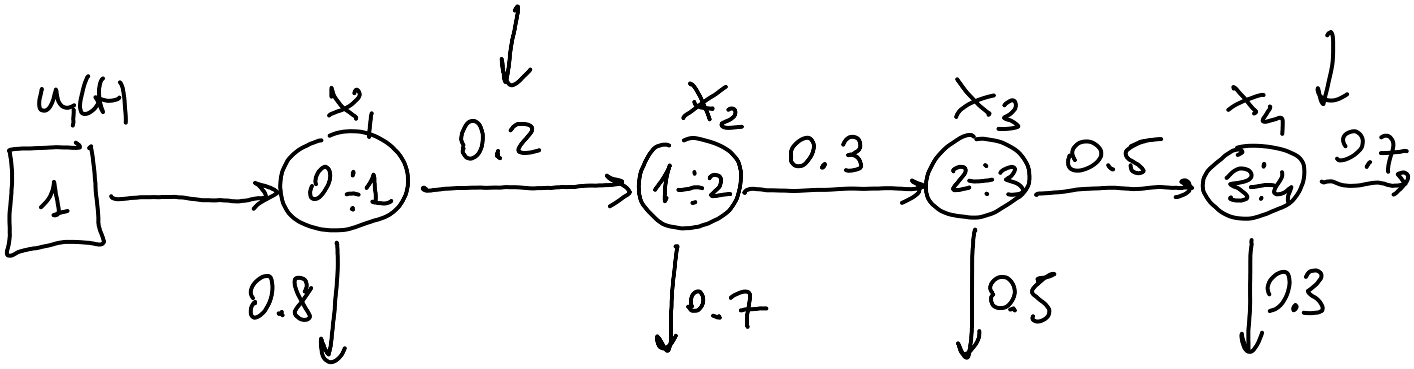
ES. 17



$$\rightarrow x_1(k+1) - \cancel{x_1(k)} = u_1(k) - 0.8\,\cancel{x_1(k)} - \cancel{0.2\,x_1(k)}$$

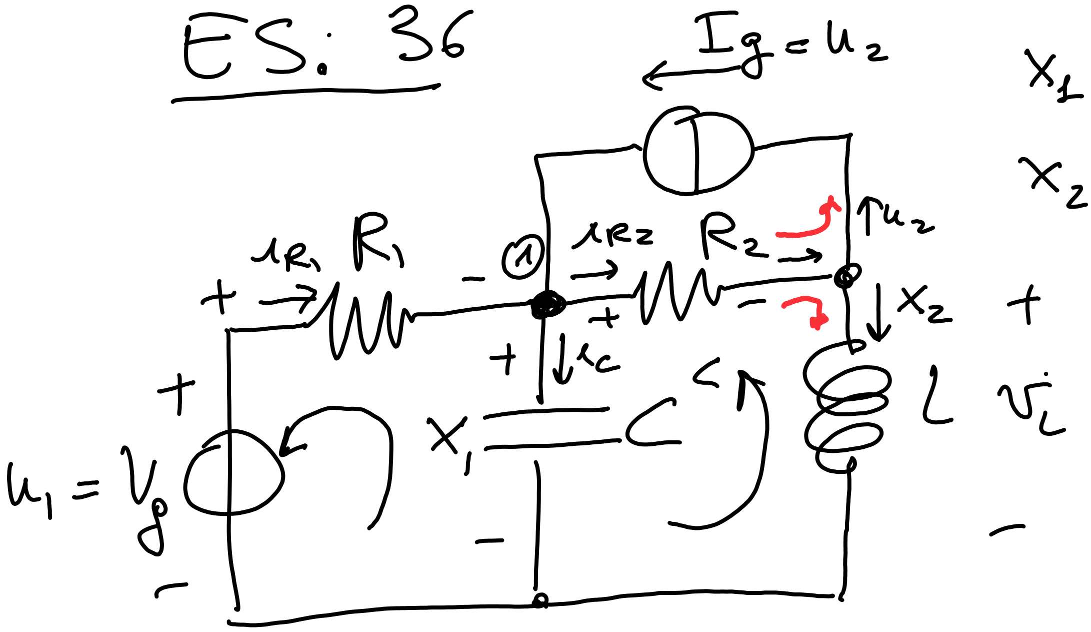
$$x_2(k+1) - \cancel{x_2(k)} = 0.2\,x_1(k) - \cancel{0.7\,x_2(k)} - \cancel{0.3\,x_2(k)}$$

$$x_3(k+1) - \cancel{x_3(k)} = 0.3\,x_2(k) - \cancel{0.5\,x_3(k)} - \cancel{0.5\,x_3(k)}$$

$$x_4(k+1) - \cancel{x_4(k)} = \underline{0.5\,x_3(k)} - \cancel{0.3\,x_4(k)} - \cancel{0.7\,x_4(k)}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

