MODELLI DI TRASFERIMENTO DI RISORSE

TEMPO DISCRETO ; MODELLI DI DECISIONE

Comportamento i $\qquad x_i(k)$

$$x_i(k+1) - x_i(k) = \underbrace{q_i^{in}(k)}_{} - \underbrace{q_i^{out}(k)}_{}$$

$$q_i^{in}(k) = \sum_e \beta_{ei}\, u_e(k) + \sum_{j \neq i} \alpha_{ji}\, x_j(k) + \sum_{j \neq i} \gamma_{ji}\, x_j(k) + \gamma_{ii}\, x_i(k)$$

$$\overset{u_e(k)}{\boxed{e}} \xrightarrow{\beta_{ei}} \overset{x_i(k)}{i}$$

$$\overset{x_j(k)}{j} \xrightarrow{\alpha_{ji}} \overset{x_i(k)}{i}$$

$$i \xrightarrow{\gamma_{ii}} i$$

$$j \xrightarrow{\gamma_{ji}} i$$

$$q_i^{out}(k) = \sum_h \beta_{ih}\, u_h(k) + \sum_{j \neq i} \alpha_{ij}\, x_i(k)$$

$$i \longrightarrow \boxed{h}$$

$$i \longrightarrow j$$

$$\underline{x_i(k+1)} = \Big(\underbrace{1 + \gamma_{ii} - \sum_{j\neq i} \alpha_{ij}}_{a_{ii}}\Big)\underline{x_i(k)}$$

$$+ \sum_{j \neq i} \big(\underbrace{\alpha_{ji} + \gamma_{ji}}_{a_{ij}}\big)\underline{x_j(k)}$$

$$+ \sum_{\ell} \big(\underbrace{\beta_{\ell i} - \beta_{i\ell}}_{b_{i\ell}}\big)\underline{u_\ell(k)}$$

$$X(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} \qquad A = \begin{bmatrix} a_{ij} \end{bmatrix} \qquad B = \begin{bmatrix} B_{i\ell} \end{bmatrix}$$

