Università degli Studi di Padova

Lecture 01

Foundations of Signals and Systems: an introduction

Tomaso Erseghe

1.1 Signals

What do we mean by signal?

- Definition
- From functions to signals
- Signal examples
- The four classes of interest to us

Welcome!

to Foundations of Signals & Systems

- One lecturer, tomaso.erseghe@unipd.it
- 16 **Lectures** on the **theoretical** aspects
- 3 **MatLab** lectures for a more practical understanding
- **Exercises** with solution
- **Homeworks** (with solutions)

Signal

Definition www.merriam-webster.com

[…]

4

a : an object used to transmit or convey **information** beyond. the range of human voice

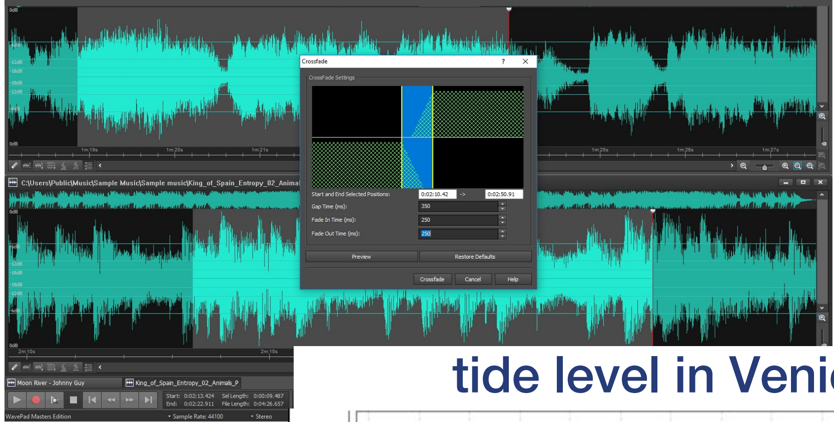
b : the **sound or image** conveyed in telegraphy, telephony, radio, radar, or television

c : a detectable physical quantity or impulse (as a voltage, current, or magnetic field strength) by which messages or **information** can be transmitted

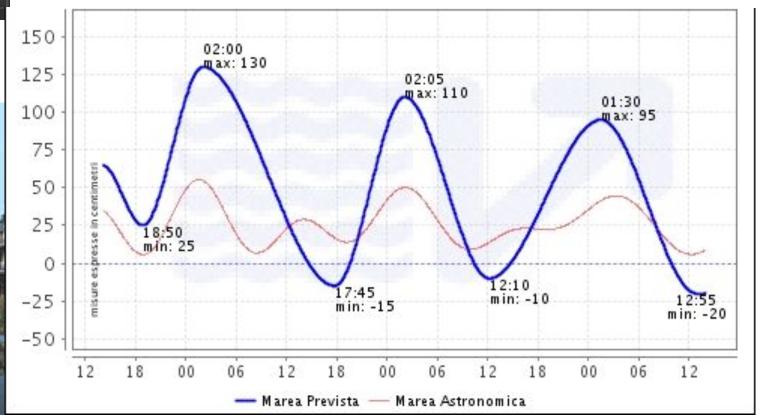
Some examples

From physics

audio wave (acoustic pressure)

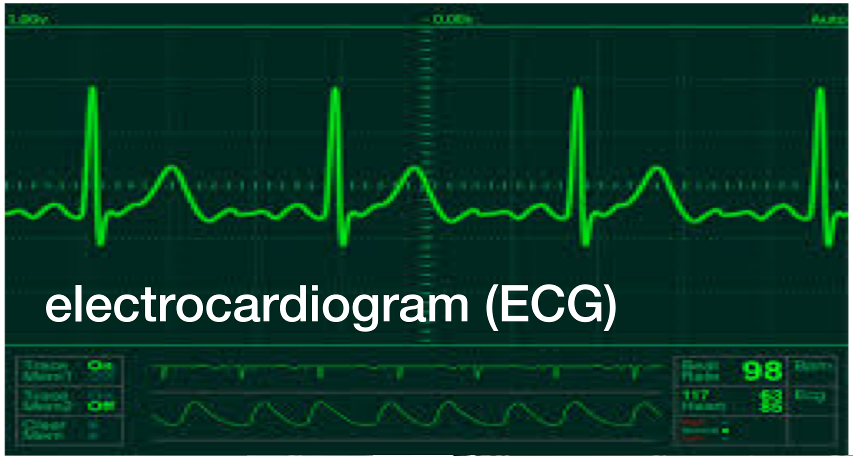


tide level in Venice

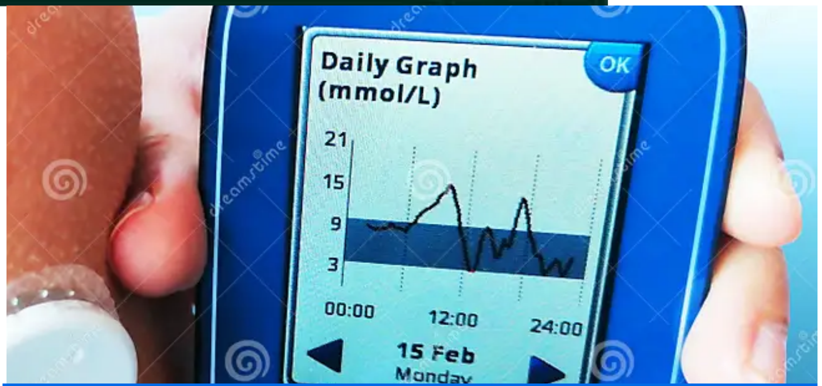


More examples

From medicine



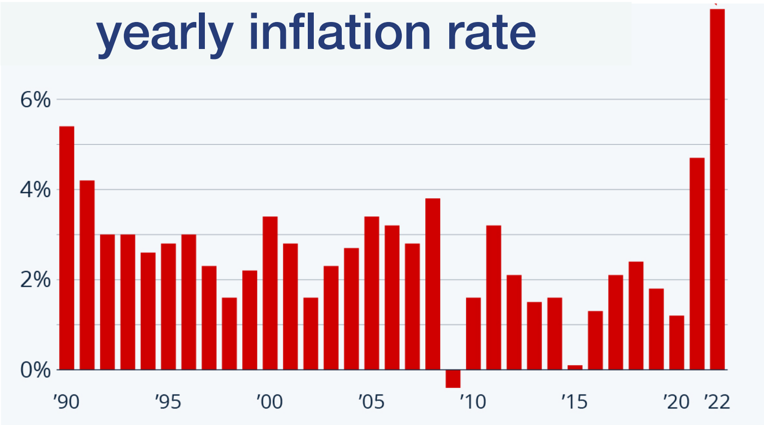
electrocardiogram (ECG)



insuline levels in blood

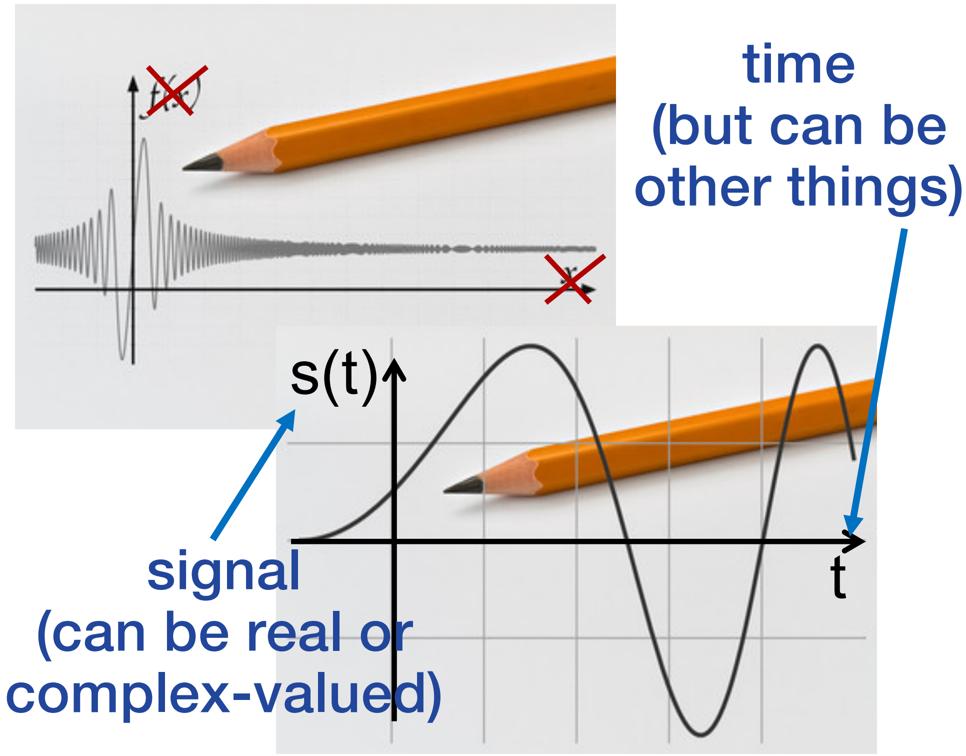
More examples

From economy



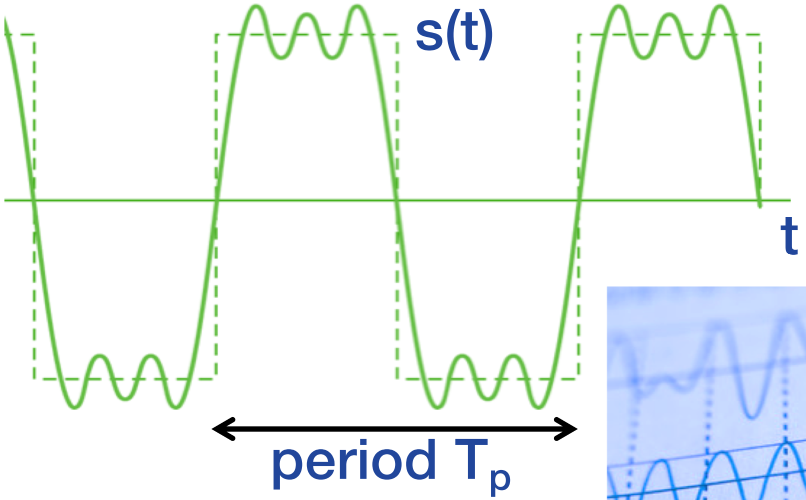
Standard notation

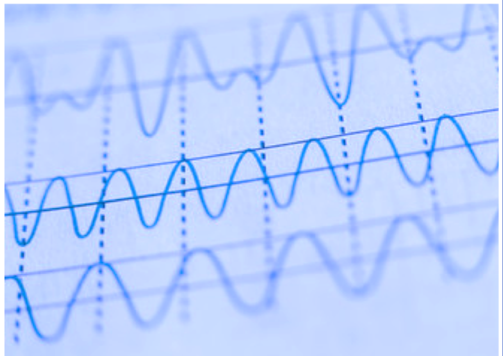
From functions to signals



Periodic signals

With a repeating shape





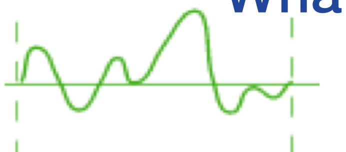
With the fundamental property

$s(t+T_p) = s(t)$

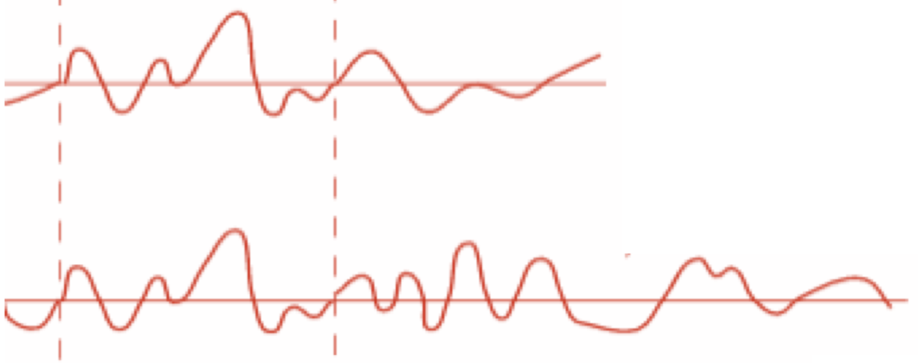
Periodic signals

As a model on a finite time window

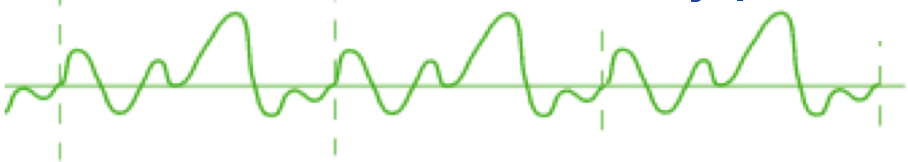
What we observed



... can be extended by arbitrary shapes

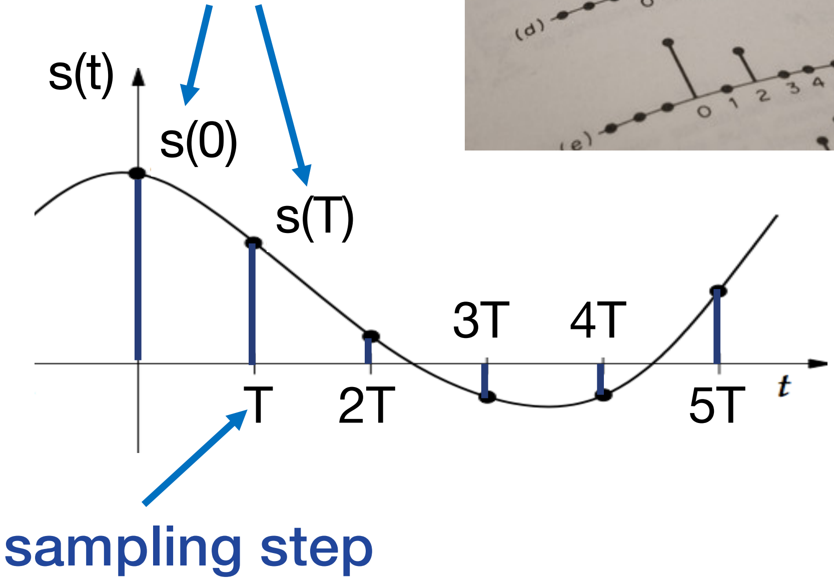


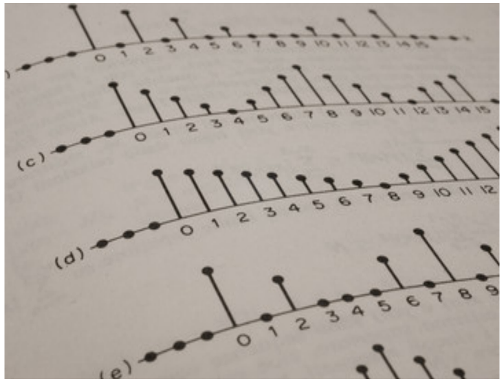
... but better to extend it by periodicity



Discrete-time signals

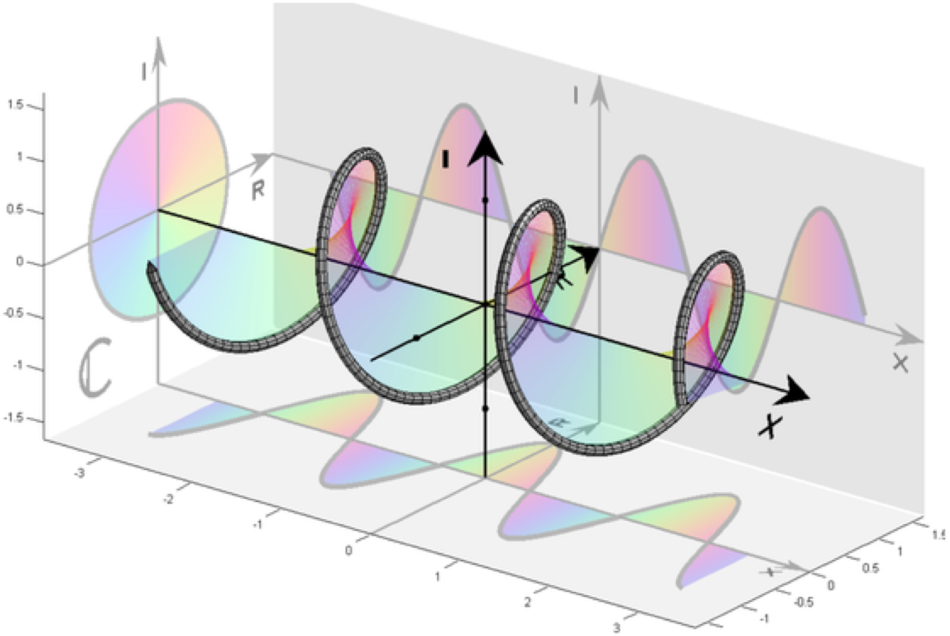
E.g., obtained by sampling





Complex-valued signals

A mathematical abstraction?



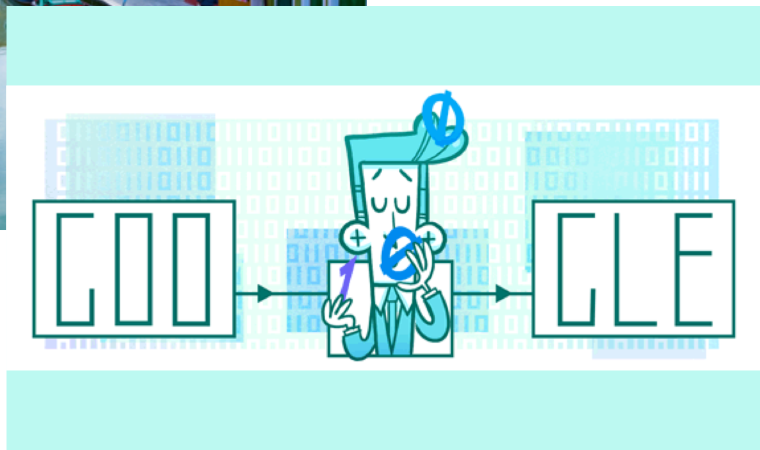
$$\underline{s}(t) = [\ \mathcal{R}[s(t)], \mathcal{I}[s(t)]\ ]$$

Multidimensional signals

In multimedia processing

$\underline{s}(x,y) = [\ r(x,y),\ g(x,y),\ b(x,y)\ ]$

color image

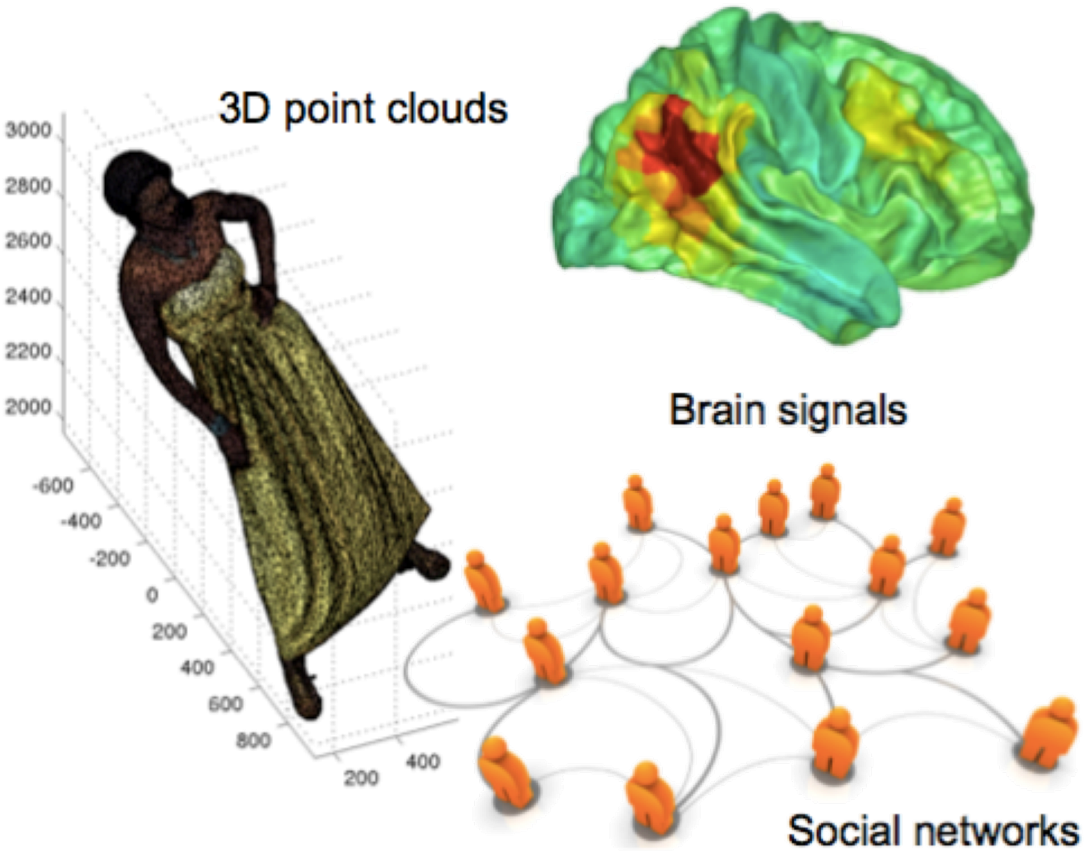


video

$\underline{s}(x,y,t) = [\ r(x,y,t),\ g(x,y,t),\ b(x,y,t)\ ]$

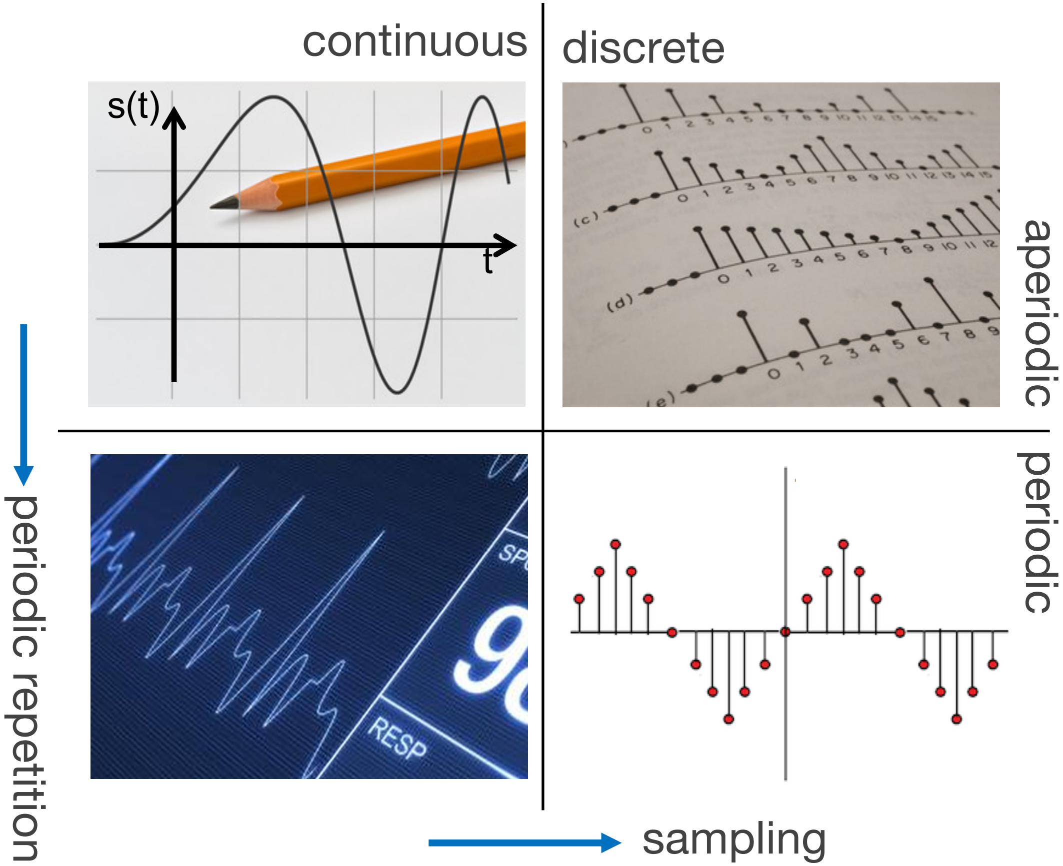
Signals on graphs

What we use today



Four classes of signals

What we deal with



Exercises

On complex values

Complex-valued signals are the key players in this course, and we expect you to already know complex values from previous mathematics courses.

Refresh your knowledge of **complex-values** with the set of introductory exercises provided.

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Lecture 01

Foundations of Signals and Systems: an introduction

Tomaso Erseghe

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1.3 Systems

What is a system?