ES: 36



$x_1 = v_C$

$x_2 = i_L$

$V_g = u_1$

$I_g = u_2$

$$i_C = C \frac{d}{dt} v_C = C \dot{x}_1$$

$$v_L = L \frac{d}{dt} i_L = \underline{L \dot{x}_2}$$

$$v_{R_1} = R_1 i_{R_1}$$

$$v_{R_2} = R_2 i_{R_2}$$

$$x_1 + v_{R_1} - u_1 = 0$$

① $I_g + i_{R_1} - i_{R_2} - i_C = 0$

$$i_{R_1} = \frac{v_{R_1}}{R_1} = \frac{u_1 - x_1}{R_1}$$

$$\boxed{\cancel{u_2} + \frac{u_1 - x_1}{R_1} - (\cancel{u_2} + x_2) - C\dot{x}_1 = 0}$$

$$\boxed{i_{R_2} = u_2 + x_2}$$

② $-v_C + v_L + v_{R_2} = 0$

$$\boxed{-x_1 + L\dot{x}_2 + R_2(u_2 + x_2) = 0}$$

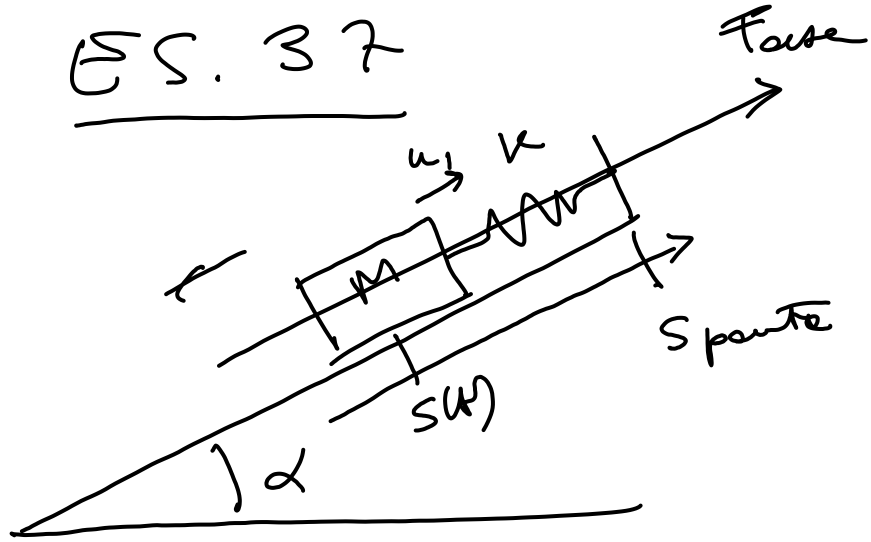
$$\dot{x}_1 = \frac{1}{C}\left[\cancel{u_2} + \frac{u_1}{R_1} - \frac{x_1}{R_1} - \cancel{u_2} - x_2\right]$$

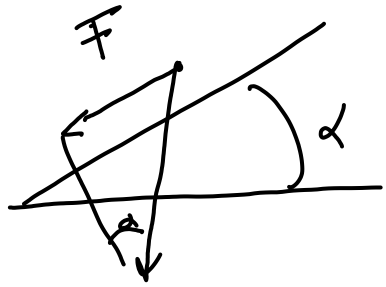
$$= -\frac{1}{R_1 C} x_1 - \frac{1}{C} x_2 + \frac{1}{R_1 C} u_1$$

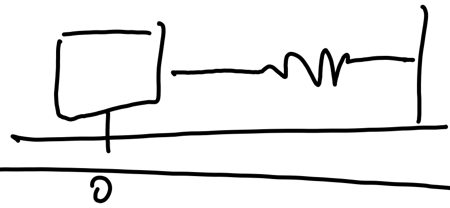
$$\dot{x}_2 = \frac{1}{L}\left[x_1 - R_2 x_2 - R_2 u_2\right] = \frac{1}{L} x_1 - \frac{R_2}{L} x_2 - \frac{R_2}{L} u_2$$

$$A = \begin{bmatrix} -\frac{1}{R_1 C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2}{L} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{R_1 C} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix}$$