$$X(k+1) = A\,X(k) + B\,u(k)$$

$$X(k) = ? \qquad \underline{u(k)} \qquad \underline{x(0)}$$

$$X(1) = A x(0) + B u(0)$$

$$X(2) = A x(1) + B u(1)$$

$$= A^2 x(0) + AB\,u(0) + B u(1)$$

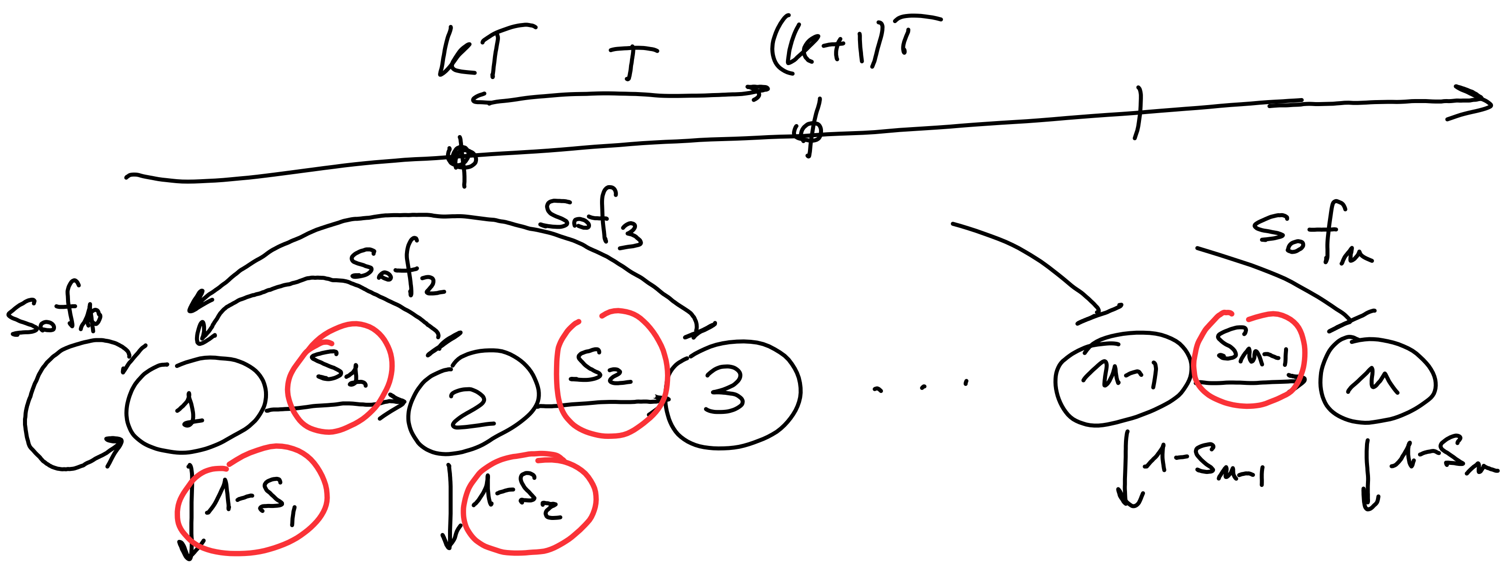
Esempio: Modelli a struttura d'età

$x_1(k)$ = popolazione di età $\in [0, T[$ all'istante k

$x_2(k)$ = popolazione di età $\in [T, 2T[$

$\vdots$

$x_n(k)$ = popolazione di età $\in [(n-1)T, +\infty)$



s_i tasso sopravvivenza

f_i tasso di fertilità

s_0 tasso di sopravvivenza alla nascita

$i \in \{2, \dots, n-1\}$

$$x_i(k+1) - \cancel{x_i(k)} = s_{i-1}\, x_{i-1}(k) - \cancel{(1-s_i)}\, x_i(k) - \cancel{s_i\, x_i(k)}$$

$$\boxed{x_i(k+1) = s_{i-1}\, x_{i-1}(k)}$$

$$\to x_1(k+1) = \cancel{x_1(k)} - \cancel{s_1 x_1(k)} - \cancel{(1-s_1)}\, x_1(k)$$

$$+ s_0 f_1 x_1(k) + s_0 f_2 x_2(k) + \cdots + s_0 f_n x_n(k)$$

$$x_n(k+1) = \cancel{x_n(k)} - \cancel{(1}-s_n)\, x_n(k) + s_{n-1} x_{n-1}(k)$$

$$x_n(k+1) = s_{n-1} x_{n-1}(k) + s_n x_n(k)$$

$$A = \begin{bmatrix} s_0 f_1 & s_0 f_2 & \cdots & & s_0 f_n \\ s_1 & & & & \\ & s_2 & & & \\ & & \ddots & & \\ & & & s_{n-1} & s_n \end{bmatrix}$$

$$x(k+1) = A\, x(k)$$

$$N(k) = x_1(k) + \cdots + x_n(k)$$ Popolazione totale

$$s_i(k) = \frac{x_i(k)}{N(k)}$$ frazione di popolazione di età i

$$S(k) = \begin{bmatrix} s_1(k) \\ \vdots \\ s_n(k) \end{bmatrix}$$ Struttura della popolazione

S_{eq} struttura di equilibrio

Se $\boxed{x(k) = N(k)\, S_{eq}}^{**} \Rightarrow \boxed{x(k+1) = N(k+1)\, S_{eq}}^{*}$