- Definition
- Linear time-invariant systems
- Transforms
- An application example
- Course overview

Transformation (systems)

Definition

www.merriam-webster.com

[…]

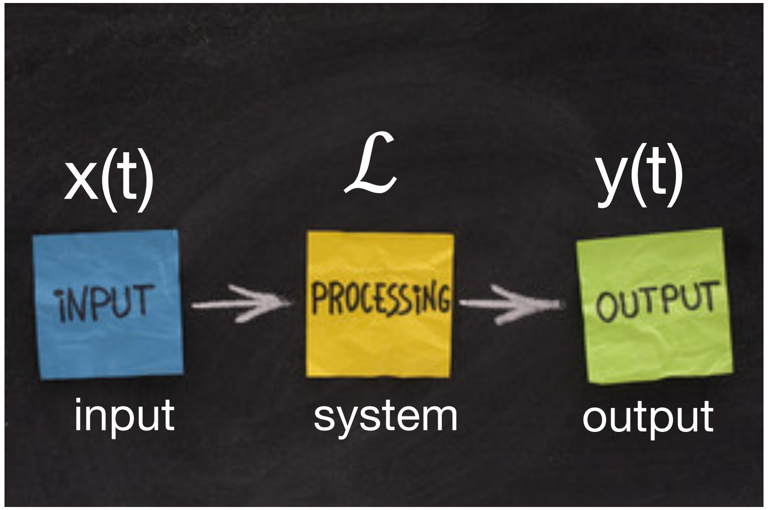
3

a (1) : the **operation of changing** (as by rotation or mapping) one configuration or expression into another **in accordance with a mathematical rule**; […]

a (2) : the **formula** that effects a transformation

b : a **mathematical correspondence** that assigns exactly one element of one set to each element of the same or another set

LTI systems



Linearity = Superposition principle

Time-Invariance = Repeatability principle

Convolution systems

LTI property

$$y(t) = \int_{-\infty}^{+\infty} x(u)h(t-u)du$$

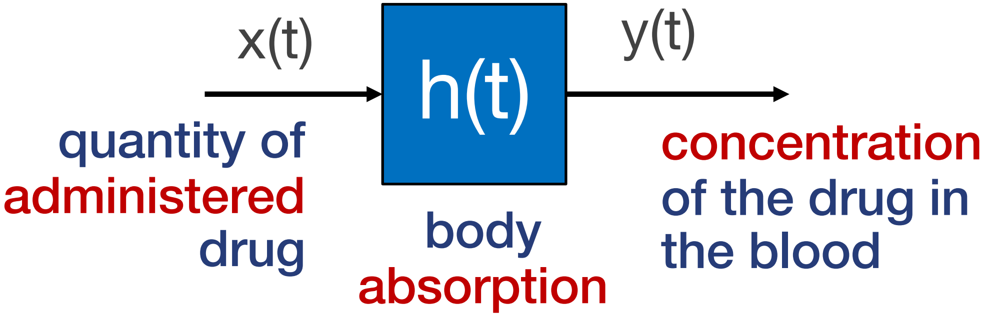
$$y_n = \sum_{k=-\infty}^{+\infty} x_k h_{n-k}$$

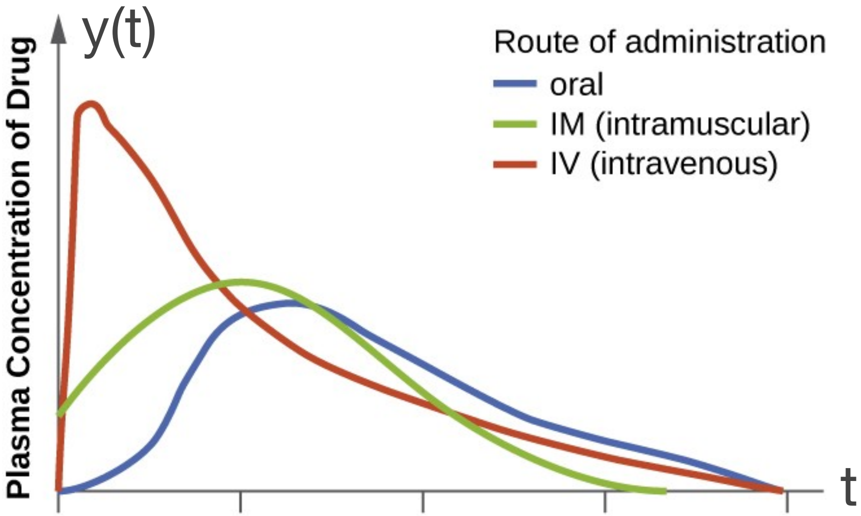
impulse response
(it characterises
the system)

The action of an LTI system is a convolution

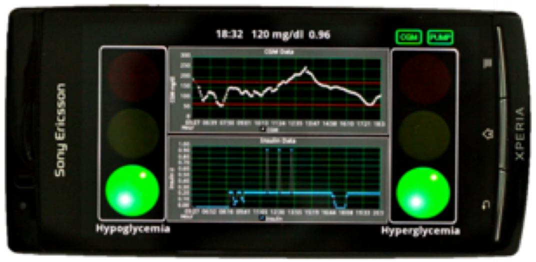
A pharmacology example

On drugs absorption

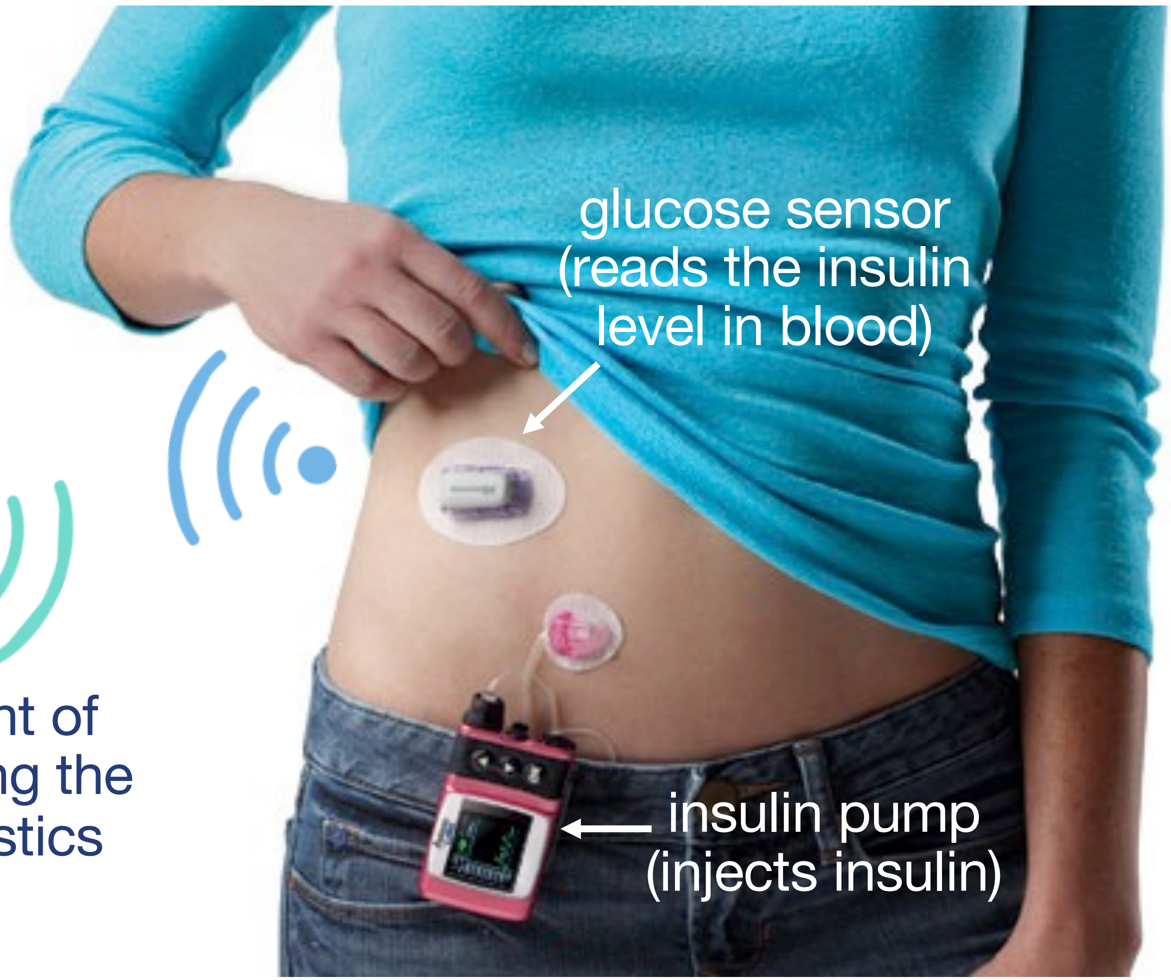




Artificial pancreas system

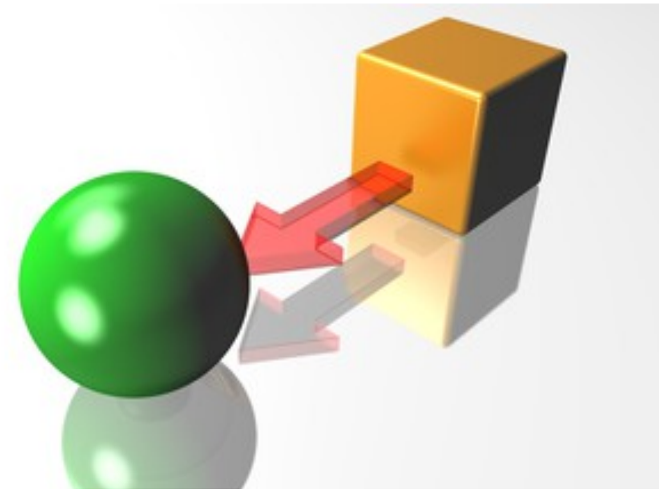


remote device:
identifies the correct amount of
insulin to be injected, knowing the
body absorption characteristics
(LTI system)



Transforms

Alternative signal representations



integrals

Fourier transforms

$$S(j\omega) = \int_{-\infty}^{\infty} s(t)e^{-j\omega t}\, dt$$

$$S(e^{j\theta}) = \sum_{n=-\infty}^{\infty} s(n)e^{-j\theta n}$$

Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

series

complex exponentials

Main property

time-domain representation

transform representation

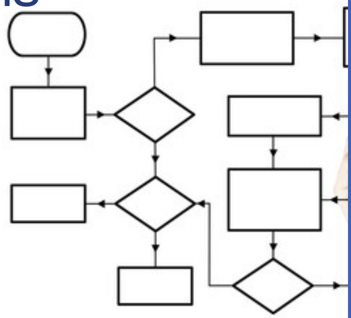
x*h(t) → **trans form** → X(s) H(s)

convolution

product

difficult to deeply understand its action, especially in complex systems

easy interpretable effect, can deal with complex systems



Course outline

time domain

Signals (4 lectures)
- Duration, area, mean value, energy, power, periodicity, complex exponentials, sinusoids, time-reversal, -shift, and -scale, periodic repetition, ideal impulses

Systems (3 lectures)
- Memory, stability, linearity, time invariance, **convolution** and its properties, **filters**, parallel and series connections

MatLab (2 lectures)
- Signal representation, **convolution** and **filters**

transform domain

Fourier transforms (6 lectures)
- Fourier series, DFT, Fourier transform, discrete-time Fourier transform, their properties and the **sampling/periodic repetition relation** among them

Sampling theorem (1 lecture)
- Reconstructing a signal from its samples

MatLab (1 lecture)
- Fourier representation and **filters**

Laplace + Z transforms (2 lectures)
- Laplace + Z transforms and their properties, application to **differential equations**

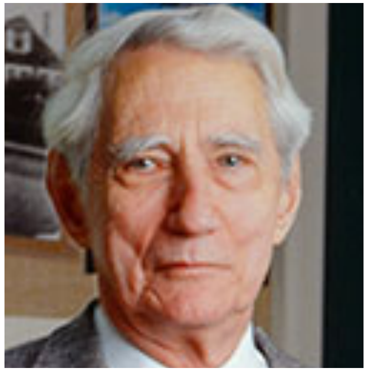
Thanks to

Pierre-Simon Laplace
Beaumont-en-Auge 23/3/1749
Paris 5/3/1827

Mathematician, physicist,
astronomer and nobleman

Jean Baptiste Joseph Fourier
Auxerre 21/3/1768
Paris 16/5/1830

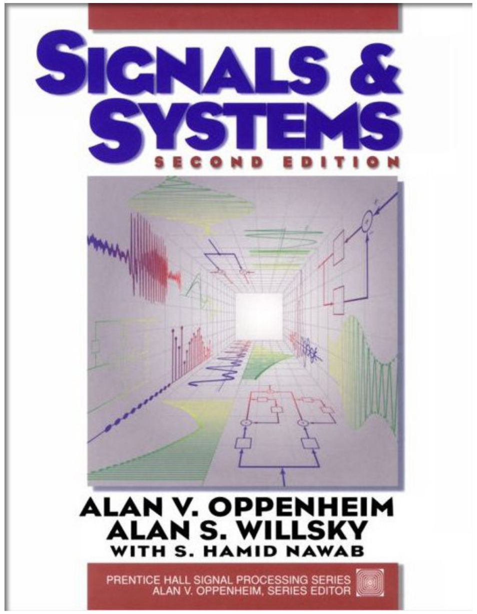
Mathematician and physicist



Claude Elwood Shannon
Petoskey (MI) 30/4/1916
Medford (MA) 24/2/2001

US engineer and mathematician

A worldwide reference



Signals and Systems, 2nd Edition
Oppenheim, Willsky, Nawab
published by Pearson

Exercises

On integrals and series

Complex-valued integrals and series are the key tools needed to understand systems. We expect you to already know how to handle them from previous mathematics courses, at least at a basic level.

Refresh your knowledge of **integrals and series** with particular care in fully understanding the **geometric series** and **primitives of complex exponentials**.

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Lecture 02

Signals: area, mean value, energy, power for aperiodic signals

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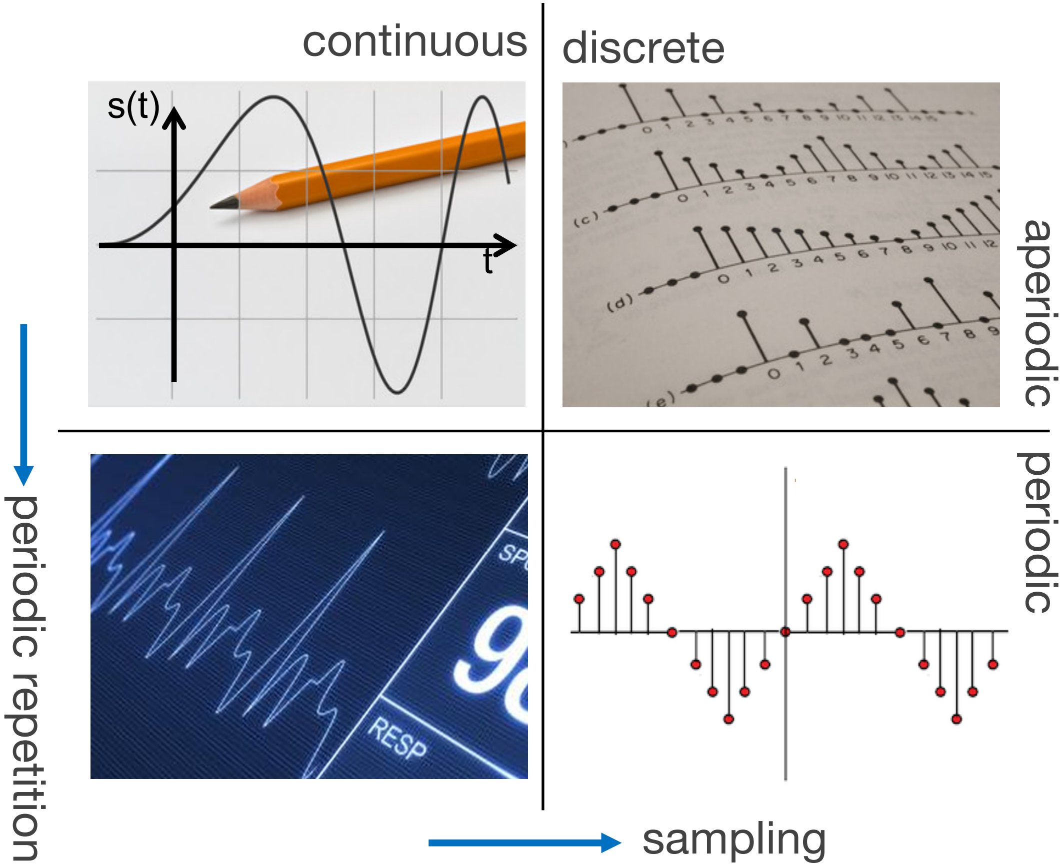
2.1 Continuous-time signals

Aperiodic case

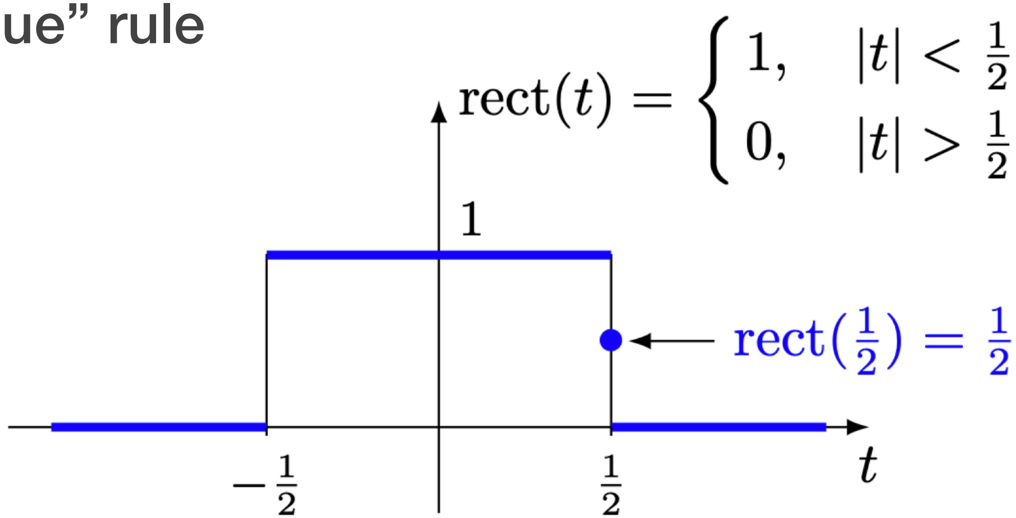
- Discontinuous signals
- Duration and extension
- Causality
- Area and mean value
- Energy and power

Four classes of signals



Continuity and discontinuity

The "semi-value" rule



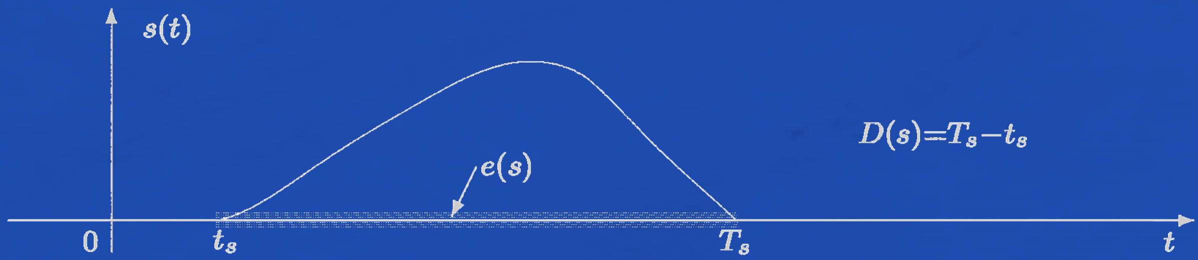
if $s(t)$ has a discontinuity at $t = t_i$ we assign to $s(t_i)$ the average value of the limits approached from right and left, respectively,

$$s(t_i) \triangleq \frac{1}{2}\left[s(t_i+) + s(t_i-)\right]$$

Extension and duration

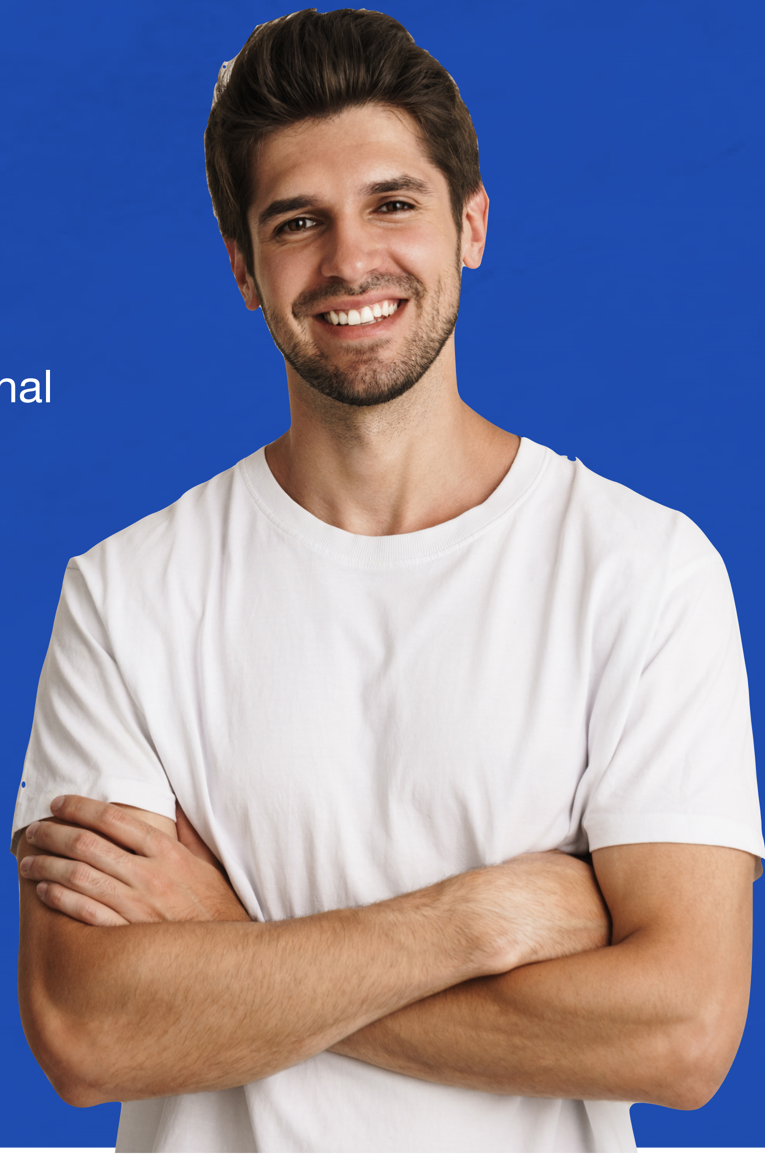
of a continuos-time aperiodic signal

extension e(s) = smaller interval **$[t_s, T_s]$** in which the signal s(t) is active – also union of intervals, with $\{t | s(t) \neq 0\} \subseteq e(s)$