ES. 37







$$-M\ddot{s}(t) - k\,s(t) + u_1(t) - Mg\sin(\alpha) = 0$$

$u_2 \overset{def}{=} \sin\alpha$

$F = -Mg\sin(\alpha)$ $\qquad \alpha = 0$

$x_1(t) = s(t) \qquad x_2(t) = \dot{s}(t)$

$\dot{x}_1(t) = \dot{s}(t) = x_2(t)$ ←

u_2

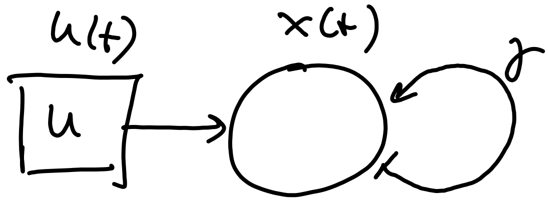
$\dot{x}_2(t) = \ddot{s}(t) = \frac{1}{M}\left[-k\,x_1(t) + u_1 - Mg\,\boxed{\sin\alpha}\right]$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & \end{bmatrix} \qquad B = \begin{array}{cc} u_1 & u_2 \\ \multicolumn{2}{c}{\begin{bmatrix} 0 & 0 \\ \frac{1}{M} & -g \end{bmatrix}} \end{array}$$