$$S_i(k+1) = \frac{x_i(k+1)}{N(k+1)} = \frac{x_i(k)}{N(k)} = S_i(k)$$

$$x(k+1) \overset{*}{=} N(k+1)\, S_{eq}$$

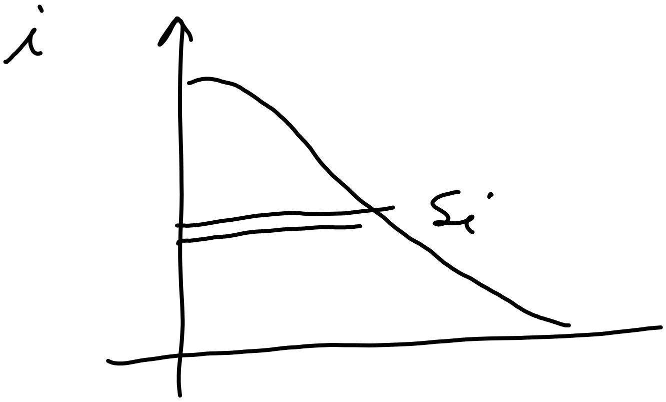
$$= A\, x(k) \overset{**}{=} A\, N(k)\, S_{eq}$$

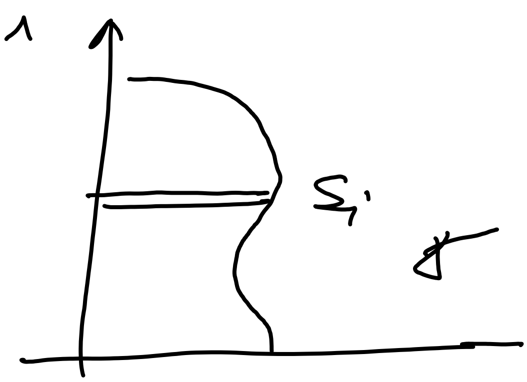
$$N(k+1)\, S_{eq} = A\, N(k)\, S_{eq}$$

$$\underline{A\, S_{eq}} = \underbrace{\frac{N(k+1)}{N(k)}}_{\lambda} S_{eq}$$

λ autovalore $\qquad$ S_{eq} autovettore di A

$$\frac{N(k+1)}{N(k)} = \lambda \qquad N(k+1) = \lambda\, N(k)$$





Se esiste i tale che

$$S_1^{eq}(k) = \underbrace{\frac{x_1(k)}{N(k)}} < S_{i+1}^{eq}(k) = \overset{S^{eq}}{\frac{x_{i+1}(k)}{N(k)}}.$$

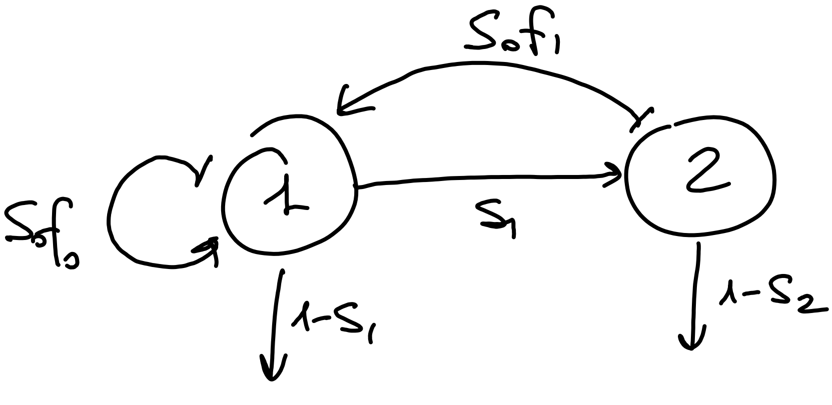
e

$$S_{i+1}^{eq}(k+1) = \frac{x_{i+1}(k+1)}{N(k+1)} = \underbrace{\frac{s_i x_1(k)}{\lambda N(k)}}$$

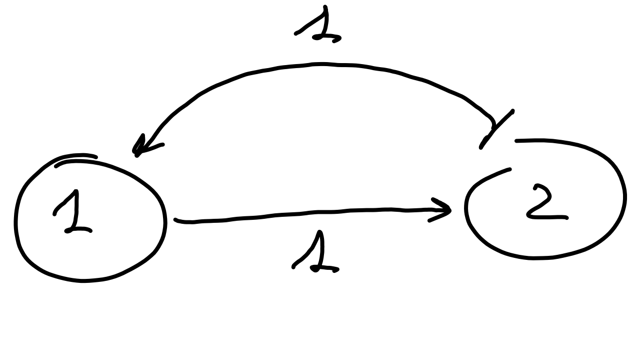
$$\frac{x_1 \cancel{(k)}}{\cancel{N(k)}} < \frac{s_i \quad x_1 \cancel{(k)}}{\lambda \cancel{N(k)}} \qquad 1 < \frac{s_i}{\lambda}$$

$$\lambda < s_{i-1} \leq 1$$

Esempi: a due classi di età



Esempio 1: Serie di Fibonacci



$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = x_2(k) + x_1(k) \end{cases}$$