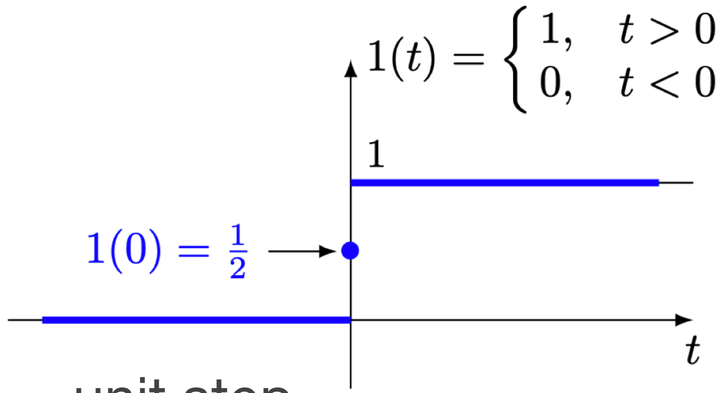


duration D(s) = measure **$T_s - t_s$** of the extension

can be finite (as in the example) or infinite

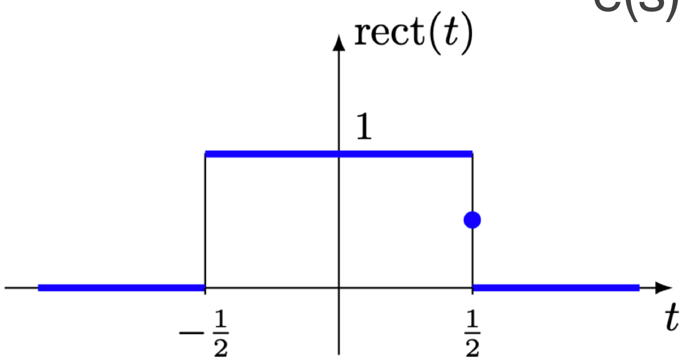


Examples



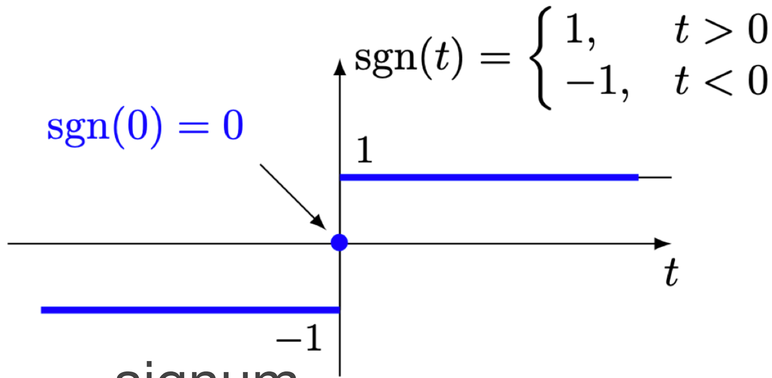
unit step

e(s) = [0,∞), D(s) = ∞



rectangle

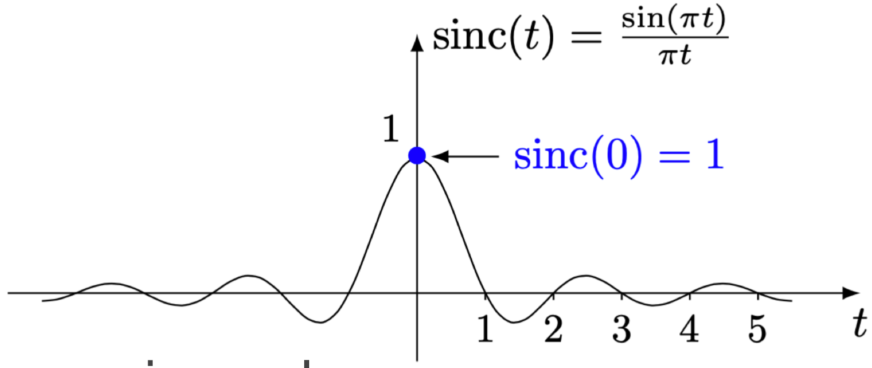
e(s) = [-½,½], D(s) = 1



signum

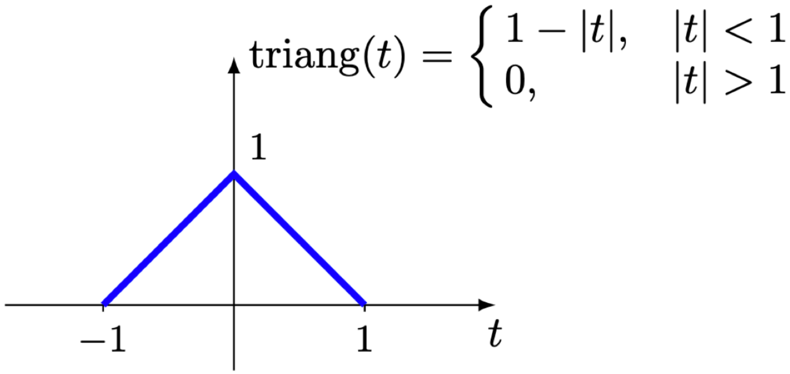
e(s) = (-∞,∞), D(s) = ∞

More examples



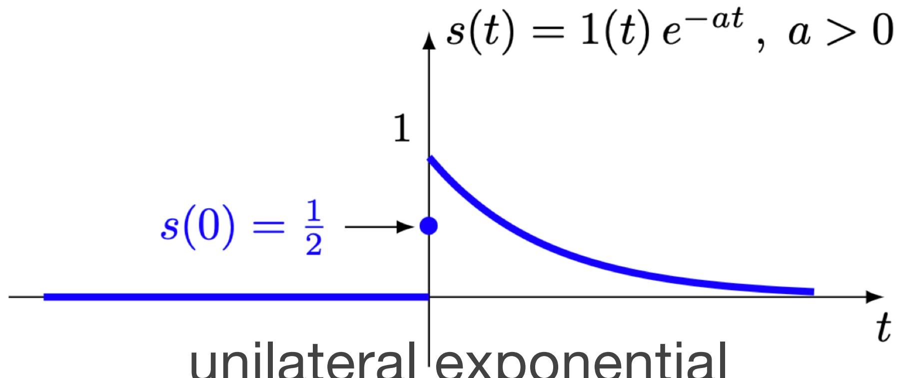
sinc pulse

e(s) = (-∞,∞), D(s) = ∞



triangular pulse

e(s) = [-1,1], D(s) = 2

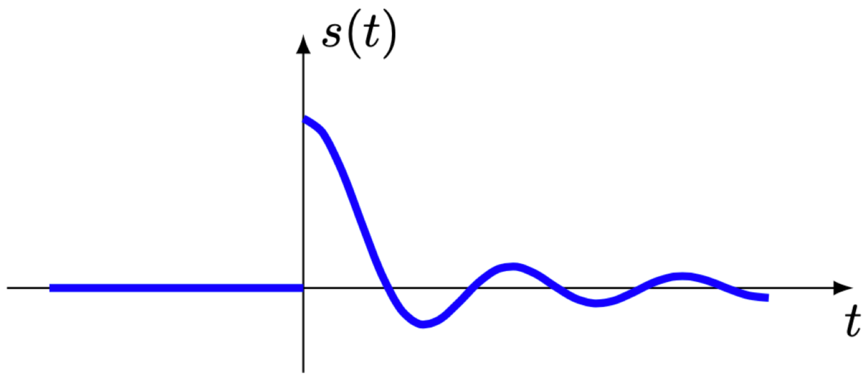


unilateral exponential

e(s) = [0,∞), D(s) = ∞

Causal signals

Causality property



A signal is said to be **causal** if active only in the positive time axis

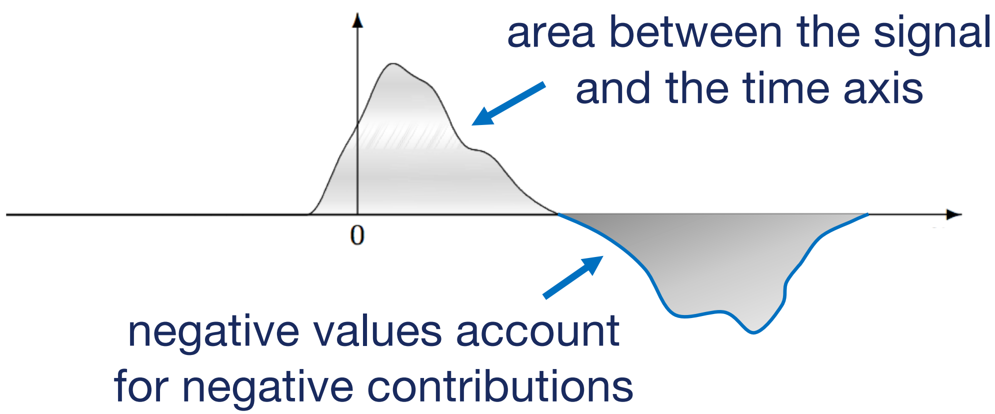
$$s(t) = 0 \quad \text{for} \quad t < 0$$

$$e(s) \subset [0, +\infty)$$

Area

Of a continuos-time aperiodic signal

$$\text{area}\,(s) = \int_{-\infty}^{+\infty} s(t)\,dt \;=\; \lim_{T\to\infty} \int_{-T}^{T} s(t)\,dt$$



Meaning: balance between positive and negative signal values, takes a complex value for complex valued signals

Mean value

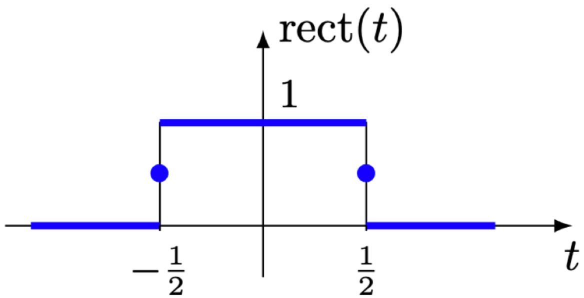
Area

$$\text{area}\,(s) = \int_{-\infty}^{+\infty} s(t)\; dt$$

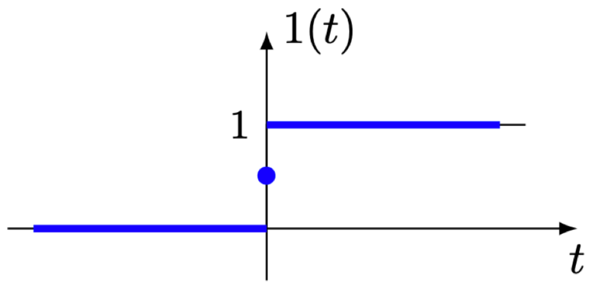
Mean (average value)

$$m_s = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s(t)\; dt$$

They capture a similar information for different classes of signals (only one at a time is meaningful)



$A_s = 1$, $m_s = 0$



$A_s = \infty$, **$m_s = ½$**

Main properties

of area and mean value

linearity $s(t) = B\,x(t) + C\,y(t)$ → $A_s = B\,A_x + C\,A_y$

$m_s = B\,m_x + C\,m_y$

finite area → zero mean value

finite (non-zero) mean value → infinite area

$$m_s = \lim_{T\to\infty} \frac{1}{2T} \overbrace{\int_{-T}^{T} s(t)\, dt}^{\text{the limit is } A_s}$$

the limit is ∞

Energy and power

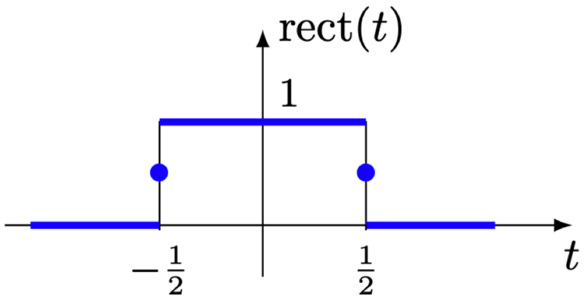
Energy

$$E_s = \lim_{T\to\infty} \int_{-T}^{T} |s(t)|^2 \, dt$$

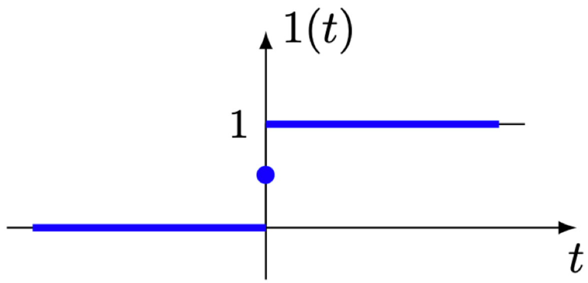
Power

$$P_s = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |s(t)|^2 \, dt$$

They correspond to the area and the mean value of $|s(t)|^2$ hence take only **real positive values**



E_s = 1, P_s = 0

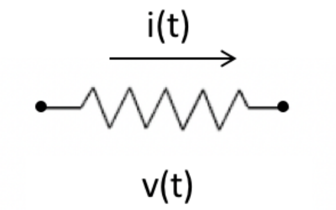


$E_s = \infty$, **P_s = ½**

Meaning

Of energy and power

These definition are inspired by power and energy dissipated in a resistor



They measure the **level of activity** of a signal

Exercises

On continuos-time aperiodic signals

Calculating **area, mean value, energy, and power** is a matter of correctly solving integral expressions.

You need to get acquainted with the basic tricks and tips. Remember that the primitive of e^{at} is e^{at}/a, irrespective of the fact that a is real or complex valued.

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Lecture 02

Signals: area, mean value, energy, power for aperiodic signals

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2.4 Discrete-time signals

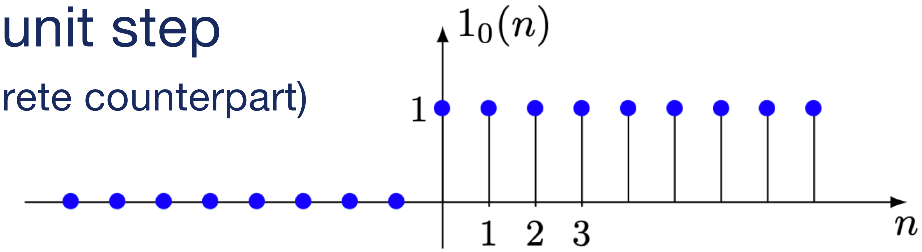
Aperiodic case

- Extension
- Causality
- Area and mean value
- Energy and power

Discrete-time signals

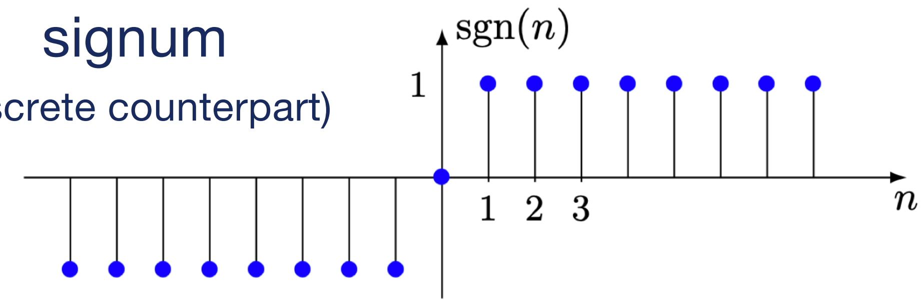
unit step

(discrete counterpart)



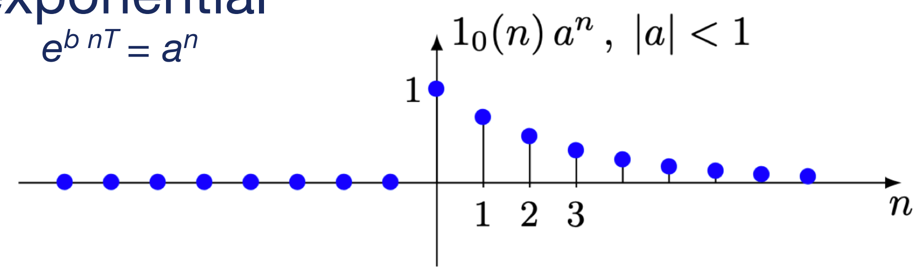
signum

(discrete counterpart)



exponential

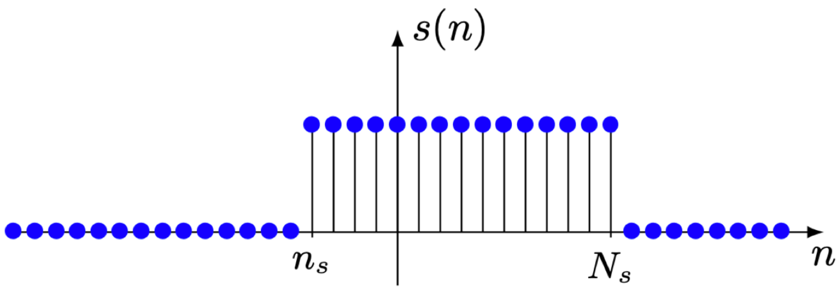
$e^{b\,nT} = a^n$



Extension and causality

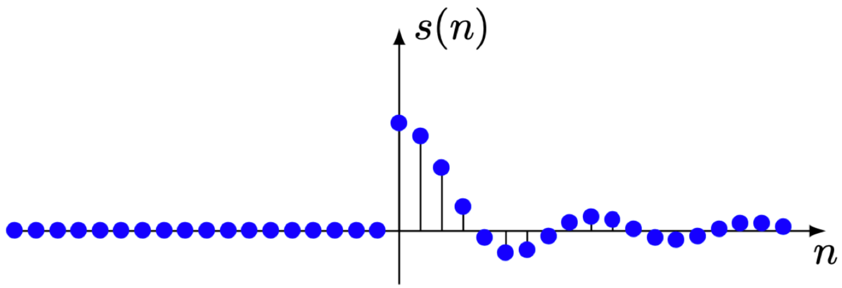
Extension

The **smallest interval** $[n_s, N_s]$ containing the active signal samples



Causality

A signal is called causal if its extension is contained in $[0,\infty)$, or, equivalently, if **$s(n)=0$ for $n<0$**



Fundamental measures

Where a sum replaces the integral

Area and mean value

$$A_s = \sum_{n=-\infty}^{\infty} s(n)$$

$$m_s = \lim_{N\to\infty} \frac{1}{1+2N} \sum_{n=-N}^{N} s(n)$$

the number of samples is $1+2N$

Energy and power

$$E_s = \sum_{n=-\infty}^{\infty} |s(n)|^2$$

$$P_s = \lim_{N\to\infty} \frac{1}{1+2N} \sum_{n=-N}^{N} |s(n)|^2$$

Main properties

linearity

$s(n) = B\,x(n) + C\,y(n) \rightarrow A_s = B\,A_x + C\,A_y$

$m_s = B\,m_x + C\,m_y$

finite area → zero mean value

finite (non-zero) **mean value** → infinite area

$$m_s = \lim_{N\to\infty} \underbrace{\frac{1}{1+2N}}_{\text{the limit is } \infty} \overbrace{\sum_{n=-N}^{N} s(n)}^{\text{the limit is } A_s}$$

Exercises

On discrete-time aperiodic signals

Calculating **area, mean value, energy, and power** is a matter of correctly solving series.

You need to get acquainted with the basic tricks and tips. The most important result to have in mind is the **geometric series** either in its full or truncated form.

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Lecture 03

Signals: periodicity, area, mean value, energy, power for periodic signals

Tomaso Erseghe

Università degli Studi di Padova

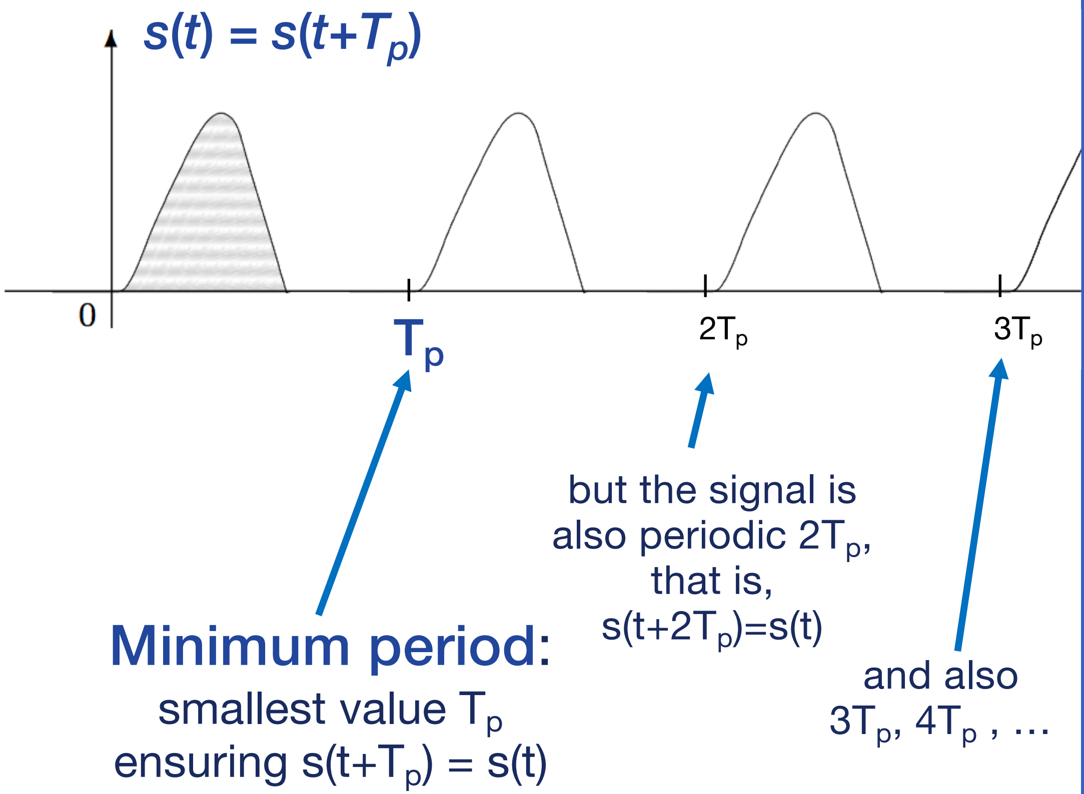
3.1 Periodic signals

The continuous-time case

- The minimum period
- Complex exponentials and sinusoids
- Compositions of periodic signals
- Area, mean value, energy, and power

Periodic signals

Main property and minimum period

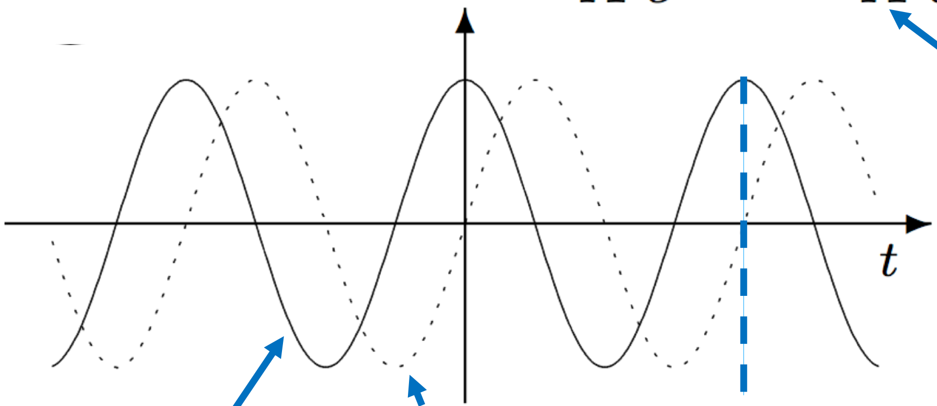


Complex exponentials

With linear phase

$$A\,e^{j\omega_0 t} = A\,e^{j2\pi f_0 t}$$

real and positive



minimum period $T_p = 1/|f_0|$

$$Ae^{j2\pi f_0 t} = \underbrace{A\cos(2\pi f_0 t)}_{\text{real part}} + \underbrace{jA\sin(2\pi f_0 t)}_{\text{imaginary part}}$$

$$\boxed{\begin{aligned} &\cos(2\pi f_0(t+T_p)) \\ &\quad = \cos(2\pi f_0 t + 2\pi f_0/|f_0|) \\ &\quad = \cos(2\pi f_0 t \pm 2\pi) \\ &\quad = \cos(2\pi f_0 t) \end{aligned}}$$

$s(t) = s(t+T_p)$

Signal composition

Identifying a common period

$$s(t) = \underbrace{2\cos(18\pi t + 1)}_{T_1 = \frac{1}{9}} - \underbrace{\sin(12\pi t + 7)}_{T_2 = \frac{1}{6}}$$

periodic of period mT_1

periodic of period kT_2

We look for the mcm $T_p = mT_1 = kT_2$

$$\frac{m}{k} = \frac{T_2}{T_1} = \frac{f_1}{f_2} = \frac{9}{6} = \frac{3}{2}$$

we simplify the rational expression

$\boxed{T_p = m\,T_1 = 3/9 = 1/3}$

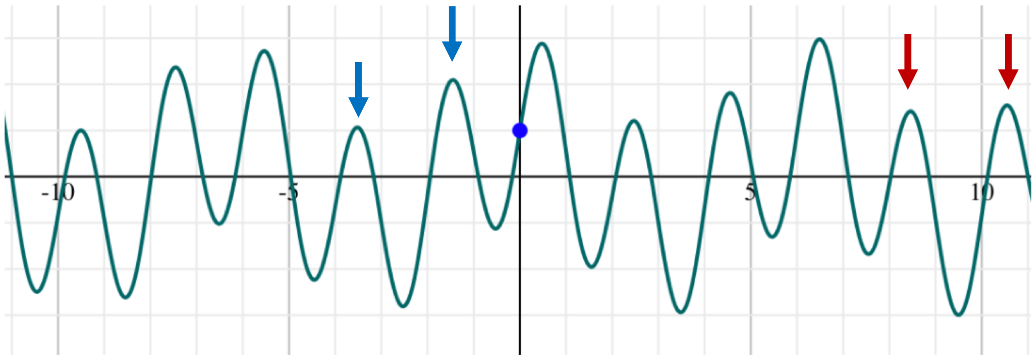
Not necessarily the minimum period!

General rule

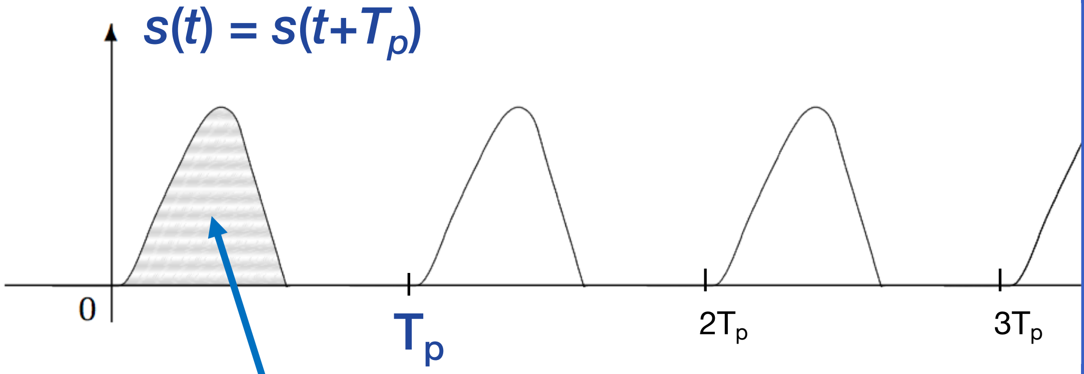
The composition of periodic signals is periodic if and only if the periods are in a rational relation

The composition of sinusoids is periodic if and only if their frequencies/pulsations are in a rational relation

Therefore cos(t) + 2 sin(πt) is NOT periodic



Area of a periodic signal



$$A_s(T_p) = \int_{t_0}^{t_0+T_p} s(t)dt$$

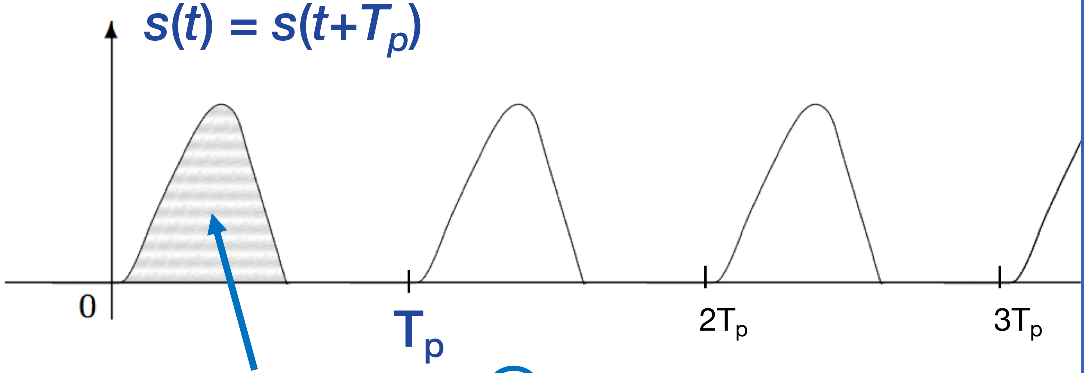
area in a period

It does not make any sense!

$$A_s = \lim_{N\to\infty} \int_{-NT_p}^{NT_p} s(t)\,dt = \lim_{N\to\infty} 2NA_s(T_p) = \infty$$

Mean of a periodic signal

$s(t) = s(t+T_p)$



$$A_s(T_p) = \int_{t_0}^{t_0+T_p} s(t)dt$$

area in a period

This is consistent!

$$m_s = \lim_{N\to\infty} \frac{1}{2NT_p} \int_{-NT_p}^{NT_p} s(t)\ dt$$

$$= \lim_{N\to\infty} \frac{2NA_s(T_p)}{2NT_p} = \frac{A_s(T_p)}{T_p}$$

Meaningful expressions

For periodic signals

Area (in a period) and mean value

$$A_s(T_p) = \int_{t_0}^{t_0+T_p} s(t)dt \qquad m_s = \frac{A_s(T_p)}{T_p} = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} s(t)dt$$

Energy (in a period) and power

$$E_s(T_p) = \int_{t_0}^{t_0+T_p} |s(t)|^2 \, dt \qquad P_s = \frac{E_s(T_p)}{T_p} = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} |s(t)|^2 dt$$

Mean and power carry the relevant information, they are correct even in case T_p is NOT the minimum period

Exercises

On continuous-time periodic signals

Calculating **mean value and power** is particularly important for compositions of complex exponentials and sinusoids.