SISTEMI NONLINEARI

Esempio — Modello di Popolazione

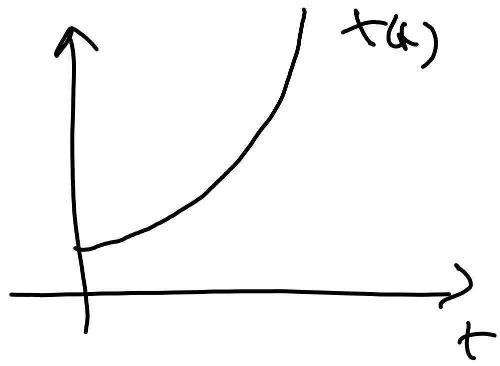
$x(t)$ n° individui



$$\dot{x}(t) = \gamma\, x(t) + u(t)$$

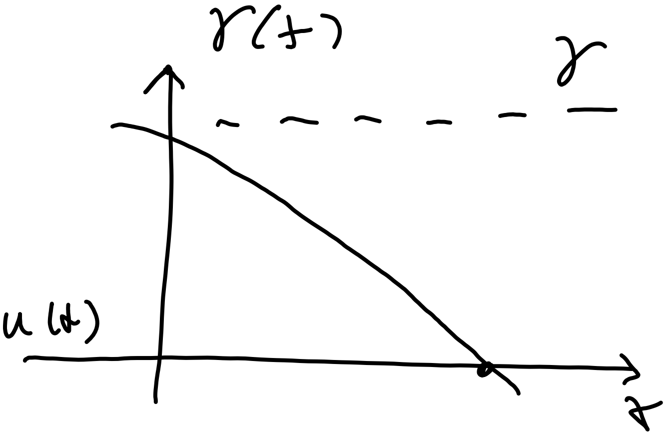
$u(t) = 0$

$\gamma > 0$



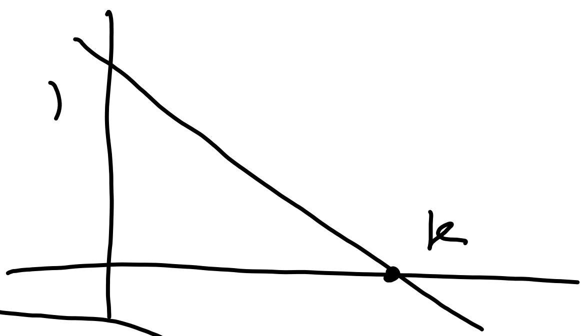
$$x(t) = e^{\gamma t} x(0)$$

$\gamma(x)$

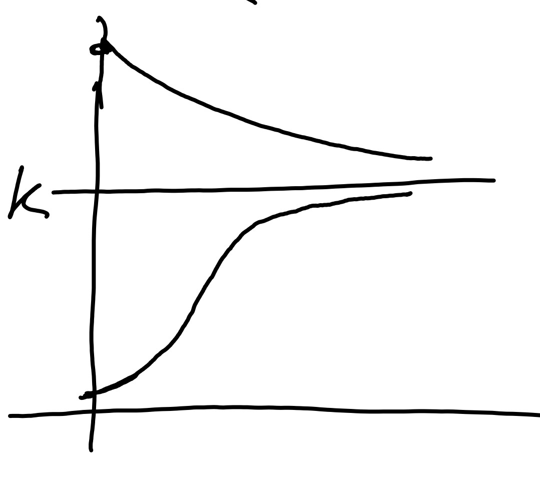


$$\dot{x}(t) = \gamma(x(t))\, x(t) + u(t)$$

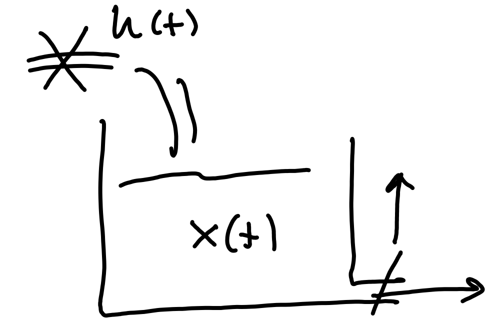
$$\gamma(x) = a\left(1 - \frac{x}{K}\right)$$

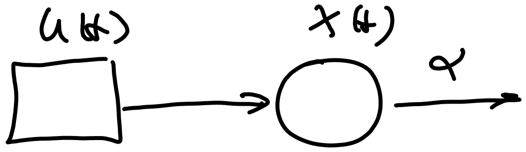


$$\boxed{\dot{x}(t) = a\left(1 - \frac{x(t)}{K}\right) x(t) + u(t)}$$



Esempio Serbatoio





$$\dot{x}(t) = u(t) - \underbrace{\alpha x(t)}_{f_{out}(x)}$$

$$f_{out}(x) = \alpha \sqrt{x}$$

$$\dot{x}(t) = u(t) - \alpha \sqrt{x(t)}$$

Esempio : pendolo

$$-J\,\ddot{\theta}(t) - b\,\dot{\theta} + u(t) - mgl\sin\theta(t) = 0$$

$$J = ml^2 \qquad x_1 = \theta \qquad x_2 = \dot{\theta}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{\theta} = \frac{1}{ml^2}\left[-b\dot{\theta} + u - \overline{mgl\sin\theta}\right]$$

$$= -\frac{g}{l}\sin(x_1) - \frac{b}{ml^2}x_2 + \frac{1}{ml^2}u$$

Sistemi nonlineari

$x_1(t) \dots x_n(t)$ stati

$u_1(t) \dots u_m(t)$ ingressi

lineare

$$\dot{x}_1(t) = a_{11} x_1(t) + a_{12} x_2(t) + \dots + b_{11} u_1(t) + b_{12} u_2(t) + \dots$$

non lineare (tempo continuo)

$$\dot{x}_1(t) = f_1 \overbrace{[x_1(t), x_2(t), \dots, x_n(t), u_1(t), u_2(t), \dots u_m(t)]}^{n+m}$$

$$\dot{x}_2(t) = f_2[\qquad]$$

$$\vdots \qquad \vdots$$

$$\dot{x}_n(t) = f_n[x_1(t) \dots \qquad]$$

Esempio

popolazione $\qquad \dot{x} = \overbrace{a\left(1 - \frac{x}{k}\right)x + u}^{f(x,u)}$

serbatoio $\qquad \dot{x} = \underbrace{-\alpha\sqrt{x} + u}_{f(x,u)}$

Pendolo $\qquad \dot{x}_1 = -x_2 \qquad f_1(x_1, x_2, u) = x_2$

$$\dot{x}_2 = \underbrace{-\frac{1}{\ell}\sin(x_1) - \frac{b}{m\ell^2} x_2 + \frac{1}{m\ell^2} u}_{f_2(x_1, x_2, u)}$$

Tempo discreto

Lineare

$$x_1(k+1) = a_{11} x_1(k) + a_{12} x_2(k) + \cdots + b_{11} u_1(k) + \cdots$$

Non lineare

$$x_1(k+1) = f_1[x_1(k), x_2(k), \ldots, x_n(k), u_1(k), \ldots, u_m(k)]$$

$$x_2(k+1) = f_2[\qquad\qquad]$$

Analisi degli equilibri

Equilibri $\equiv$ soluzioni costanti del sistema dinamico

Trovare soluzioni costanti

$x_1(t) = \bar{x}_1 \quad \forall t \qquad \bar{x}_1 = ?$

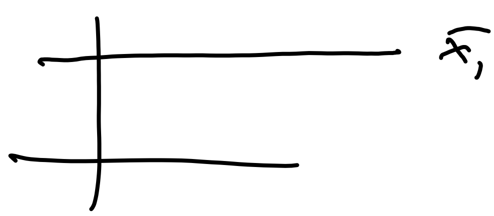
$x_2(t) = \bar{x}_2 \quad \forall t \qquad \bar{x}_2 = ?$