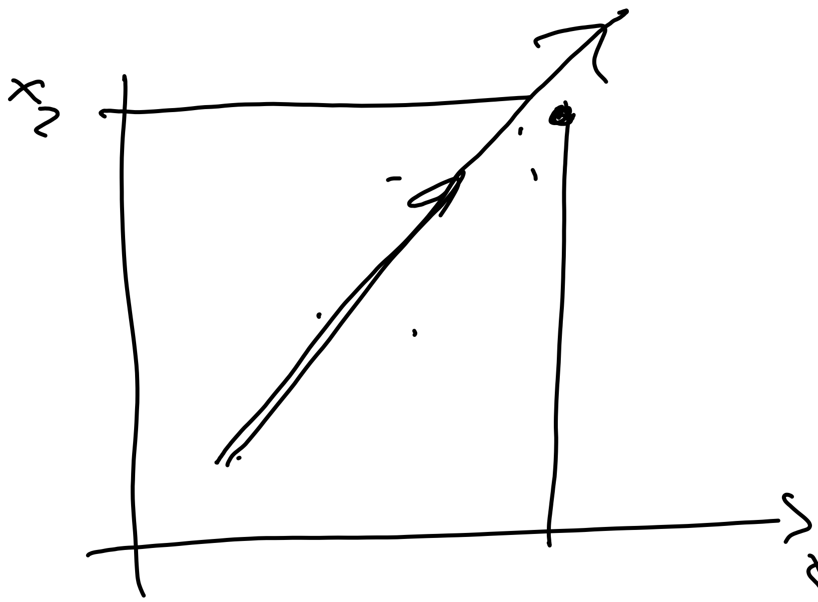
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$N(k+2) = X_1(k+2) + X_2(k+2)$$

$$= X_2(k+1) + X_1(k+1) + X_2(k+1)$$

$$= N(k+1) + X_2(k) + X_1(k) = N(k+1) + N(k)$$



$$\begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix}$$

$S_1(k) \qquad S_2(k)$

$$\begin{bmatrix} S_1(k) \\ S_2(k) \end{bmatrix}' \longrightarrow \begin{bmatrix} S_{eq} \\ S_{eq} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad N(k+1) \cong \lambda N(k)$$

$$\det(zI - A) = 0 \qquad \det \begin{bmatrix} z & -1 \\ -1 & z-1 \end{bmatrix} = z(z-1) - 1$$

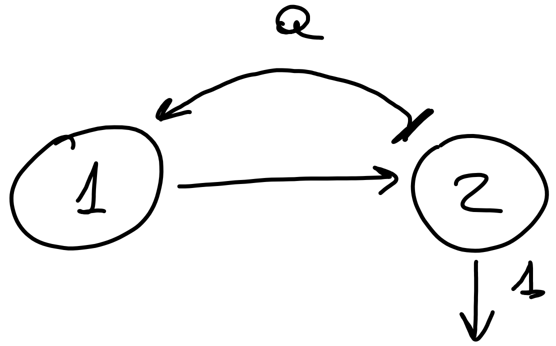
$$= z^2 - z - 1$$

$$z_{12} = \frac{1 \pm \sqrt{5}}{2} \longrightarrow \frac{1+\sqrt{5}}{2}$$

$$\begin{bmatrix} \frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{1+\sqrt{5}}{2}-1 \end{bmatrix} \begin{bmatrix} S_{eq,1} \\ S_{eq,2} \end{bmatrix} = 0$$