The most important result to learn is that complex exponentials and sinusoids have **zero mean** while **powers** take the forms **$|A|^2$** and **$\frac{1}{2} A^2$**, respectively, with A a multiplying constant. In their composition, powers simply sum.

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Lecture 03

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3.4 Periodic signals

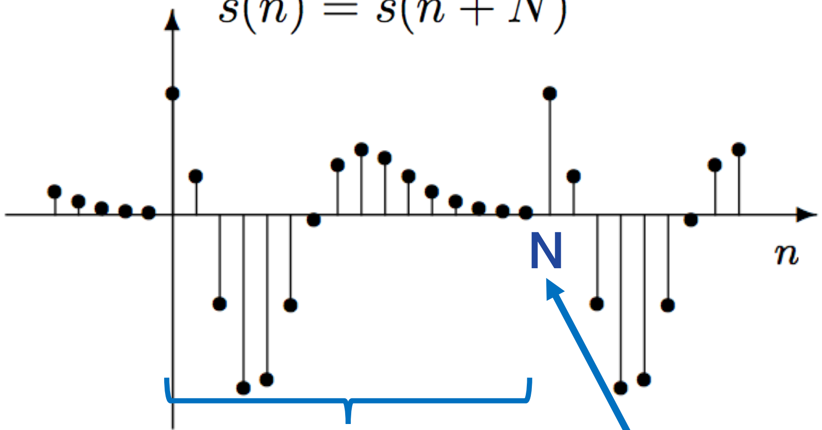
The discrete-time case

- Complex exponentials and sinusoids
- Periodicity issues
- Area, mean value, energy, and power

Periodic signals

Main property and minimum period

$$s(n) = s(n+N)$$



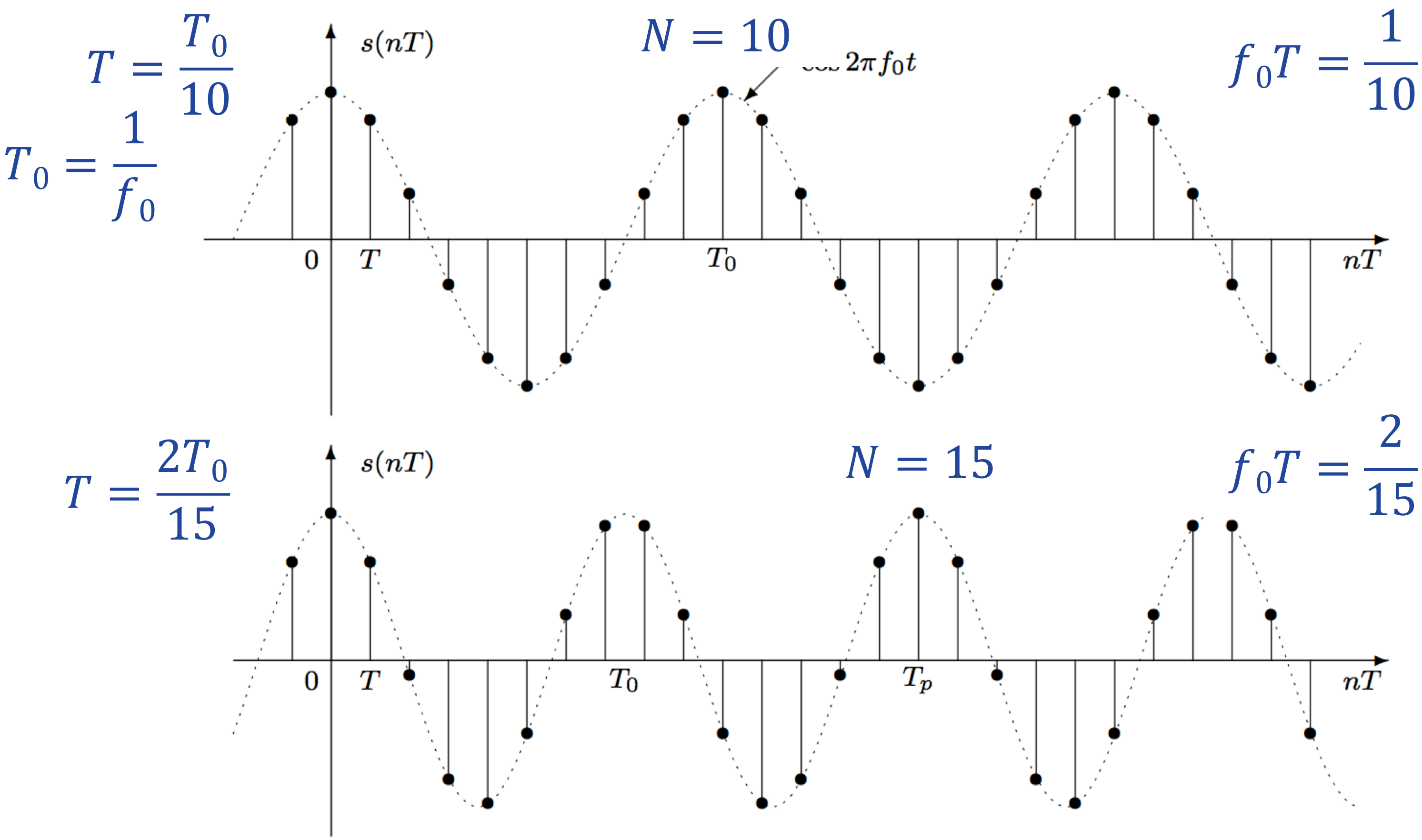
the signal is uniquely defined by the N samples in the period (vector)

Minimum period:

smallest value N ensuring s(n+N) = s(n) but the signal is also periodic 2N, 3N, …

Sampled sinusoids $A_0 \cos(2\pi f_0 nT + \varphi_0)$

What is their period?



Sampled sinusoids

Identifying a period

$$s(n) = \cos(2\pi f_0 n T)$$

We must have

$$s(n) = s(n+N)$$

$$\cos(2\pi f_0 n T + 2\pi f_0 N T) = \cos(2\pi f_0 n T)$$

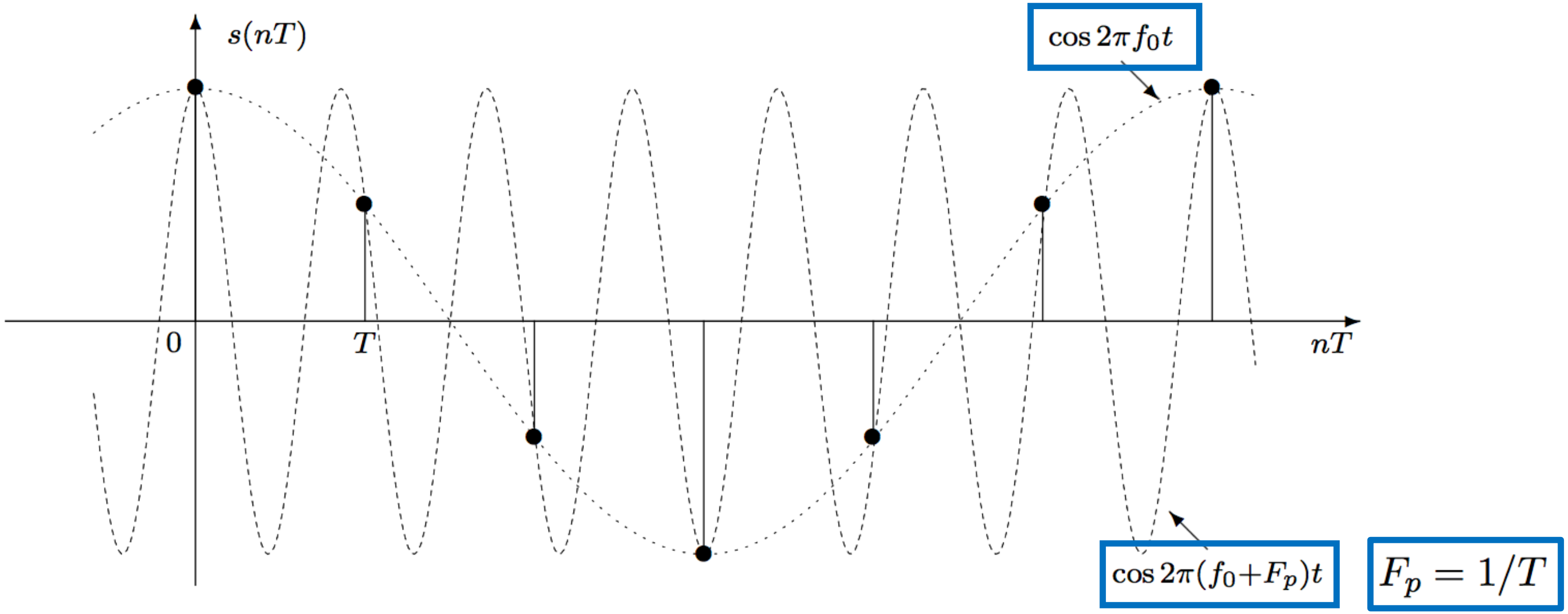
$$\Downarrow$$

$$f_0 T = \frac{k}{N}$$

A non-rational value f_0T implies that the sampled sinusoid is NOT periodic

Ambiguity of frequency

In sampled sinusoids



$$\cos(2\pi(f_0 + F_p)nT) = \cos(2\pi f_0 nT + 2\pi n) = \cos(2\pi f_0 nT)$$

Mean and power

For periodic signals

Area (in a period) and mean value

$$m_s = \frac{A_s(N)}{N}$$

$$A_s(N) = \sum_{n=n_0}^{n_0+N-1} s(n)$$

Energy (in a period) and power

$$P_s = \frac{E_s(N)}{N}$$

$$E_s(N) = \sum_{n=n_0}^{n_0+N-1} |s(n)|^2$$

Exercises

On discrete-time periodic signals

Calculating **mean value and power** is particularly important for compositions of complex exponentials and sinusoids.

The most important result to learn is that complex exponentials and sinusoids have **zero mean** while **powers** take the forms $|A|^2$ and $\frac{1}{2} A^2$, respectively, with A a multiplying constant. In their composition, powers simply sum.

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Lecture 04

Signals: time-reversal, -shift, and -scale, symmetries

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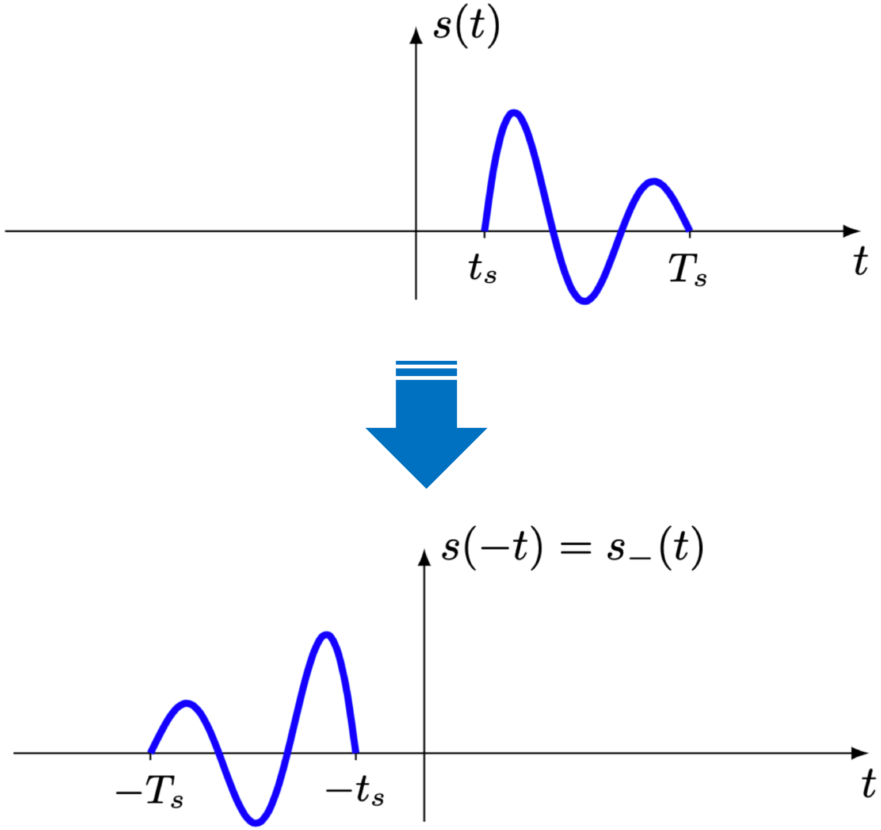
4.1 Basic transformations

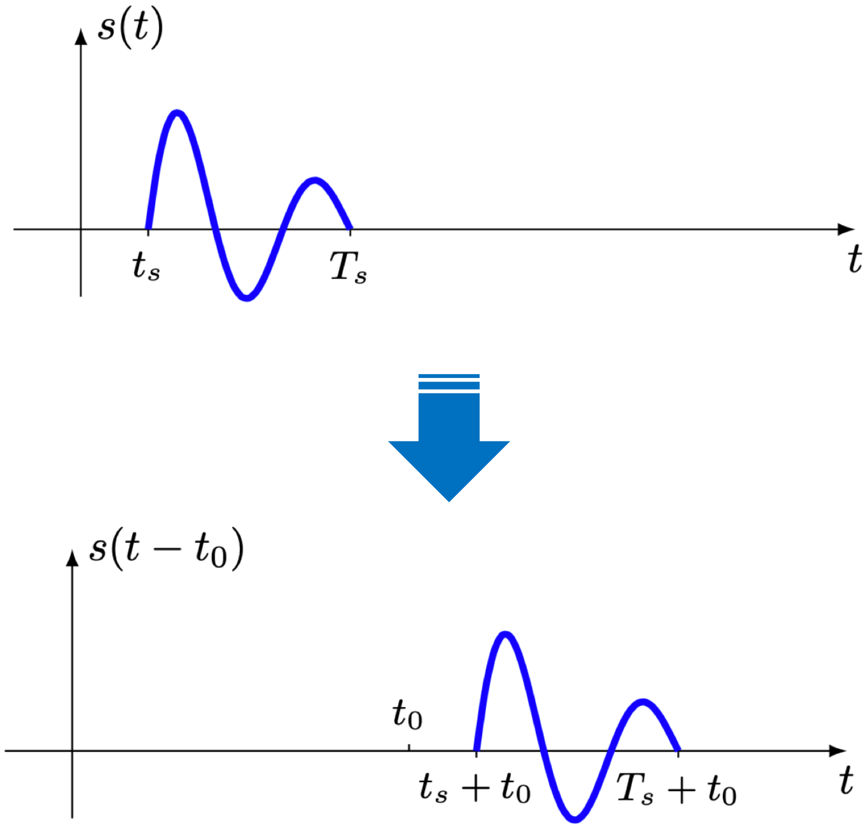
In continuous and discrete-time aperiodic domains

- Time reversal, time shift, time scaling
- Combining transformations
- Symmetries
- Periodic repetition

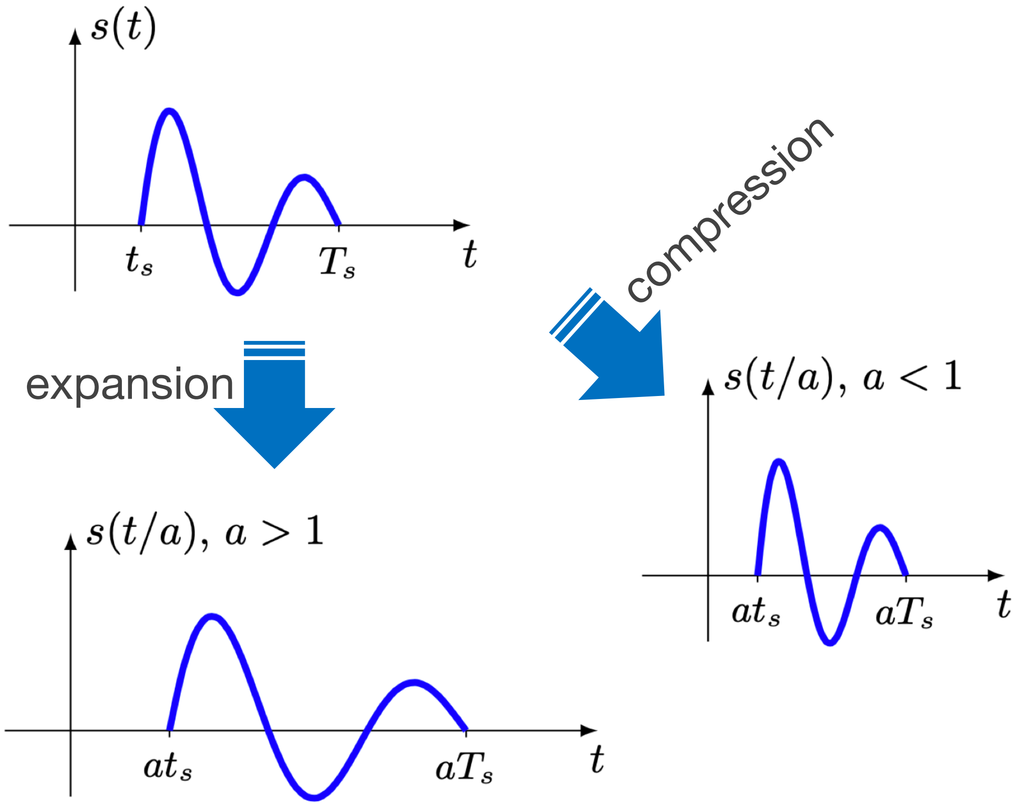
Time reversal



Time shift

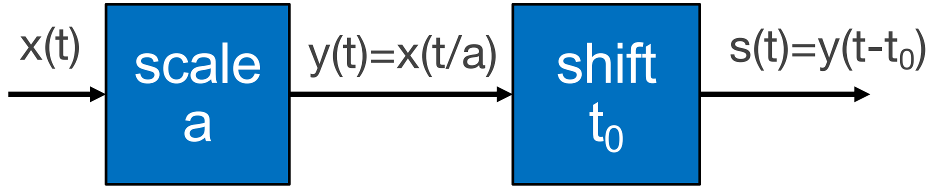


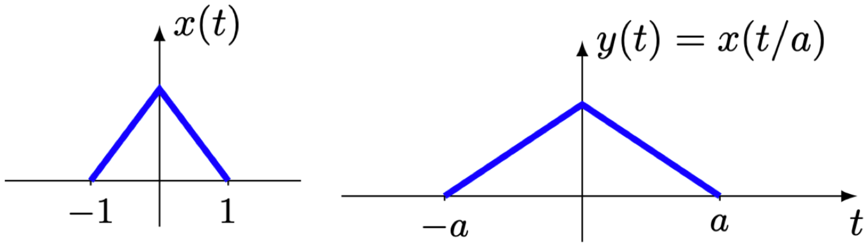
Time scale

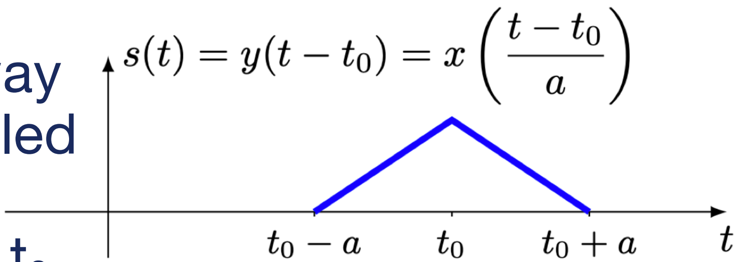


$s(t)$
t_s
T_s
t
expansion
compression
$s(t/a),\ a > 1$
at_s
aT_s
t
$s(t/a),\ a < 1$
at_s
aT_s
t

Time scale + time shift



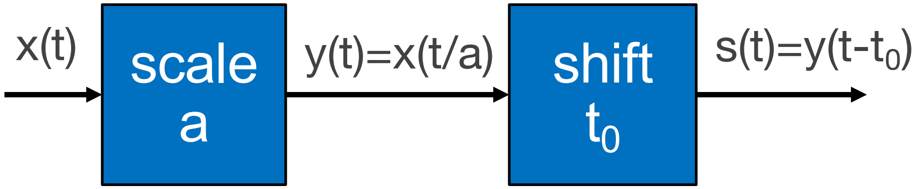




The natural way to write a scaled and shifted signal, where t_0 is the true signal position

Swapping shift and scale

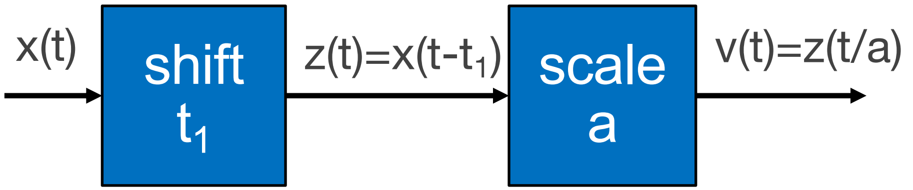
Which is the one to prefer?



Scale + shift

$$s(t) = x\left(\frac{t - t_0}{a}\right)$$

The preferred one,
where t_0 is the true shift

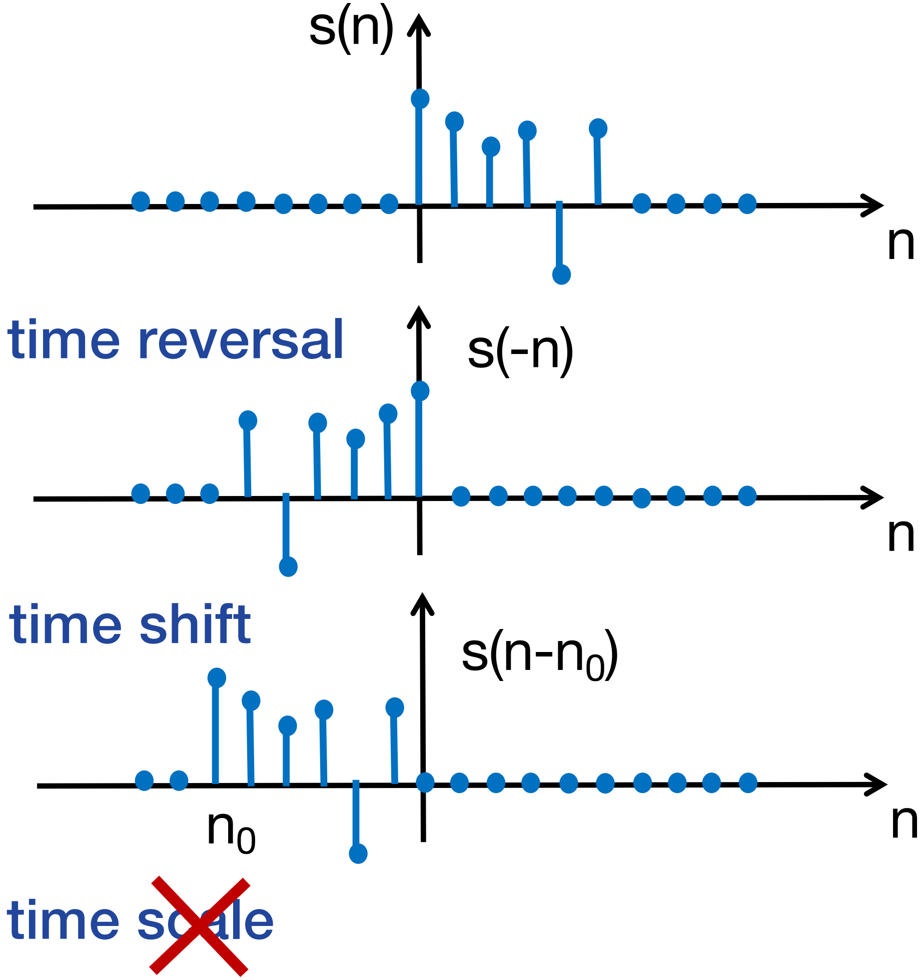


Shift + scale

$$v(t) = x\left(\frac{t}{a} - t_1\right) = x\left(\frac{t - at_1}{a}\right)$$

They correspond
when $at_1=t_0$

Discrete-time signals



Exercises

On basic transformations

Understanding **time reversal, shift, and scale** may seem too easy, but you need to familiarize with these transformations as they are the basis for expressing signals in a compact form.

Bear in mind the meaning of $s((t-t_0)/a)$ as this is fundamental in all that follows.