$\vdots$

$x_n(t) = \bar{x}_n \quad \forall t \qquad \bar{x}_n = ?$

$u_1(t) = \bar{u}_1 \quad \forall t$

$u_m(t) = \bar{u}_m \quad \forall t$



$\dot{x}_1 = f_1[x_1 \dots x_n\ u_1 \dots u_m]$ $\qquad \dot{x}_1(t) = 0$

$0 = f_1[\bar{x}_1, \dots, \bar{x}_n, \bar{u}_1, \dots \bar{u}_m]$

equazione algebrica con incognite $\underline{\bar{x}_i}$, $\underline{\bar{u}_i}$

$\dot{x}_2 = f_2[x_1 \dots x_n\ u_1 \dots u_n] \leftarrow$ $x_i(t) = \bar{x}_i$, $u_i(t) = \bar{u}_i$

$0 = f_2[\bar{x}_1 \dots \bar{x}_n\ \bar{u}_1 \dots \bar{u}_n]$

n equazioni in $n+m$ incognite

Ad esempio fisso $\underline{\bar{u}_1 \dots \bar{u}_m} \xrightarrow{\text{trovo}} \underline{\bar{x}_1 \dots \bar{x}_n}$

<u>Caso tempo continuo</u>

$\dot{x} = f(x, u)$ sistema dinamico non lineare

Equilibrio $x(t) = \bar{x}$ $\quad u(t) = \bar{u}$

$0 = f(\bar{x}, \bar{u})$ equazione degli equilibri

<u>Tempo discreto</u>

Equilibrio $\Longleftrightarrow$ soluzioni costanti

$x_1(k+1) = f_1[x_1(k) \dots x_n(k), u_1(k) \dots]$

$x_i(k) = \bar{x}_i \quad \forall k \qquad u_i(k) = \bar{u}_i \quad \forall k$

$\bar{x}_1 = f_1[\bar{x}_1 \ldots \bar{x}_n, \bar{u}_1 \ldots \bar{u}_m]$

n equazioni in n+m incognite

$$\bar{x} = f(\bar{x}, \bar{u})$$

Esempio Serbatoio

$x(t) = \bar{x}$

$u(t) = \bar{u}$

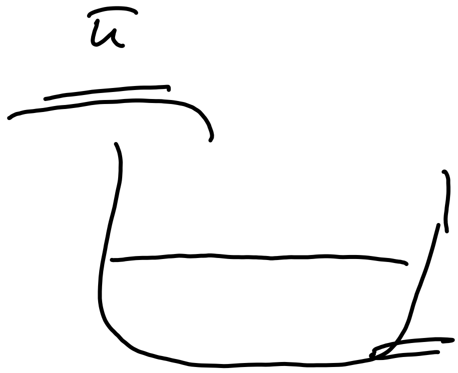
$$\dot{x} = -\alpha\sqrt{x} + u$$

$$0 = -\alpha\sqrt{\bar{x}} + \bar{u} \qquad \bar{u} = \alpha\sqrt{\bar{x}}$$

Fisso $\bar{x} \Longrightarrow \bar{u} = \alpha\sqrt{\bar{x}}$

$$\bar{u} \longrightarrow \boxed{\frac{\bar{u}^2}{\alpha^2}} \longrightarrow \bar{x}$$

Fisso $\bar{u} \longrightarrow \bar{x} = \dfrac{\bar{u}^2}{\alpha^2} \leftarrow$



Esempio Popolazione

$x(t) = \bar{x}$

$u(t) = \bar{u}$

$$\dot{x} = a\left(1 - \frac{x}{k}\right)x + u$$

$$0 = a\left(1 - \frac{\bar{x}}{k}\right)\bar{x} + \bar{u}$$

Fisso $\bar{x}$ $\longrightarrow$ $\bar{u} = -a\left(1-\frac{\bar{x}}{k}\right)\bar{x}$

Fisso $\bar{u}$ $\longrightarrow$ equaz. di II grado in $\bar{x}$

$$\underbrace{a\left(1-\frac{\bar{x}}{k}\right)\bar{x}}_{f(\bar{x})} = -\bar{u}$$

$\bar{u}$