$$S_{eq,1} + S_{eq,2} = 1$$

2° Esempio



$$x_1(k+1) = a\, x_2(k)$$

$$x_2(k+1) = x_1(k)$$

$$A = \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}$$

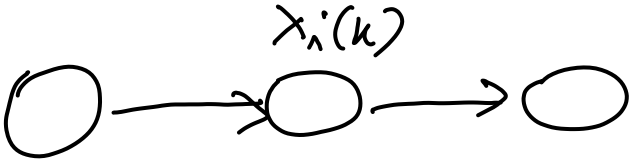
$$S_{eq} = \begin{bmatrix} S_{eq,1} \\ S_{eq,2} \end{bmatrix}$$

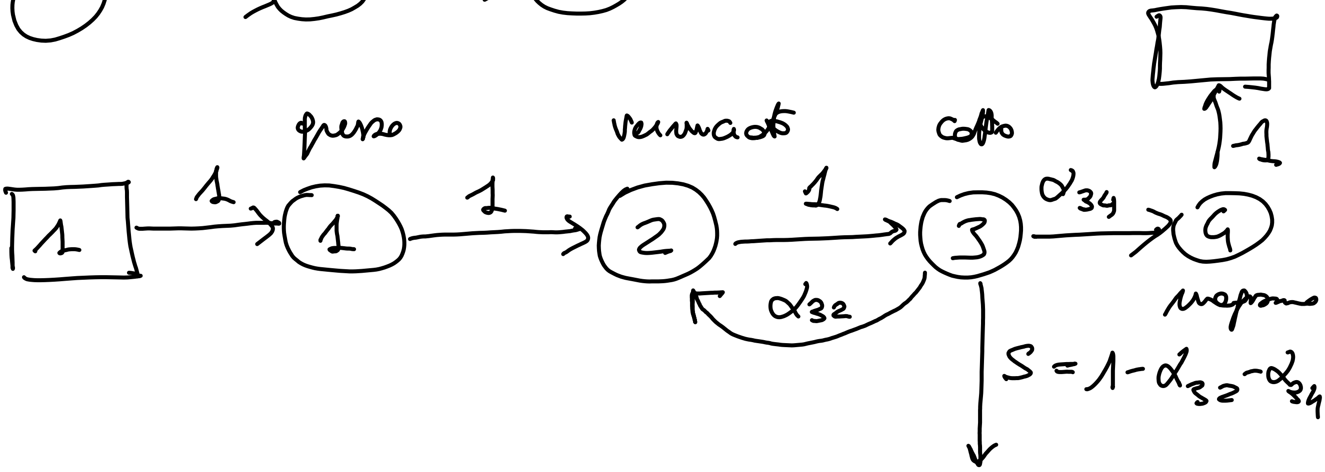
$$\det(zI - A) = \det\begin{bmatrix} z & -a \\ -1 & z \end{bmatrix} = z^2 - a \qquad \lambda = \sqrt{a}$$

$$\begin{bmatrix} \sqrt{a} & -a \\ \boxed{-1 \quad \sqrt{a}} \end{bmatrix} \begin{bmatrix} S_{eq,1} \\ S_{eq,2} \end{bmatrix} = 0$$

$$-S_{eq,1} + \sqrt{a}\, S_{eq,2} = 0 \qquad S_{eq,1} = \sqrt{a}\, S_{eq,2}$$

Esempio: Catena di produzione con test di qualità





$$x_1(k+1) = \underline{u_1(k)}$$

$$x_2(k+1) = \underline{x_1(k)} + \alpha_{32}\, x_3(k)$$

$$x_3(k+1) = x_3(k) + \underline{x_2(k)} - \alpha_{32}\, x_3(k)$$

$$- \alpha_{34}\, x_3(k) - S\, x_3(k)$$

$$= \cancel{(1-\alpha_{32}-\alpha_{34}-1+\alpha_{32}+\alpha_{34})\, x_3(k)}$$

$$\underline{x_2(k)}$$

$$x_4(k+1) = -u_2(k) + \alpha_{34}\, x_3(k)$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & \alpha_{32} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha_{34} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$