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Lecture 04

Signals: time-reversal, -shift, and -scale, symmetries

Tomaso Erseghe

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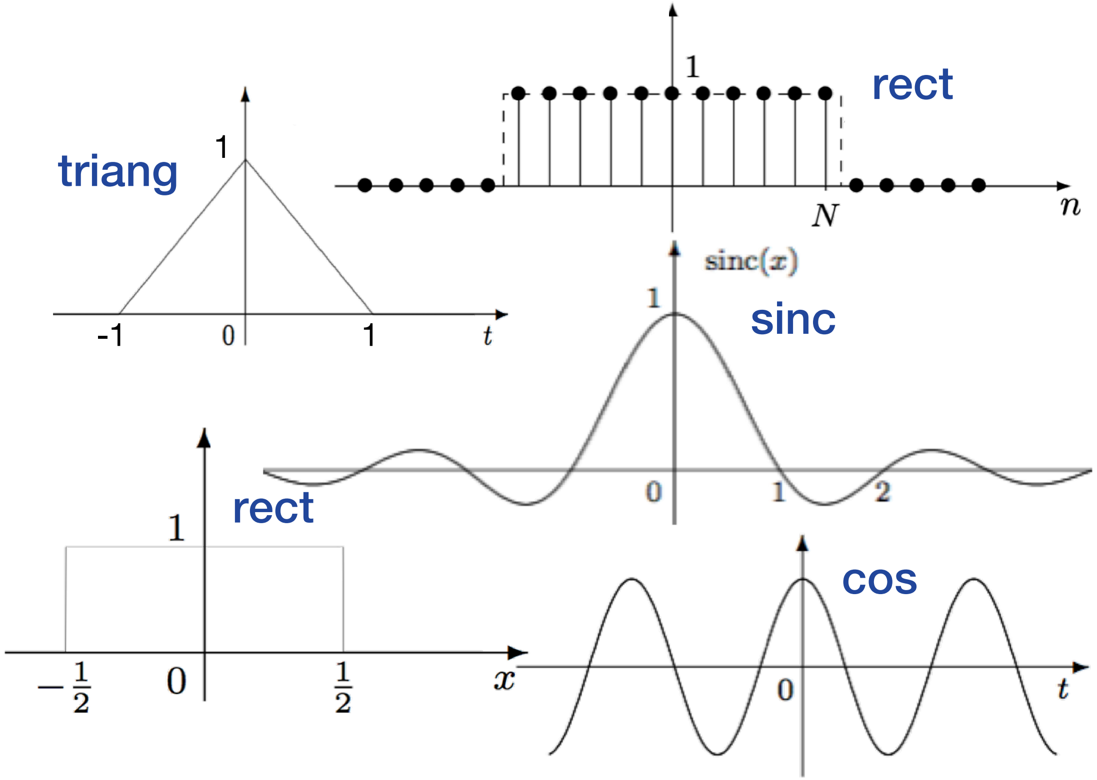
4.4 Symmetries in signals

Signal invariance to a transformation

- Even and odd symmetry
- Real and imaginary symmetry
- Hermitian and anti-Hermitian symmetry

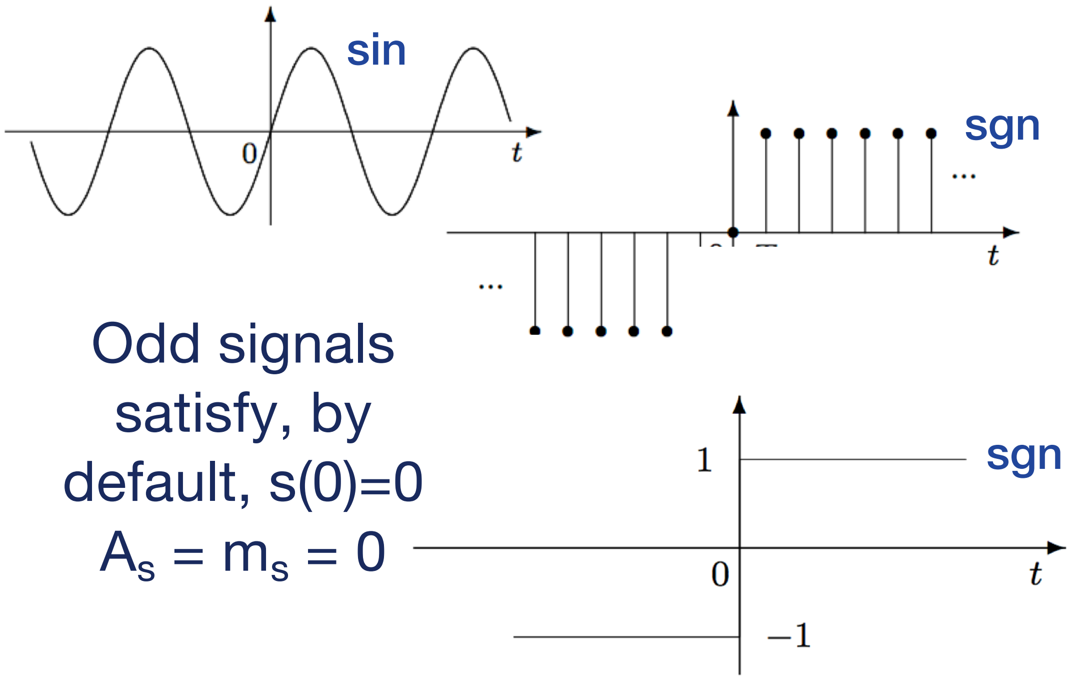
The even symmetry

A signal is said to be **even symmetric** if it is invariant to time-reversal, that is, if $s(t) = s(-t)$ or $s(n) = s(-n)$



The odd symmetry

A signal is said to be odd symmetric if
$s(t) = -s(-t)$ or $s(n) = -s(-n)$



Odd signals satisfy, by default, $s(0)=0$

$A_s = m_s = 0$

Even and odd components

The same applies in discrete-time

Every signal can be (uniquely) decomposed into its even and odd parts $s(t) = s_e(t) + s_o(t)$

where

$$s_e(t) = \tfrac{1}{2}\, s(t) + \tfrac{1}{2}\, s(-t) = s_e(-t)$$

$$s_o(t) = \tfrac{1}{2}\, s(t) - \tfrac{1}{2}\, s(-t) = -s_o(-t)$$

$$1(t) = \underbrace{\tfrac{1}{2}}_{\text{even part}} + \underbrace{\tfrac{1}{2}\, \mathrm{sgn}(t)}_{\text{odd part}}$$

Other symmetries of interest

Real/imaginary

real $s(t) = s^*(t)$
imaginary $s(t) = -s^*(t)$

components
$s_{re}(t) = \frac{1}{2}\, s(t) + \frac{1}{2}\, s^*(t)$
$s_{im}(t) = \frac{1}{2}\, s(t) - \frac{1}{2}\, s^*(t)$

Hermitian/anti-Hermitian

Hermitian $s(t) = s^*(-t)$
anti-Hermitian $s(t) = -s^*(-t)$

components
$s_h(t) = \frac{1}{2}\, s(t) + \frac{1}{2}\, s^*(-t)$
$s_a(t) = \frac{1}{2}\, s(t) - \frac{1}{2}\, s^*(-t)$

On the Hermitian symmetry

Understanding its meaning

$s(t) = a(t) + j\,b(t)$

Hermitian $s(t) = s^*(-t)$
means
$a(t) + j\,b(t) = a(-t) - j\,b(-t)$
even real-part
odd imaginary-part

anti-Hermitian $s(t) = -s^*(-t)$
means
odd real-part
even imaginary-part

Exercises

On symmetries

Understanding **symmetries** is fundamental for appreciating some important features of the Fourier transform.

They are straightforward, but need to be deeply understood.

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Lecture 05

Signals: periodic repetition, ideal impulses, generalized derivatives

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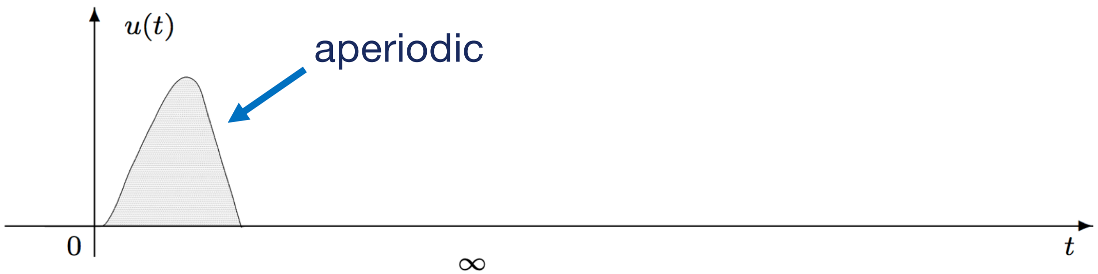
5.1 Periodic repetition

In continuous and discrete-time

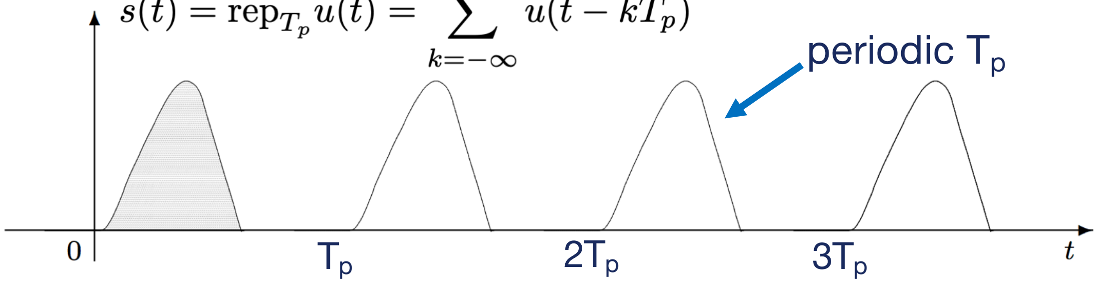
- The periodic repetition
- Compactly expressing periodic signals
- The presence of aliasing

Periodic repetition

From aperiodic to periodic



$$s(t) = \mathrm{rep}_{T_p} u(t) = \sum_{k=-\infty}^{\infty} u(t - kT_p)$$



Periodic repetition

Proof of periodicity

$$s(t) = \sum_{n=-\infty}^{+\infty} u(t - nT_p) \triangleq \mathrm{rep}_{T_p}\ u(t)$$

proof

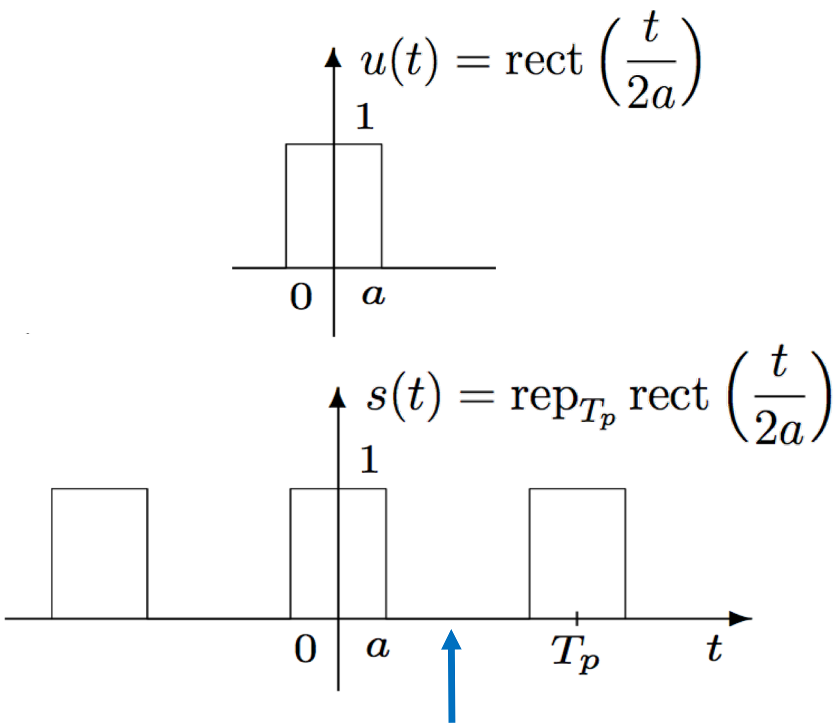
$$s(t + T_p) = \sum_{n=-\infty}^{+\infty} u(t + T_p - nT_p)$$

$$= \sum_{n=-\infty}^{+\infty} u(t - \underbrace{(n-1)}_{m} T_p)$$

$$= \sum_{m=-\infty}^{+\infty} u(t - mT_p) = s(t)$$

Square wave

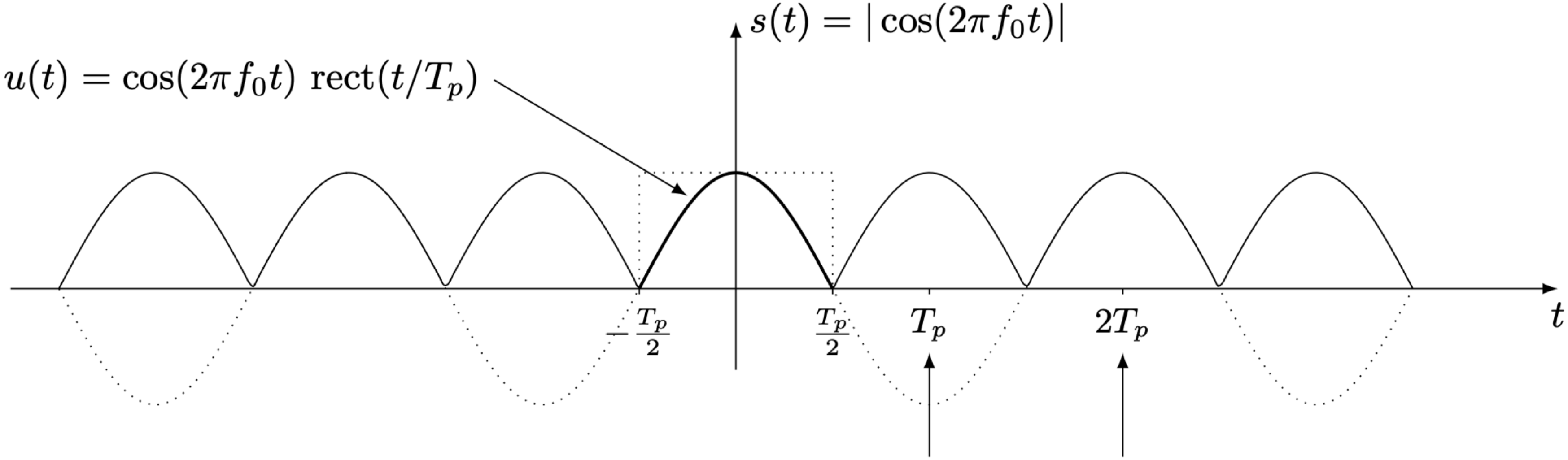
In the form of a periodic repetition



duty cycle = fraction of the period T_p where the signal is active **$d = 2a/T_p$**

Rectified sinusoid

In the form of a periodic repetition



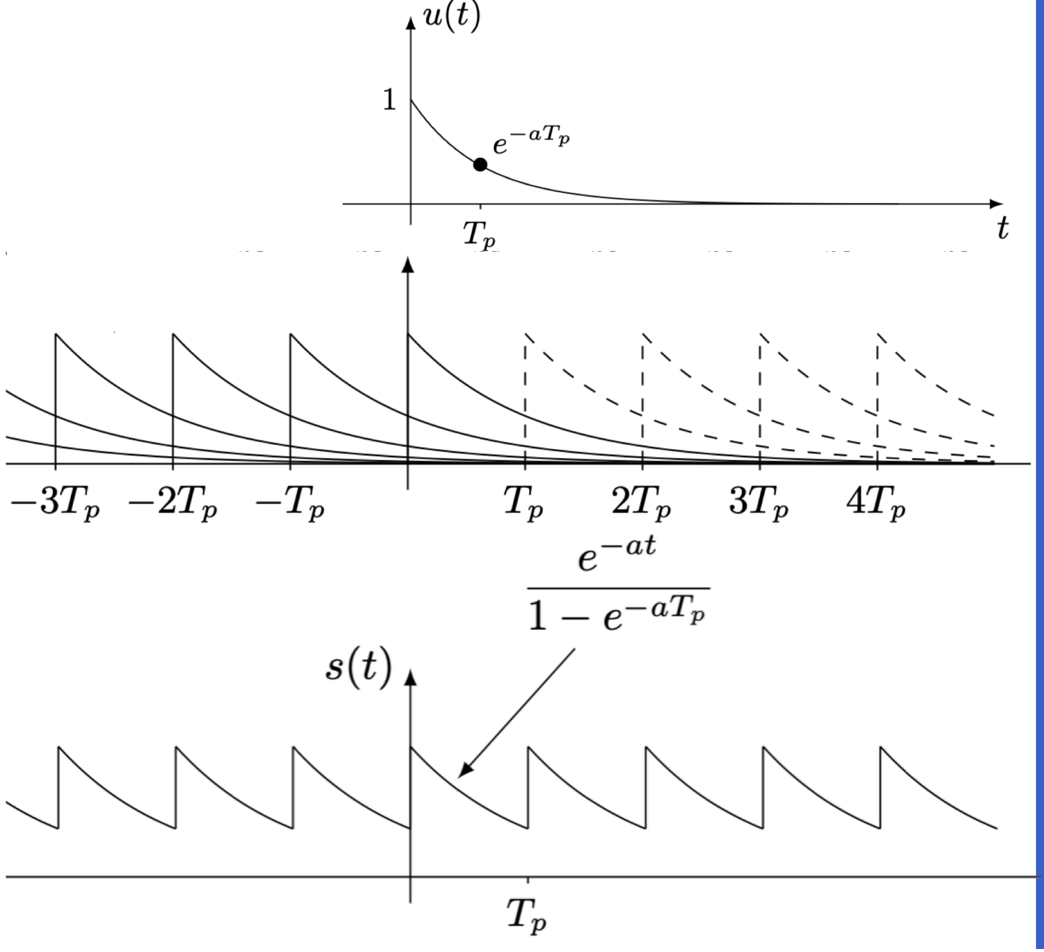
$$\boxed{s(t) = \mathrm{rep}_{T_p} u(t)}$$

$T_p = \frac{1}{2|f_0|}$ rectified cosinus period

$T_1 = \frac{1}{|f_0|}$ cosinus period

The presence of aliasing

In a periodic repetition



Properties

Of the periodic repetition

linearity

$$\mathrm{rep}_{T_p} a(t) + b(t) = \mathrm{rep}_{T_p} a(t) + \mathrm{rep}_{T_p} b(t)$$

time-reversal

$$\mathrm{rep}_{T_p} u(-t) = s(-t) \qquad \boxed{s(t) = \mathrm{rep}_{T_p} u(t)}$$

time-shift

$$\mathrm{rep}_{T_p} u(t - t_0) = s(t - t_0)$$

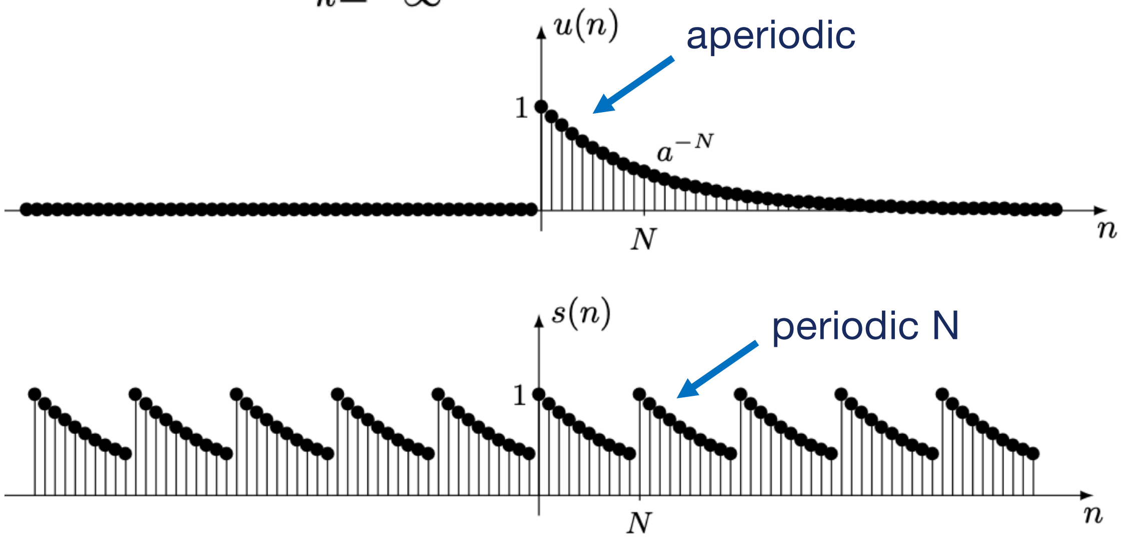
time-scale

$$\mathrm{rep}_{aT_p} u(t/a) = s(t/a)$$

The discrete-time case

Periodic repetition

$$s(n) = \sum_{k=-\infty}^{\infty} u(n - kN)$$



With the same properties of the continuous-time case

Exercises

On the periodic repetition

Calculating **a periodic repetition** is particularly important in the presence of **aliasing** as, in this case, a full understanding of the process might not be trivial.

We mainly concentrate on the **continuos-time** domain, as this is by far the most relevant for applications, but the same rationale readily applies to the discrete-time case as well.

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Lecture 05

Signals: periodic repetition, ideal impulses, generalized derivatives

Tomaso Erseghe

Università degli Studi di Padova

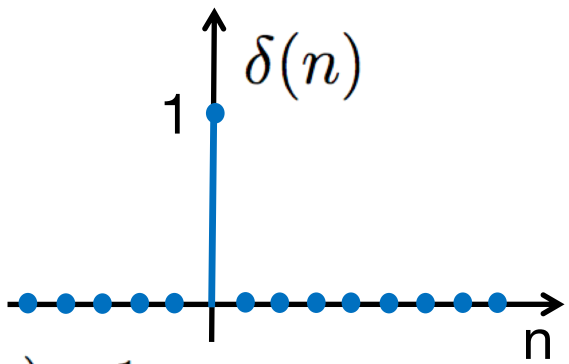
5.4 Ideal impulses

Also known as delta functions

- The Kronecker delta
- The Dirac delta
- Sifting properties
- Generalized derivatives

Kronecker delta

Signal impulse in discrete-time



extension

$$e(\delta) = \{0\}$$

area

$$A_\delta = \sum_{n=-\infty}^{\infty} \delta(n) = 1$$

even symmetry

$$\delta(n) = \delta(-n)$$

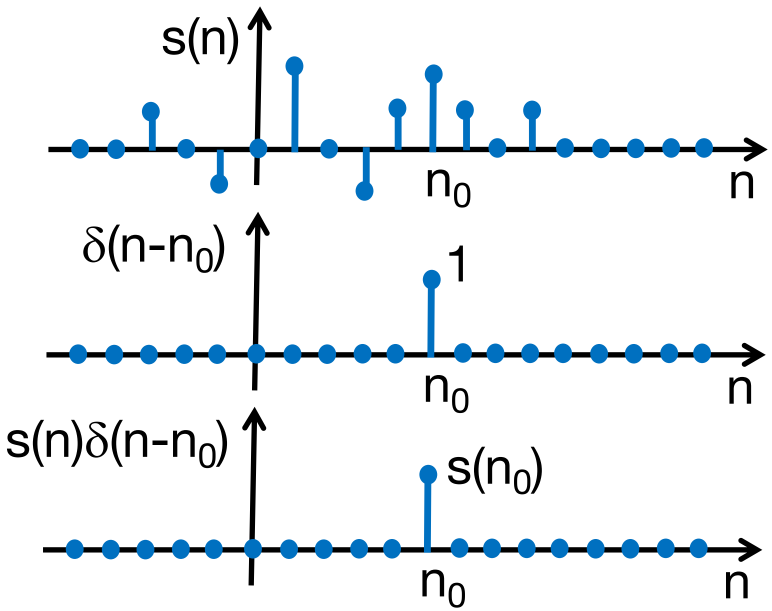
unit-step connection

$$1_0(n) = \sum_{m=-\infty}^{n} \delta(m)$$

cumulative sum

Sifting property #1

Of the Kronecker delta

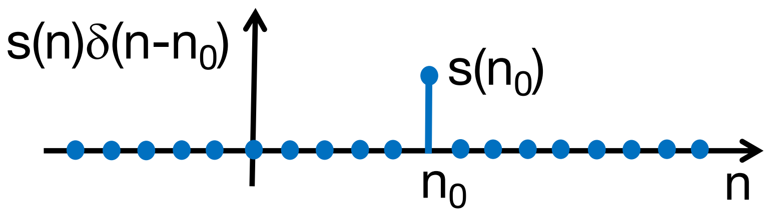


$$s(n)\delta(n-n_0) = s(n_0)\delta(n-n_0)$$

the Kronecker delta reveals the signal (where it is centred)

Sifting property #2

Of the Kronecker delta



$$\sum_{n=-\infty}^{\infty} s(n)\delta(n-n_0) = s(n_0)$$

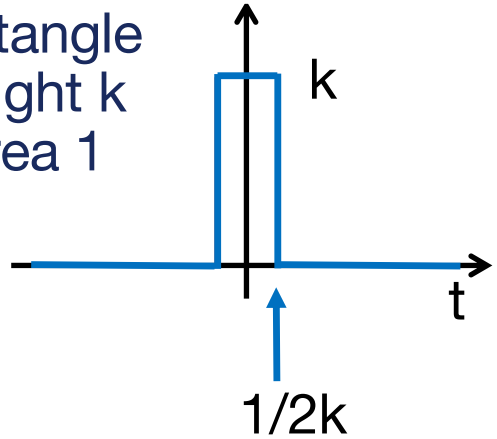
in a summation, the Kronecker delta reveals the signal value (the one where it is centred)

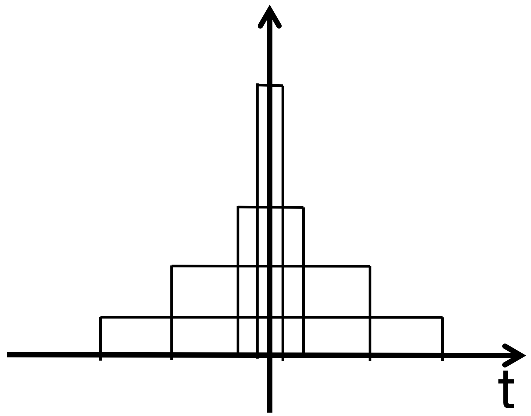
Dirac delta

Signal impulse in continuous time

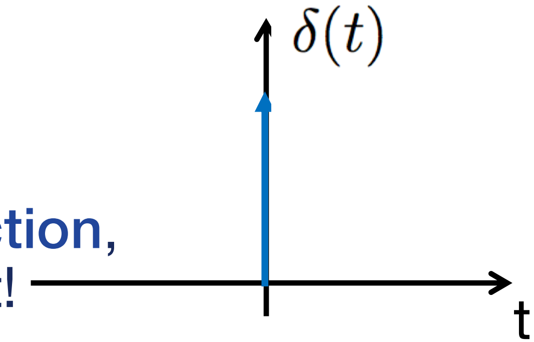
$$r_k(t) = k\,\mathrm{rect}(kt)$$

rectangle
height k
area 1





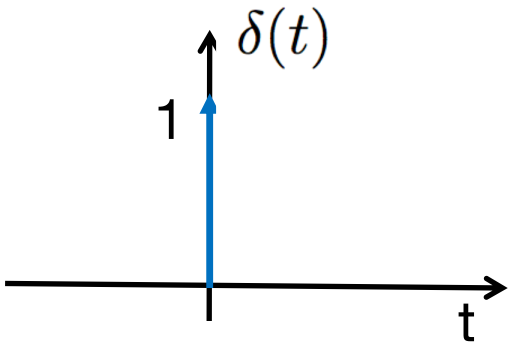
$$\delta(t) = \lim_{k\to\infty} r_k(t)$$



This is a **generalized function**,
defined through a limit!

Properties #1

Of the Dirac delta function



extension

$$e(\delta) = \{0\}$$

area

$$A_\delta = \int_{-\infty}^{\infty} \delta(t)dt = \lim_{k\to\infty} \underbrace{\int_{-\infty}^{\infty} r_k(t)dt}_{1} = 1$$

even symmetry

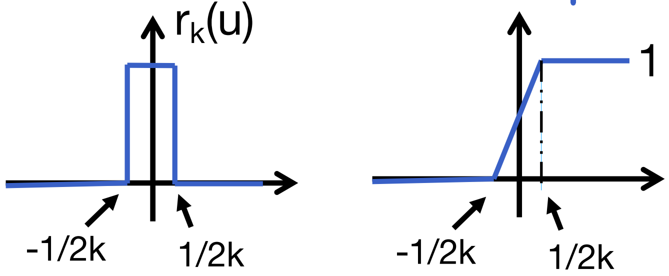
$$\delta(t) = \delta(-t)$$

Properties #2

Of the Dirac delta function

unit-step connection

$$1(t) = \int_{-\infty}^{t} \delta(u)\, du = \lim_{k \to \infty} \underbrace{\int_{-\infty}^{t} r_k(u)\, du}$$



generalized derivative

$$\delta(t) = \frac{d1(t)}{dt}$$

Sifting property #1

Of the Dirac delta

$$\int_{-\infty}^{\infty} s(t)\delta(t)dt = \lim_{k\to\infty}\int_{-\infty}^{\infty} s(t)r_k(t)dt$$

$$= \lim_{k\to\infty} \underbrace{\frac{1}{\frac{1}{k}}\int_{-\frac{1}{2k}}^{\frac{1}{2k}} s(t)dt}_{s(t_0),t_0\in[-\frac{1}{2k},\frac{1}{2k}]} = s(0)$$

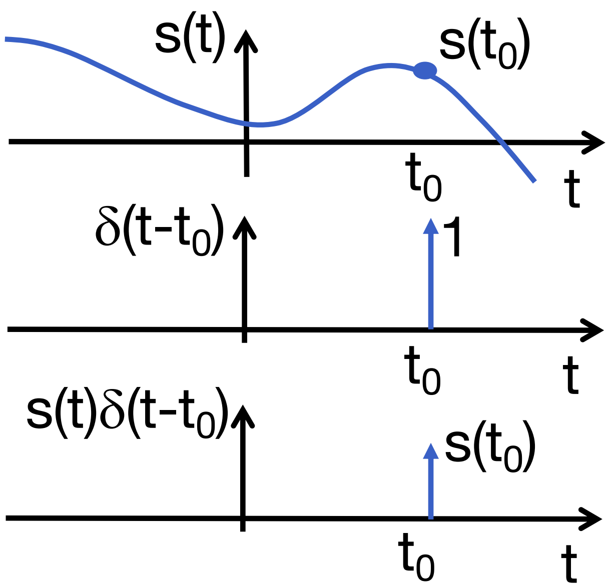
mean-value theorem

$$\int_{-\infty}^{\infty} s(t)\delta(t-t_0)dt = s(t_0)$$

in an integral, the Dirac delta reveals the signal value (the one where it is centred)

Sifting property #2

Of the Dirac delta

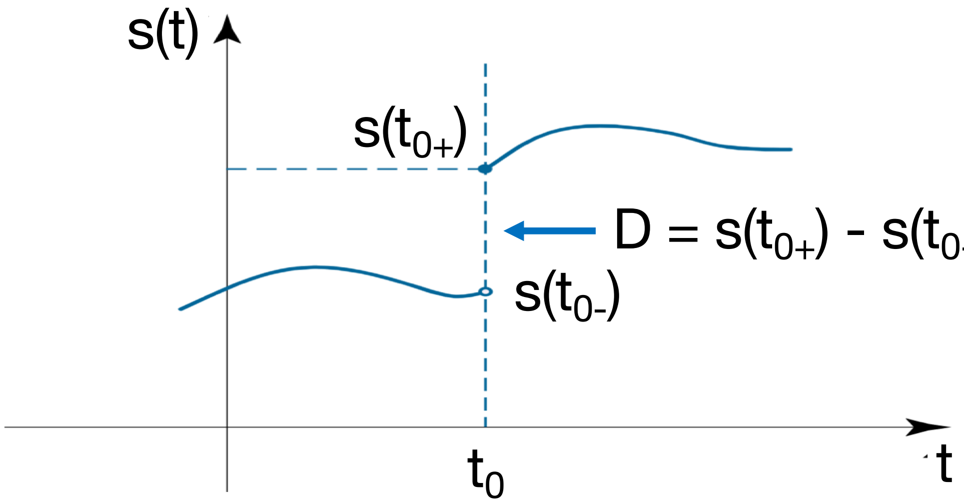


the Dirac delta reveals the signal value (where it is centred)

$$s(t)\delta(t-t_0) = s(t_0)\delta(t-t_0)$$

Generalized derivatives

By the Dirac delta

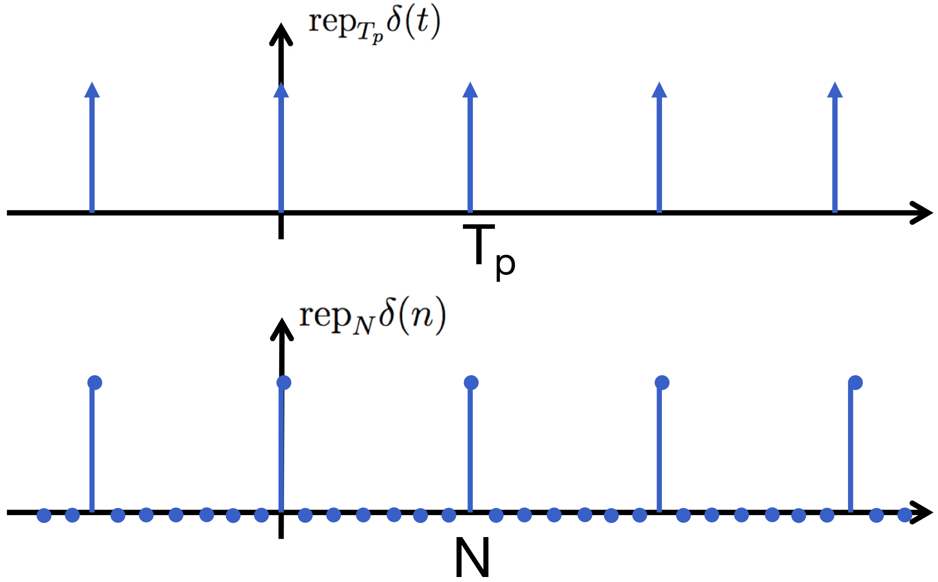


the Dirac delta expresses the derivative in a point of discontinuity

$$s'(t) = \underbrace{[s(t_0+) - s(t_0-)]}_{D}\delta(t - t_0)$$

Periodic counterparts

In continuous and discrete time



the continuous and discrete time comb signals

Exercises

On the ideal impulses

Getting acquainted with the sifting property of **ideal impulses** and with **generalized derivatives** is fundamental for correctly understanding the rest of this course.

We mostly concentrate on the **continuos-time** domain, as this is by far the most relevant for exercises, but the discrete-time case is equivalently important especially in some theoretical results that will lead to convolution.

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Lecture 06

Systems: invertibility, memory, BIBO stability, linearity, time-invariance, connection in series and parallel

Tomaso Erseghe

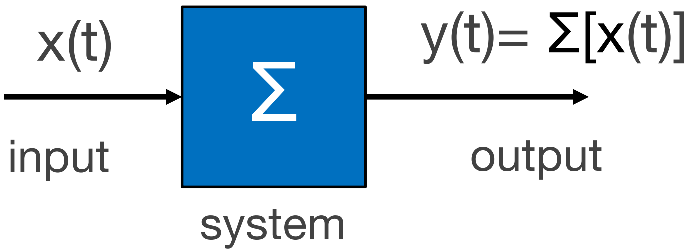
6.1 Systems properties #1

In continuous and discrete-time

- Invertibility
- Reality
- Memory
- BIBO stability

System model

In continuous or discrete-time



$$\Sigma : S_{in} \rightarrow S_{out}$$

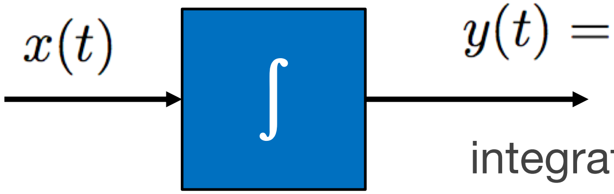
input
signal
space

output
signal
space

Signal space choice

continuous

$x(t)$

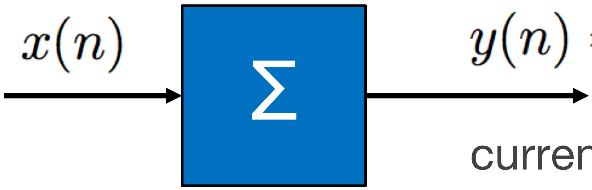


$$y(t) = \int_{-\infty}^{t} x(u)du$$

integrator

discrete

$x(n)$

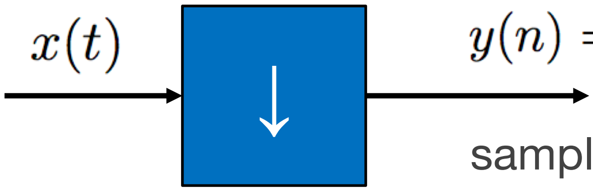


$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

current sum

hybrid

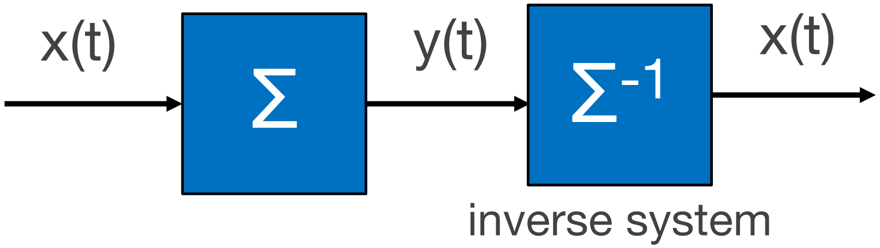
$x(t)$



$$y(n) = x(nT)$$

sampler

Invertibility



invertible

current sum $y(n) = \sum_{k=-\infty}^{n} x(k)$

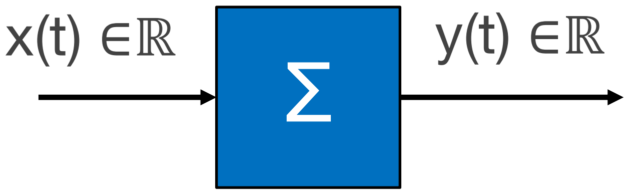
increment $x(n) = y(n) - y(n-1)$

non-invertible

absolute value $y(t) = |x(t)|$

unless S_{in} is real-valued and positive signals!!!

Reality



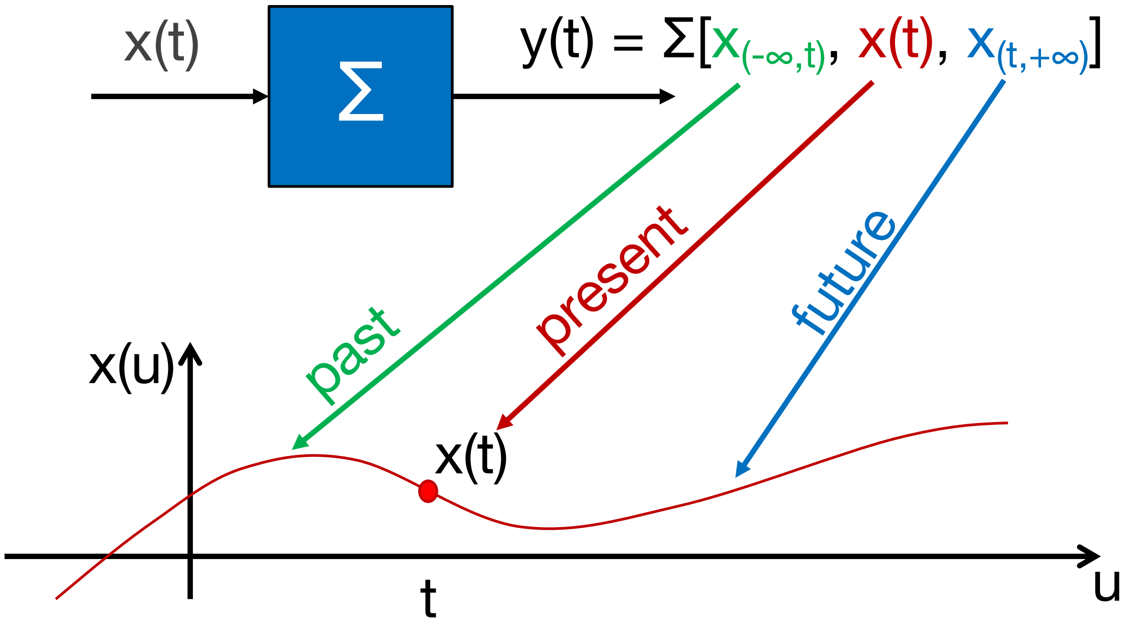
A system is **real** if for every real-valued input, the output is real-valued

current sum $y(n) = \sum_{k=-\infty}^{n} x(k)$

increment $x(n) = y(n) - y(n-1)$

running average $y(t) = \frac{1}{2T} \int_{t-T}^{t+T} x(u)du$

Memory of a system



static (or instantaneous) $y(t) = \Sigma[x(t)]$

absolute value $y(t) = |x(t)|$

Dynamic systems

causal y(t) = Σ[$x_{(-\infty,t)}$, x(t)]

current sum $y(n) = \sum_{k=-\infty}^{n} x(k)$

increment $x(n) = y(n) - y(n-1)$

integrator $y(t) = \int_{-\infty}^{t} x(u)du$

anti-causal y(t) = Σ[x(t), $x_{(t,+\infty)}$]

finite-memory y(t) = Σ[$x_{[t1,t2]}$]

running average $y(t) = \frac{1}{2T}\int_{t-T}^{t+T} x(u)du$

BIBO stability

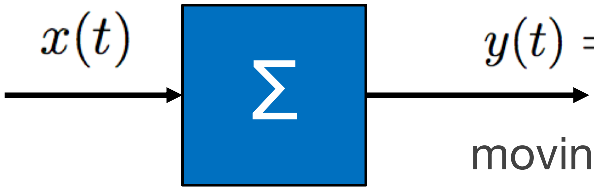
Bounded input – bounded output

A system is BIBO stable if
for every bounded input, $|x(t)|<L_x$,
the output is bounded, $|y(t)|<L_y$

or, equivalently,
$|x(n)|<L_x \rightarrow |y(n)|<L_y$

BIBO stability

Proof by construction



$$y(t) = \frac{1}{2T}\int_{t-T}^{t+T} x(u)du$$

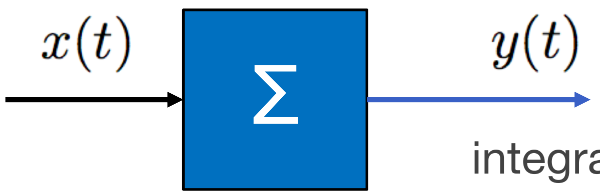
moving average

is BIBO stable

$$\boxed{\begin{aligned} |y(t)| &= \left|\frac{1}{2T}\int_{t-T}^{t+T} x(u)du\right| \\ &\leq \frac{1}{2T}\int_{t-T}^{t+T} |x(u)|du \\ &\leq \frac{1}{2T}\int_{t-T}^{t+T} L_x du = L_x \end{aligned}}$$

BIBO stability

Proof by counterexample



$$y(t) = \int_{-\infty}^{t} x(u)du$$

integration

is not BIBO stable

$$\boxed{\begin{aligned} y(t) &= \int_{-\infty}^{t} x(u)du \\ &= \int_{-\infty}^{t} 1(u)du = t \cdot 1(t) \end{aligned}}$$

Exercises

On systems

Understanding the **memory aspects and BIBO stability** in a continuous or discrete-time system is a fundamental process, than needs to be trained.

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Lecture 06

Systems: invertibility, memory, BIBO stability, linearity, time-invariance, connection in series and parallel

Tomaso Erseghe

Università degli Studi di Padova

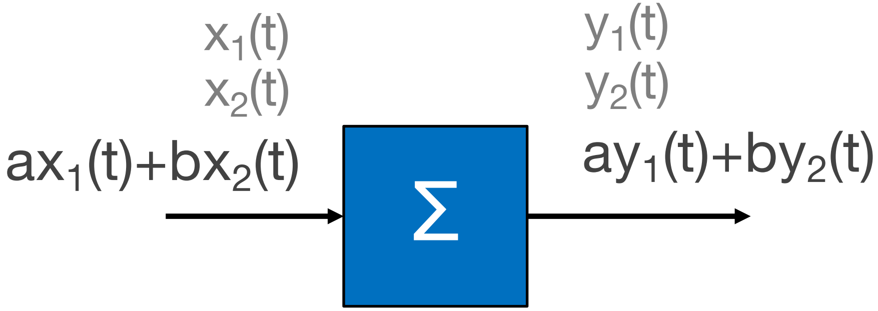
6.4 Systems properties #2

In continuous and discrete-time

- Linearity
- Time-invariance
- Eigenfunction and impulse response
- Series, parallel, and feedback systems

Linearity

Superposition principle



linearity = additivity + homogeneity

$$\boxed{\alpha x(t) \quad \overset{\Sigma}{\Longrightarrow} \quad \alpha y(t)}$$

$$\boxed{x_1(t) + x_2(t) \quad \overset{\Sigma}{\Longrightarrow} \quad y_1(t) + y_2(t)}$$

absolute value $y(t) = |x(t)|$

non homogeneous $|\alpha x(t)| \neq \alpha |x(t)|$

non additive $|x_1(t) + x_1(t)| \neq |x_1(t)| + |x_2(t)|$

Linear systems

Examples

time-shift $x(t-t_0)$

product $x(t)\,g(t)$ by any known waveform $g(t)$

product $A\,x(t)$ by a constant

integration $\int$, summation Σ, derivative d of $x(t)$

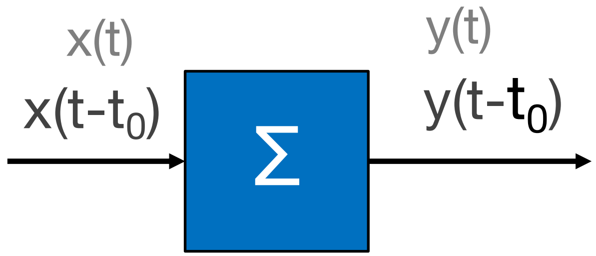
non-linear

sum $x(t) + A$ of a constant, or a waveform

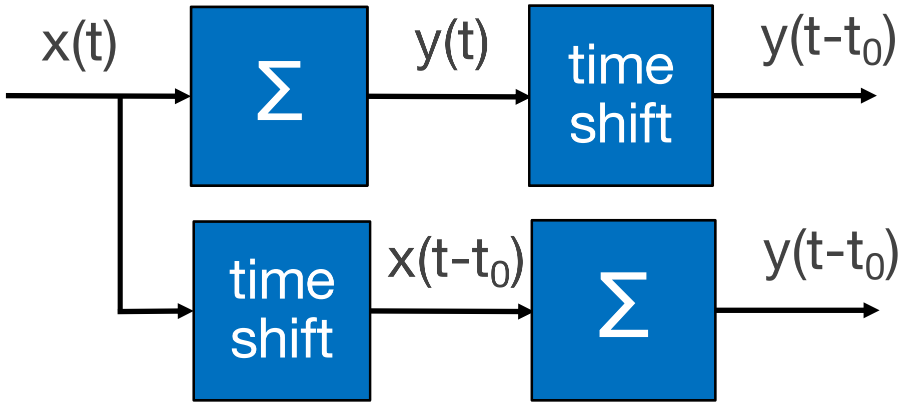
function $g(x(t))$ applied to $x(t)$

Time invariance

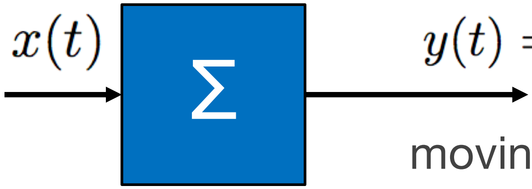
Repeatability principle



a commutativity property



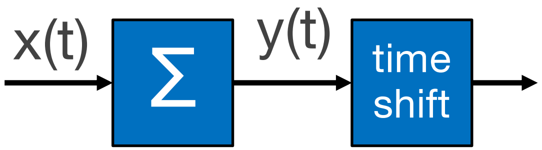
Verifying time invariance



$$y(t) = \frac{1}{2T}\int_{t-T}^{t+T} x(u)du$$

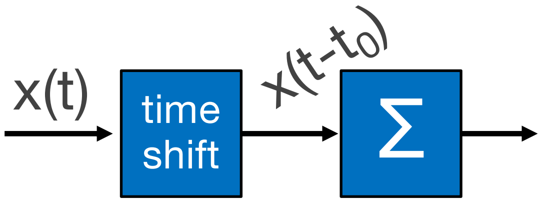
moving average

replace t → t-t_0



$$\frac{1}{2T}\int_{t-t_0-T}^{t-t_0+T} x(u)du$$

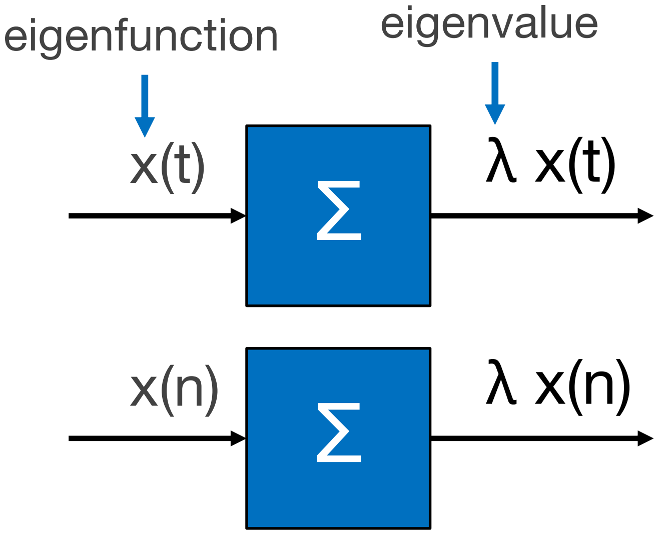
replace x(t) → x(t-t_0)



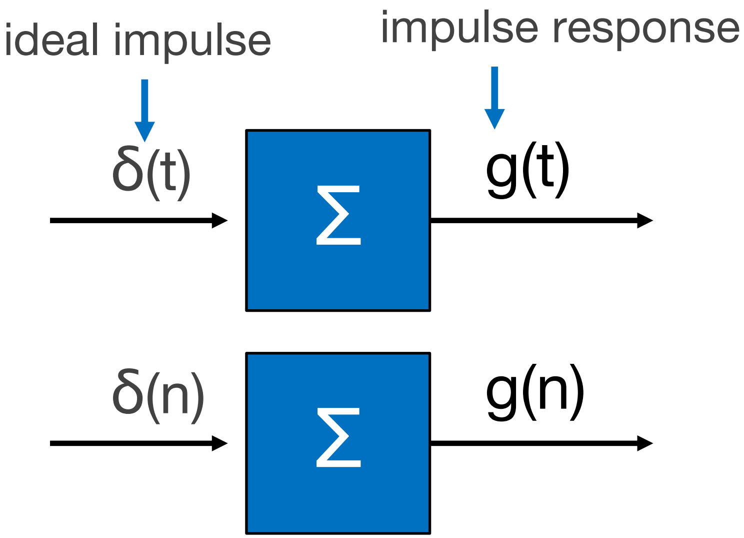
$$\frac{1}{2T}\int_{t-T}^{t+T} x(u-t_0)du$$

time-invariant: equal by a change of variable!

Eigenfunction

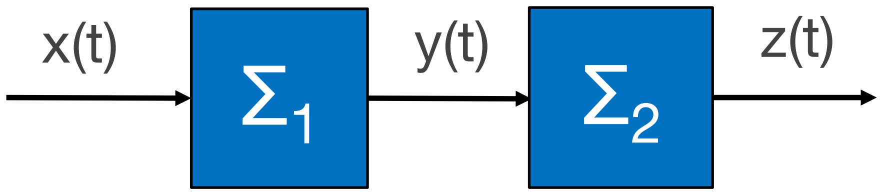


Impulse response

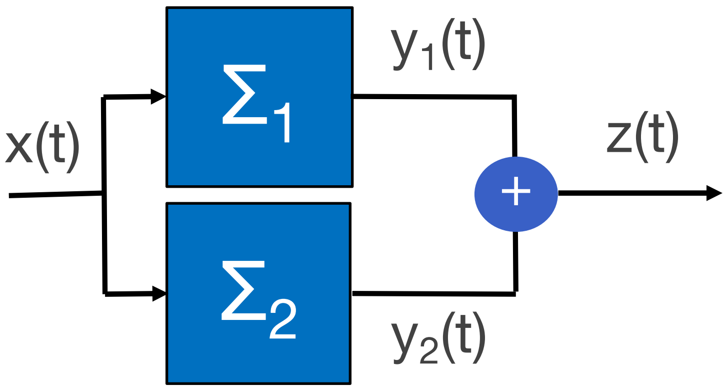


Series and parallel

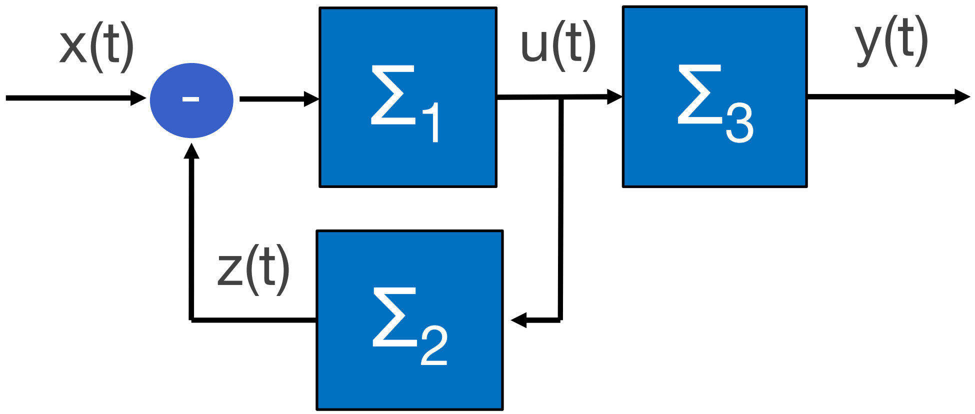
series



parallel



Feedback system



Exercises

On systems

Understanding the **linearity and time-invariant** properties of a continuous or discrete-time system is a fundamental process.

Take particular care in understanding how to correctly identify the presence of time-invariance, as this is one of the key aspects of this course

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Lecture 07

Systems: LTI systems as convolution systems, properties of convolution

Tomaso Erseghe

Università degli Studi di Padova

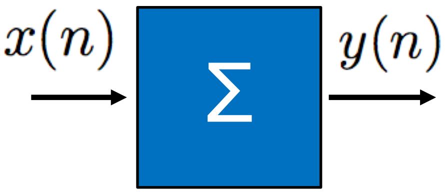
7.1 LTI systems as convolution systems

In continuous and discrete-time

- Convolution
- Graphical interpretation of convolution

LTI discrete-time system

As a convolution



LTI system

$$y(n) = \Sigma\Bigg[\underbrace{\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)}_{\text{x(n) by sifting property}}\Bigg]$$

by linearity

$$= \sum_{k=-\infty}^{\infty} x(k)\Sigma\left[\delta(n-k)\right]$$

by time invariance

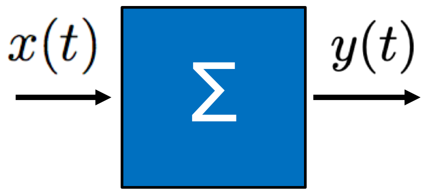
discrete-time convolution

$$= \sum_{k=-\infty}^{\infty} x(k)g(n-k)$$

impulse response

LTI continuous-time system

As a convolution



$$y(t) = \Sigma\left[\overbrace{\int_{-\infty}^{\infty} x(u)\delta(t-u)\,du}^{x(t)\text{ by sifting property}}\right]$$

by linearity

$$= \int_{-\infty}^{\infty} x(u)\Sigma[\delta(t-u)]\,du$$

continuous-time convolution

by time invariance

$$= \int_{-\infty}^{\infty} x(u)g(t-u)\,du$$

impulse response

LTI is convolution

LTI systems are also called filters

$$x(n) \longrightarrow \boxed{g(n)} \longrightarrow y(n) = \sum_{k=-\infty}^{\infty} x(k)g(n-k)$$

LTI system/filter

$$x(t) \longrightarrow \boxed{g(t)} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(u)g(t-u)\,du$$

LTI system/filter

an LTI system is uniquely identified
by its impulse response

Convolution z = x * y

As a standalone operator

$x(t)$, $y(t)$ → * →

$$z(t) = x * y(t) = \int_{-\infty}^{\infty} x(u)y(t-u)du$$

time-reversed
and time-shifted
$y_{-}(u-t)$ or $y_{-}(k-n)$

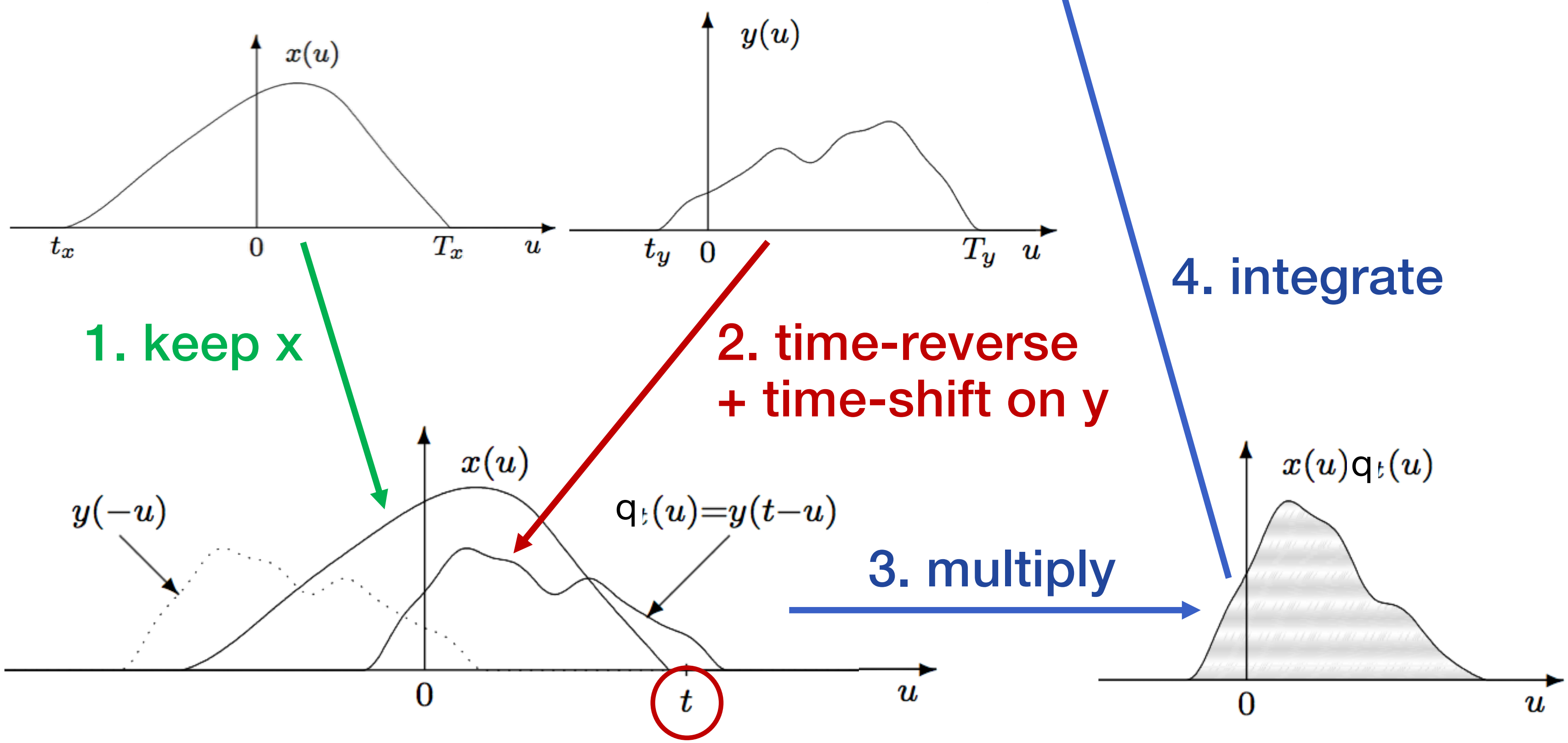
$x(n)$, $y(n)$ → * →

$$z(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$$

Graphical interpretation

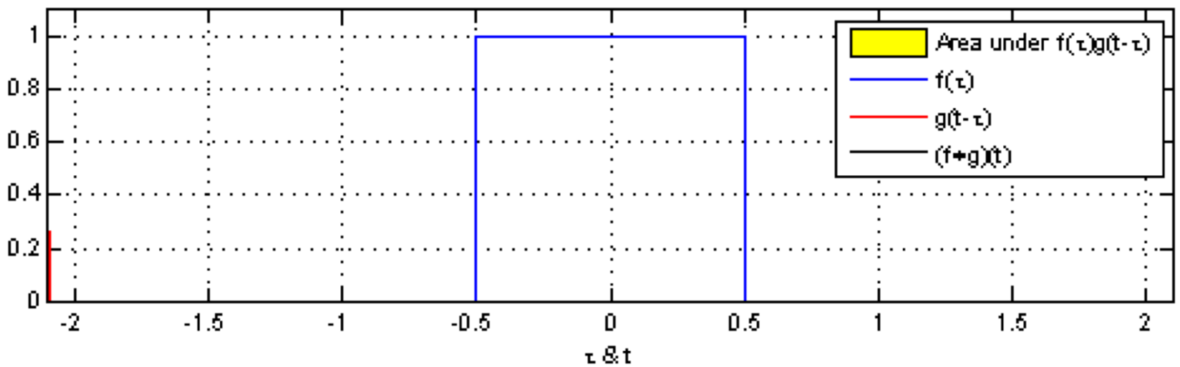
In continuous-time

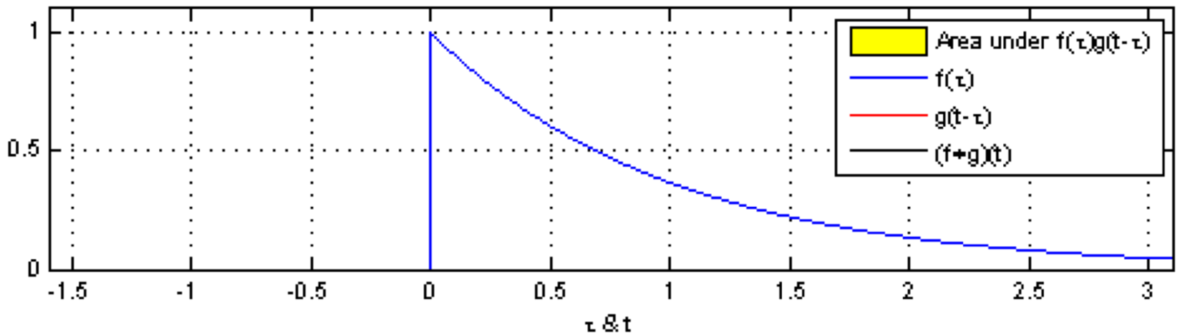
$$z(t) = \int_{-\infty}^{\infty} x(u)\, \underbrace{y(t-u)}_{y_{-}(u-t)}\, du$$



Visualizing convolution

From www.wikipedia.org





Exercises

On convolution

Understanding the **convolution** operator is a fundamental requirement of this course. You need to fully appreciate the graphical interpretation (time-reversal and time-shift) in both continuous and discrete-time.

Another fundamental point is to **recognize** when an integral or series expression is a **convolution** , as this could be useful in managing equations more easily.

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Lecture 07

Systems: LTI systems as convolution systems, properties of convolution

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Università degli Studi di Padova

7.4 Properties of convolution

In continuous and discrete-time

- Commutativity, associativity, linearity (distributivity)
- Identity element
- Extension
- Area
- Time-shifts

Properties of convolution

In both continuous and discrete-time

commutativity $x * y = y * x$

associativity $x * y * z = (y * x) * z = x * (y * z)$

$$x * y * z = z * x * y = y * z * x$$

linearity $(x + y) * z = x * z + y * z$

$x * (y + z) = x * y + x * z$

identity $x * \delta = x$

area $A_{x*y} = A_x \, A_y$

Commutativity

v = t-u

$$x * y(t) = \int_{-\infty}^{\infty} x(u)y(t-u)\,du$$
$$= \int_{-\infty}^{\infty} x(t-v)y(v)\,dv = y * x(t)$$

m = n-k

$$x * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$$
$$= \sum_{m=-\infty}^{\infty} x(n-m)y(m) = y * x(n)$$

choose the order you want!

Identity

even symmetric

$$x * \delta(t) = \int_{-\infty}^{\infty} x(u)\delta(t-u)\,du$$
$$= \int_{-\infty}^{\infty} x(u)\delta(u-t)\,du = x(t)$$

even symmetric

$$x * \delta(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$
$$= \sum_{k=-\infty}^{\infty} x(k)\delta(k-n) = x(n)$$

Area

swap integrals

$$A_{x*y} = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(u) y(t-u)\, du \right) dt$$

$$= \int_{-\infty}^{\infty} x(u) \underbrace{\left(\int_{-\infty}^{\infty} y(t-u)\, dt \right)}_{A_y} du = A_x A_y$$

swap sums

$$A_{x*y} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x(k) y(n-k) \right)$$

$$= \sum_{k=-\infty}^{\infty} x(k) \underbrace{\left(\sum_{n=-\infty}^{\infty} y(n-k) \right)}_{A_y} = A_x A_y$$

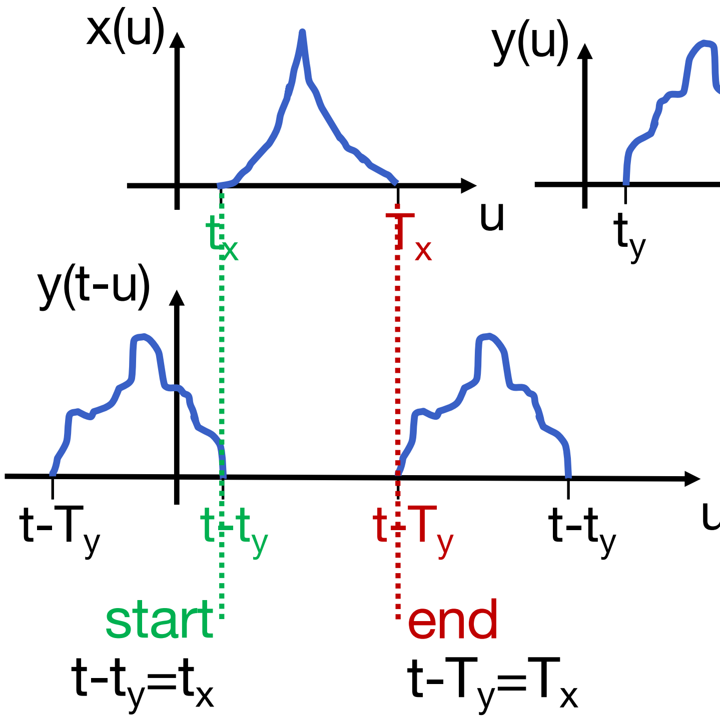
area is useful as a check!

Extension of the convolution

sums of initial/ending times

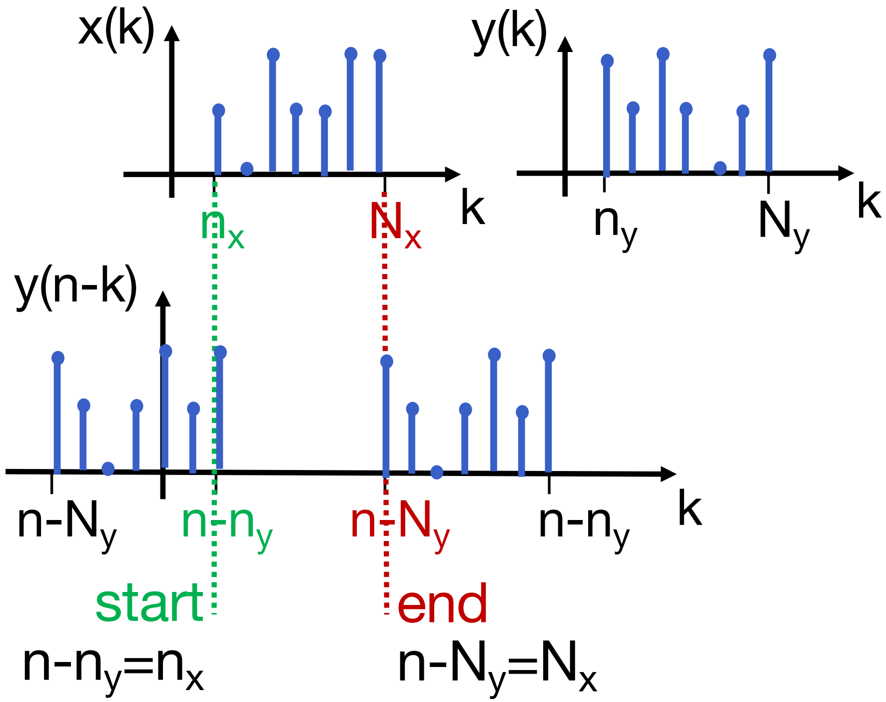
continuous-time

$e(x*y) = [t_x+t_y, T_x+T_y]$



discrete-time

$e(x*y) = [n_x+n_y, N_x+N_y]$



Time shift property

$$\delta_{t_0}(t) = \delta(t - t_0)$$

$$s(t - t_0) = s * \delta_{t_0}(t) = \int_{-\infty}^{\infty} \delta(u - t_0)s(t - u)\, du$$

$$s(n - n_0) = s * \delta_{n_0}(n) = \sum_{k=-\infty}^{\infty} \delta(k - n_0)s(n - k)$$

$$\delta_{n_0}(t) = \delta(n - n_0)$$

composition of time-shifts

$$\begin{aligned} x_{t_0} * y_{t_1}(t) &= x * \delta_{t_0} * y * \delta_{t_1}(t) \\ &= x * y * \delta_{t_0} * \delta_{t_1}(t) \\ &= x * y * \delta_{t_0+t_1}(t) \\ &= x * y(t - (t_0 + t_1)) \end{aligned}$$

Exercises

On systems

Understanding the **linearity and time-shift** properties of a continuous or discrete-time convolution is a fundamental process that will ease the calculation in many cases.

Use the extension and area properties for checking the correctness of your results.

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Lecture 08

Systems: Circular convolution, Filters

Tomaso Erseghe

Università degli Studi di Padova

8.1 Circular convolution

In continuous and discrete-time

- Definition
- Properties
- Interpretation through periodic repetition

One periodic signal

In the convolution

$$x * y(t) = \int_{-\infty}^{\infty} x(u)y(t-u)\,du$$

periodic T_p

aperiodic

periodic T_p
($t = t+T_p$)

$$x * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$$

periodic N

aperiodic

periodic N
(n = n+N)

Two periodic signals

In the convolution

$$\cancel{x * y}(t) = \int_{-\infty}^{\infty} \overbrace{x(u)y(t-u)}^{\text{periodic } T_p}\, du$$

periodic T_p periodic T_p

These diverge!

$$\cancel{x * y}(n) = \sum_{k=-\infty}^{\infty} \overbrace{x(k)y(n-k)}^{\text{periodic } N}$$

periodic N periodic N

Circular convolution

$$x *_{\text{cir}} y(t) = \int_{t_0}^{t_0+T_p} x(u)y(t-u)\,du$$

periodic T_p

$$x *_{\text{cir}} y(n) = \sum_{k=n_0}^{n_0+N-1} x(k)y(n-k)$$

periodic N

Properties

In both continuous and discrete-time

commutativity $x * y = y * x$

associativity $x * y * z = z * x * y$
$= y * z * x$

linearity $(x + y) * z = x * z + y * z$
$x * (y + z) = x * y + x * z$

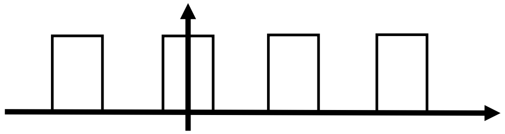
area $A_{x*y} = A_x \; A_y$

identity $\boxed{x * \mathrm{comb} = x}$

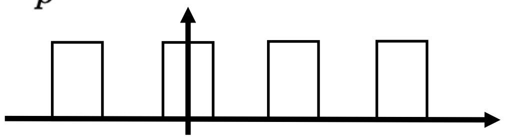
time shift $\boxed{x * \mathrm{comb}_{t_0/n_0} = x_{t_0/n_0}}$

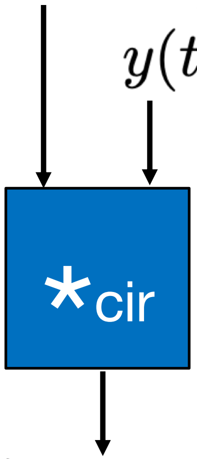
Interpreting circular conv.

$$x(t) = \mathrm{rep}_{T_p} x_0(t)$$



$$y(t) = \mathrm{rep}_{T_p} y_0(t)$$

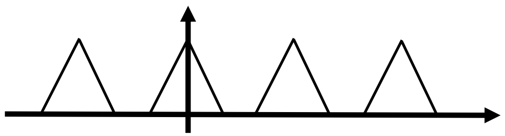




$$z(t) = x *_{\mathrm{cir}} y(t) = \mathrm{rep}_{T_p} x_0 * y_0(t)$$

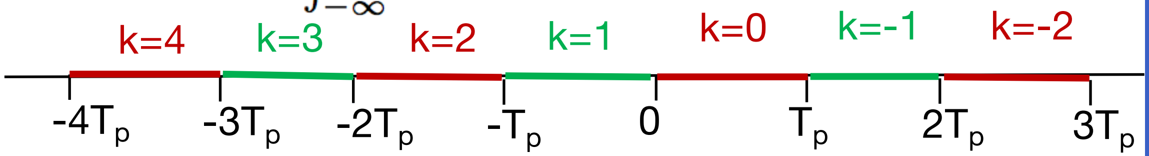
$$= x * y_0(t)$$

$$= y * x_0(t)$$



Proof – part 1

$$
\begin{aligned}
x * y(t) &= \int_0^{T_p} y(t-u)x(u)\,du \\
&= \int_0^{T_p} y(t-u) \sum_{k=-\infty}^{\infty} x_0(u-kT_p)\,du \\
&= \sum_{k=-\infty}^{\infty} \int_0^{T_p} x_0(u-kT_p)y(t-u)\,du \\
&= \underbrace{\sum_{k=-\infty}^{\infty} \int_{-kT_p}^{-(k-1)T_p}}_{\int_{-\infty}^{\infty}} x_0(v)\,\underbrace{y(t-v-kT_p)}_{y(t-v)}\,dv
\end{aligned}
$$



Proof – part 2

$$
\begin{aligned}
x * y(t) &= \int_{-\infty}^{\infty} x_0(v) y(t-v)\, dv \\
&= \int_{-\infty}^{\infty} x_0(v) \sum_{n=-\infty}^{\infty} y_0(t - v - nT_p)\, dv \\
&= \sum_{n=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} x_0(v) y_0(t - v - nT_p)\, dv}_{x_0 * y_0(t - nT_p)}
\end{aligned}
$$

with a perfectly equivalent proof
also in discrete-time

Exercises

On the circular convolution

Understanding the **circular convolution** operator is a fundamental requirement of this course. You need to fully appreciate the interpretation through periodic repetition to be able to evaluate it correctly.

Lecture 08

Systems: Circular convolution, Filters

Tomaso Erseghe

Università degli Studi di Padova

8.4 Filters

In continuous and discrete-time

- Recap
- Properties by the impulse response
- Eigenfunctions

Filters (LTI systems)

aperiodic g

$$x(n) \longrightarrow \boxed{g(n)} \longrightarrow y(n) = x * g(n)$$

aperiodic | aperiodic
periodic N | periodic N

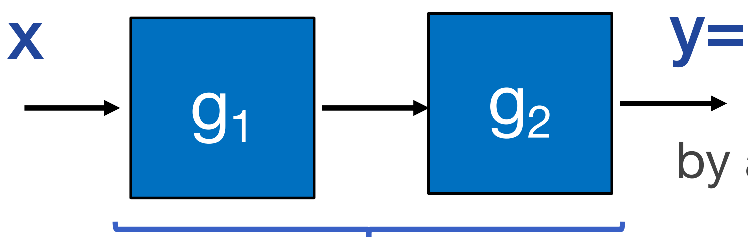
aperiodic g

$$x(t) \longrightarrow \boxed{g(t)} \longrightarrow y(t) = x * g(t)$$

aperiodic | aperiodic
periodic T_p | periodic T_p

an LTI system is uniquely identified by its impulse response

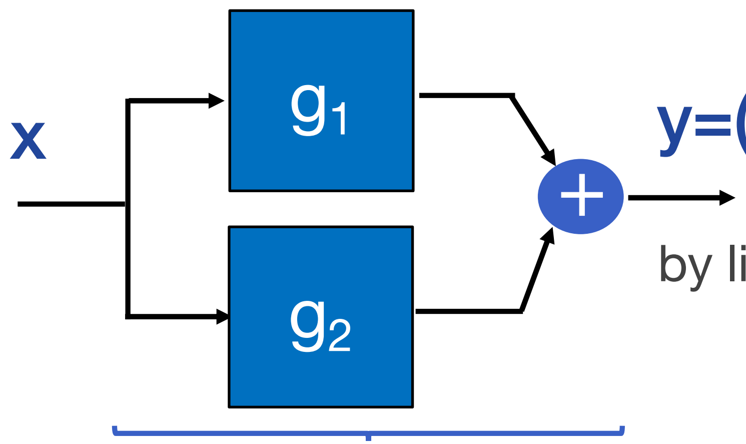
Series and parallel



$y=(g_1 * g_2) * x$

by associativity

$g = g_1 * g_2$



$y=(g_1 + g_2) * x$

by linearity

$g = g_1 + g_2$

Properties

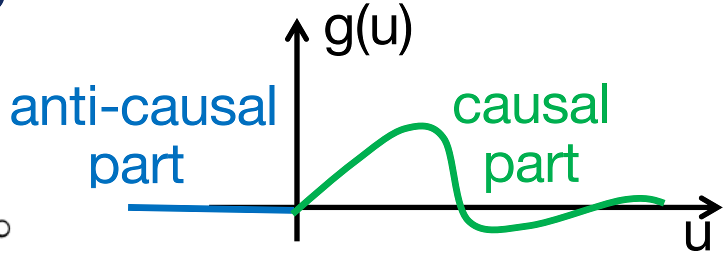
In both continuous and discrete-time

linearity	
time invariance	
reality	real-valued g
causality	causal g
BIBO stability	absolutely summable or integrable g

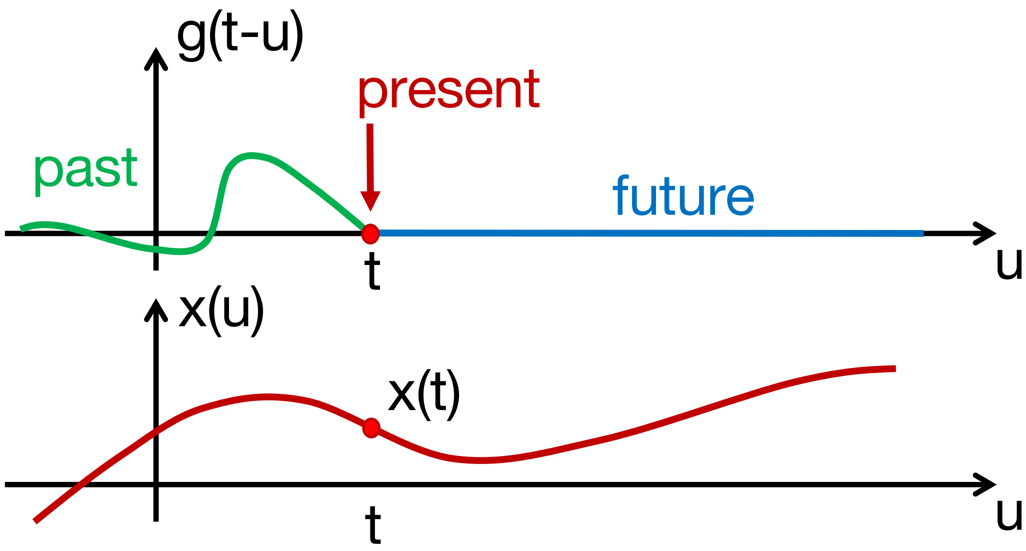
$$\boxed{L_g < \infty}$$

$$L_g = \begin{cases} \displaystyle\int_{-\infty}^{\infty} |g(t)|\, dt & \text{, continuous-time} \\ \displaystyle\sum_{n=-\infty}^{\infty} |g(n)| & \text{, discrete-time} \end{cases}$$

Causality



$$y(t) = \int_{-\infty}^{\infty} x(u) g(t-u)\, du$$



BIBO stability – part 1

If $L_g < \infty$ then filter is BIBO stable

$$
\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{\infty} x(u) g(t-u)\, du \right| \\
&\leq \int_{-\infty}^{\infty} |x(u)| \cdot |g(t-u)|\, du \\
&< \int_{-\infty}^{\infty} L_x \cdot |g(t-u)|\, du \\
&= \int_{-\infty}^{\infty} L_x \cdot |g(v)|\, dv \\
&= L_x \cdot L_g
\end{aligned}
$$

BIBO stability – part 2

If filter is BIBO stable then $L_g<\infty$

1. let $L_g=\infty$ by absurd

2. choose a bounded input $|x(t)|=1$

$$x(t) = e^{-j\varphi(-t)}\ , \qquad g(t) = |g(t)|\, e^{j\varphi(t)}$$

3. check that the output diverges at t=0

$$\begin{aligned} y(0) &= \int_{-\infty}^{\infty} x(u)g(0-u)\,du \\ &= \int_{-\infty}^{\infty} e^{-j\varphi(-t)} \cdot |g(-u)|\, e^{j\varphi(-u)}\,du \\ &= \int_{-\infty}^{\infty} |g(-u)|\,du = \int_{-\infty}^{\infty} |g(v)|\,dv = L_g \end{aligned}$$

Eigenfunctions

continuous-time

$$x(t) = \boxed{e^{j\omega_0 t}} \longrightarrow \boxed{g(t)} \longrightarrow y(t) = \lambda x(t)$$

eigenfunction

$$y(t) = \int_{-\infty}^{\infty} g(u)x(t-u)du$$

$$= \int_{-\infty}^{\infty} g(u)e^{j\omega_0(t-u)}du$$

eigenvalue

$$= \underbrace{e^{j\omega_0 t}}_{x(t)} \underbrace{\boxed{\int_{-\infty}^{\infty} g(u)e^{-j\omega_0 u}du}}_{\lambda = G(j\omega_0)}$$

$$\boxed{|G(j\omega_0)| \leq \int_{-\infty}^{\infty} |g(u)|du}$$

converge for BIBO stable filters

discrete-time

$$x(n) = \boxed{e^{j\varphi_0 n}} \longrightarrow \boxed{g(n)} \longrightarrow y(n) = \lambda x(n)$$

eigenfunction

$$y(n) = \sum_{k=-\infty}^{\infty} g(k)x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} g(k)e^{j\varphi_0(n-k)}$$

eigenvalue

$$= \underbrace{e^{j\varphi_0 n}}_{x(n)} \underbrace{\boxed{\sum_{k=-\infty}^{\infty} g(k)e^{-j\varphi_0 k}}}_{\lambda = G(e^{j\varphi_0})}$$

$$\boxed{|G(e^{j\varphi_0})| \leq \sum_{k=-\infty}^{\infty} |g(k)|}$$

Exercises

On filters

Identifying the **impulse response** and deriving the main properties from its expressions is a fundamental process that helps in investigating LTI systems.

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Lecture 9

Introduction to MatLab

Tomaso Erseghe

9.1 MatLab

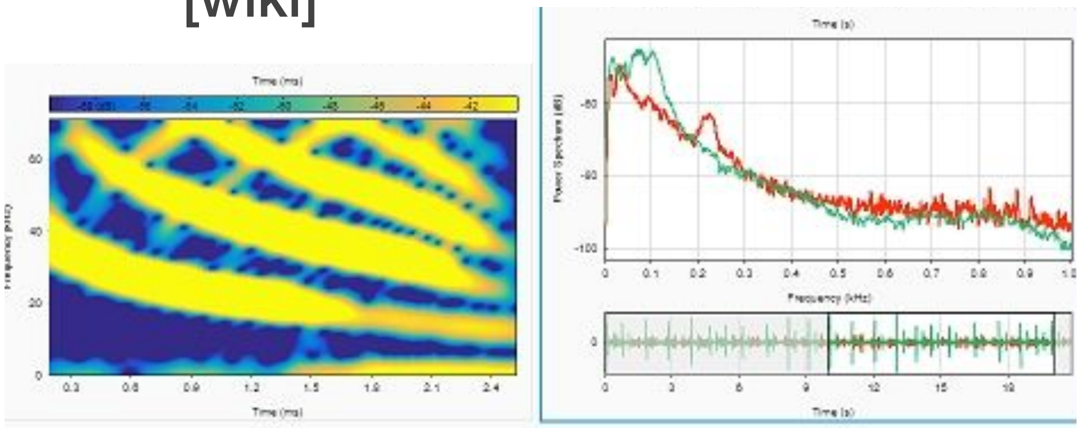
An introduction

- Install MatLab
- The four MatLab windows
- Variables and functions in MatLab
- Saving and loading data

MatLab

MATrix LABoratory by MathWorks

MATLAB "is a numerical computer environment which allows matrix manipulations, plotting of functions and data, implementation of algorithms" [wiki]



MatLab free unipd license

https://asit.unipd.it/servizi/contratti-software-licenze/matlab

Home > Servizi > Contratti Software e Licenze > MATLAB

Contratti Software e Licenze

- Aule Informatiche
- Account e Accessi +
- App IO +
- Applicazioni via Terminal Server +
- Assistenza tecnica ASIT +
- Business analysis
- Consulenza +
- Contratti Software e Licenze

MATLAB

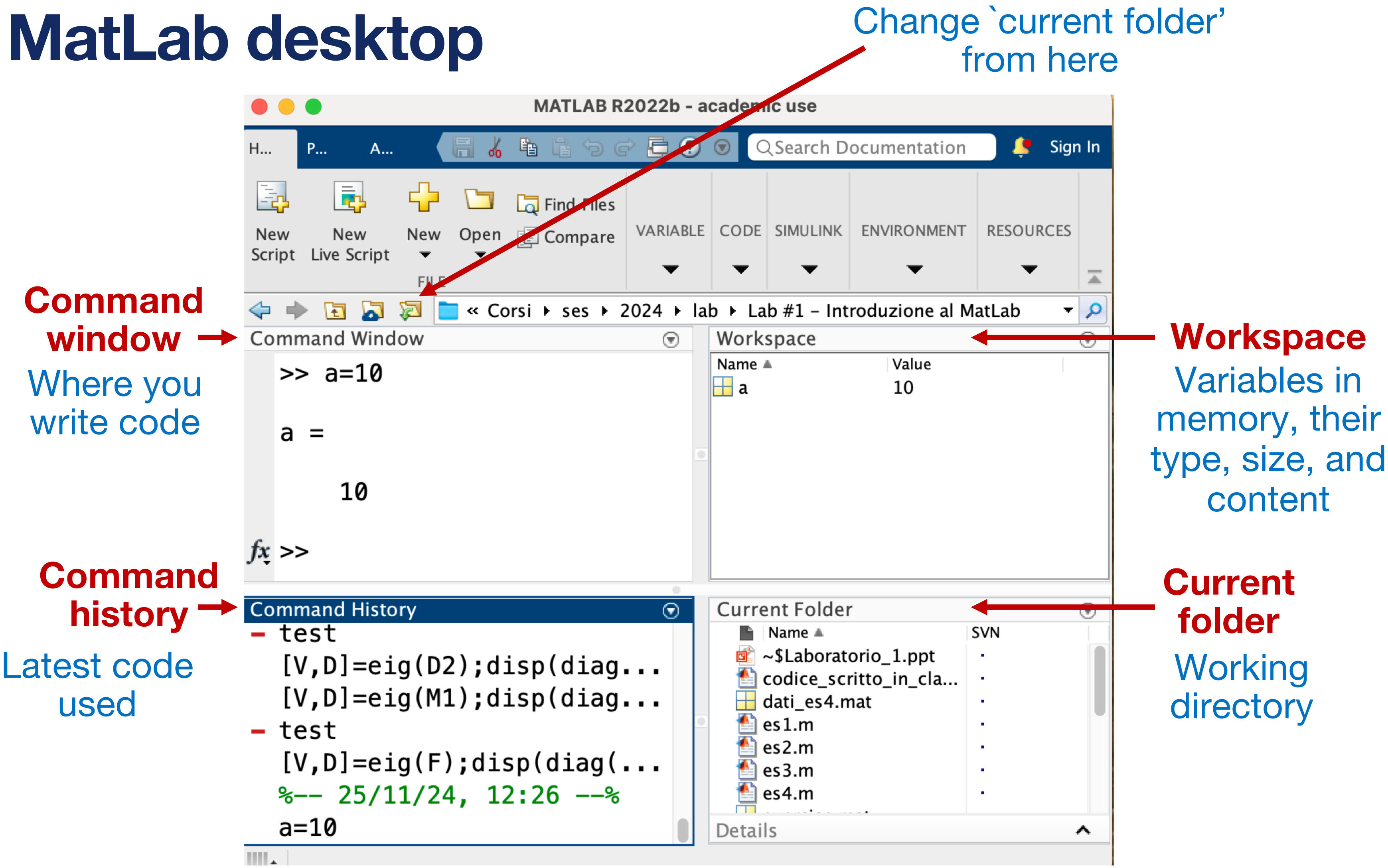
Software Download - Students, Faculty, and Staff

Students, faculty, and staff may download an individual stand-alone copy of the software from the *MATLAB Portal:* https://www.mathworks.com/academia/tah-portal/universita-degli-studi-di-padova-31194939.html

Helpful links:

- How to Install and activate MATLAB on my personal computer?
- Need installation help? Contact MathWorks Installation Support
- How I update the license after the Campus-Wide License has been renewed?

MatLab desktop



Vectors and matrices

Row vector
Separated by commas

Column vector
Separated by semicolons

Matrix
Rows separated by semicolons, elements in a row separated by commas

Command Window

```
>> v = [1, 2, 3]

v =

     1     2     3

>> w = [4; 5; 6]

w =

     4
     5
     6

>> M = [1,2,3;4,5,6;7,8,9]

M =

     1     2     3
     4     5     6
     7     8     9

fx >>
```

Simple operations

Transpose

Sum
Entry by entry

Matrix product
Needs matching dimensions to work correctly

Entrywise product
uses .*

Command Window

```
>> w'

ans =

     4     5     6

>> v+w'

ans =

     5     7     9

>> M*w

ans =

    32
    77
   122

>> v.*(w')

ans =

     4    10    18
```

Complex numbers

Imaginary unit →

Vector →
Complex valued

Hermitian →
Applies complex conjugation and transpose

Transpose →
uses .’

```
Command Window

>> 1i

ans =

   0.0000 + 1.0000i

>> v = [1+1i*2, 3-1i*4]

v =

   1.0000 + 2.0000i   3.0000 - 4.0000i

>> v'

ans =

   1.0000 - 2.0000i
   3.0000 + 4.0000i

>> v.'

ans =

   1.0000 + 2.0000i
   3.0000 - 4.0000i
```

Useful tips

operator	description
.*	elementwise multiplication
*	matrix multiplication
+	matrix sum
./	elementwise division
.^	elementwise power
^	matrix (squared) power
‘	Hermitian transform
.’	transpose
zeros(N)	N x N zero matrix
zeros(N,M)	N x M zero matrix
eye(N)	N x N identity matrix
rand(N,M)	N x M matrix with random entries

Save and load

who List of active variables (ans is the output)

save command

clears memory

load command

```
Command Window
>> who

Your variables are:

M    a    ans  v    w

>> save('tmp.mat','a','v','w','M')
>> clear all
>> who
>> load('tmp.mat')
>> who

Your variables are:

M  a  v  w

>>
```

Work
Name
a
M
v
w

For and while loops

```
N = 10;
for I = 1:N
    for J = 1:N
        A(I,J) = 1/(I+J-1);
    end
end
```

semicolon

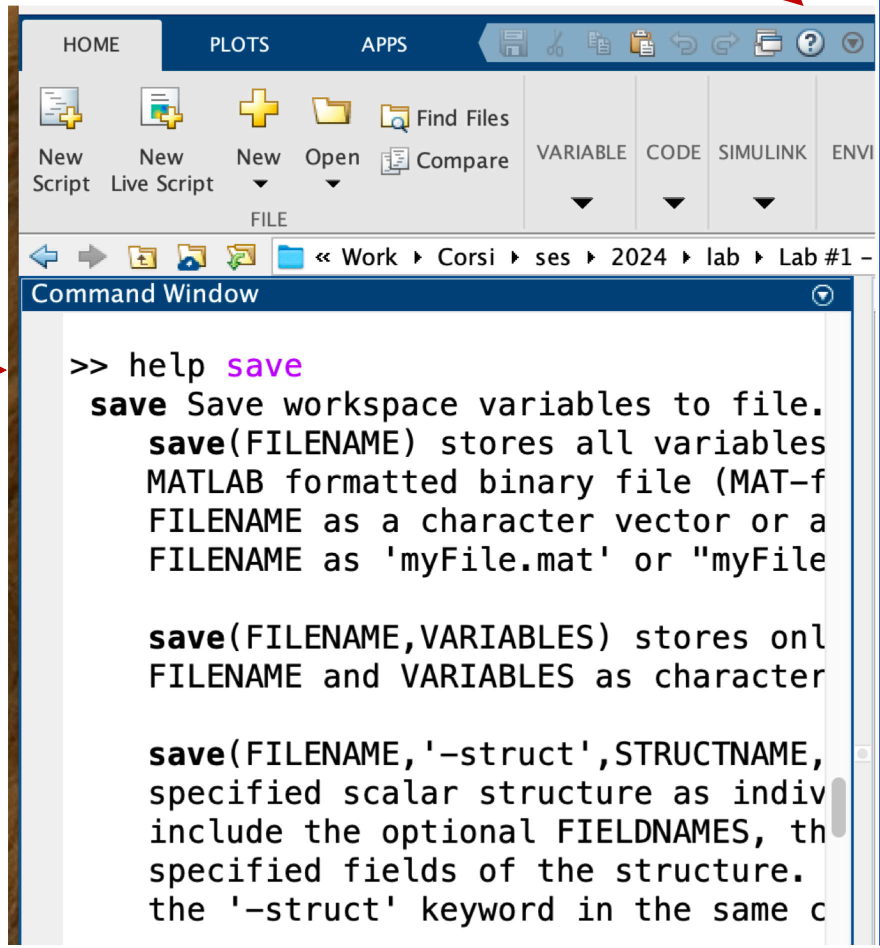
To prevent output to be printed

```
I = 1; N = 10;
while I <= N
    J = 1;
    while J <= N
        A(I,J) = 1/(I+J-
        1);
        J = J+1;
    end
    I = I+1;
end
```

MatLab help

Help button

Help command



Google Help

Exercises

On MatLab intro

We could spend an entire day introducing MatLab, but there is plenty of available resources out there!

If you are linked to your university's Campus-Wide License, you are automatically enrolled and can get started at https://matlabacademy.mathworks.com/

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Lecture 9

Introduction to MatLab

Tomaso Erseghe

Università degli Studi di Padova

9.2 Plots in MatLab

An introduction

- Defining time and signal samples
- Drawing a figure
- Controlling the figure layout
- Multiple plots

Plotting a signal

A starting example

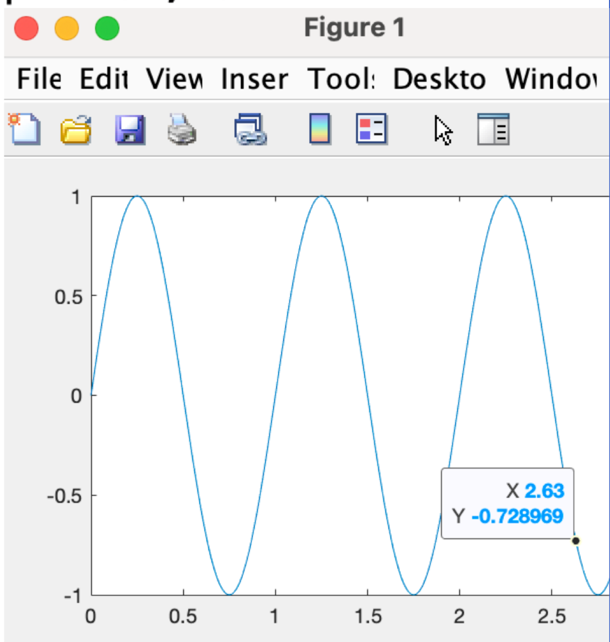
Define time samples
from 0 to 3 spaced by .01

Define signal samples
using the MatLab function sin, here pi is 3.14…

Plot

Command Window

```
>> t = 0:0.01:3;
>> s = sin(2*pi*t);
>> plot(t,s)
fx >>
```



Plotting a signal

Using a script

Open a script

must be a .m file, with no spaces or special characters in the name

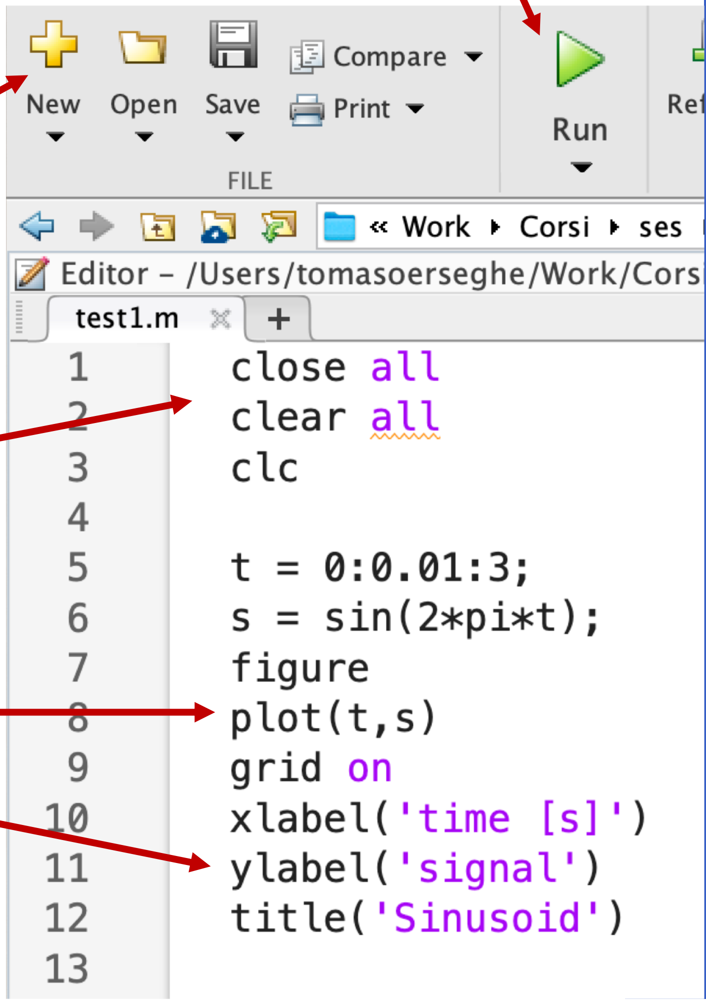
Clear memory

Close figures, clear variables and command window

Plot

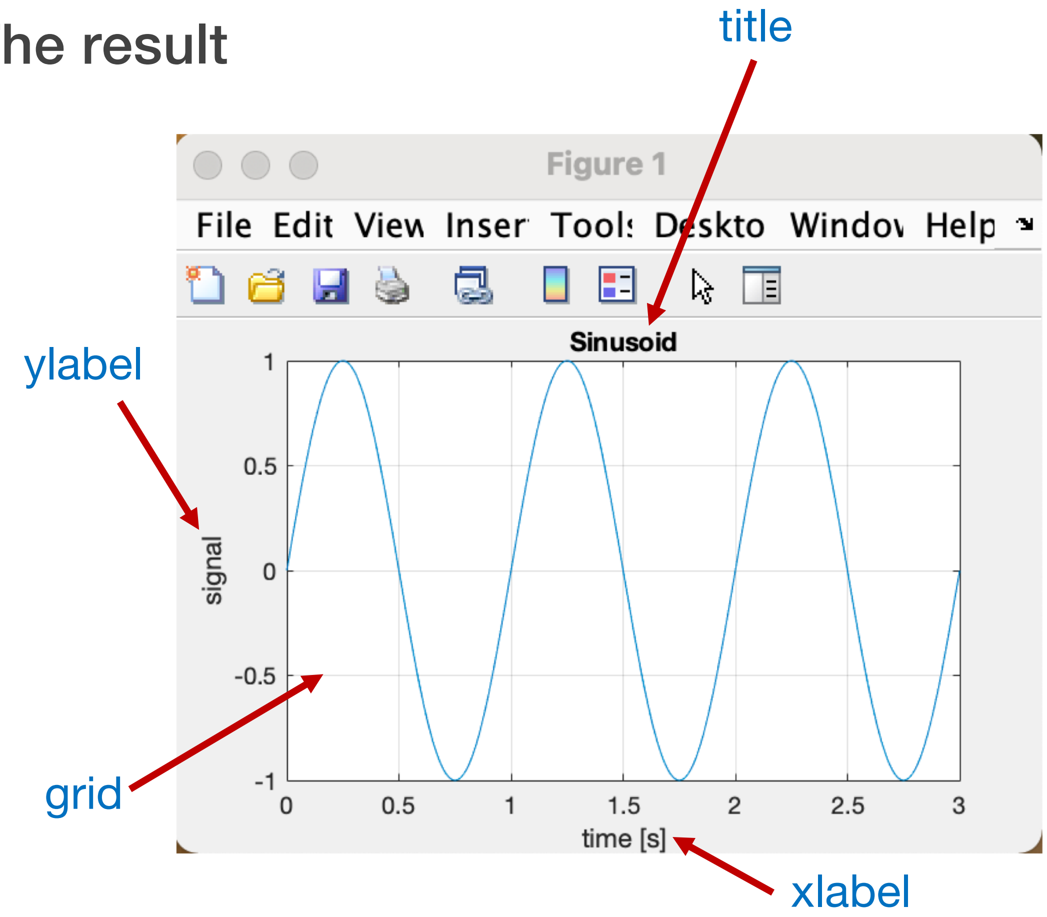
Add grid, labels, and title

Run script



Plotting a signal

The result



Multiple plots

Same window

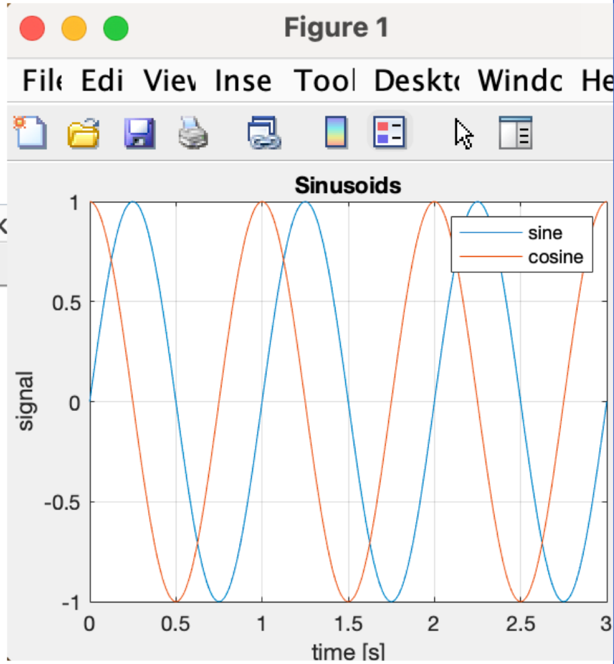
Editor – /Users/tomasoerseghe/Work

test1.m

```
close all
clear all
clc

t = 0:0.01:3;
s = sin(2*pi*t);
c = cos(2*pi*t);
figure
plot(t,s,t,c)
grid on
xlabel('time [s]')
ylabel('signal')
title('Sinusoids')
legend('sine', 'cosine')

```



Put one after another

Add legend

Multiple plots

Same window – an alternative

Editor – /Users/tomasoerseghe/Work/Corsi/s...

test1.m

```
close all
clear all
clc

t = 0:0.01:3;
s = sin(2*pi*t);
c = cos(2*pi*t);
figure
plot(t,s)
hold on
plot(t,c)
hold off
grid on
xlabel('time [s]')
ylabel('signal')
title('Sinusoids')
legend('sine', 'cosine')
```

Plot the first

Hold the result, otherwise only the last plot command will appear

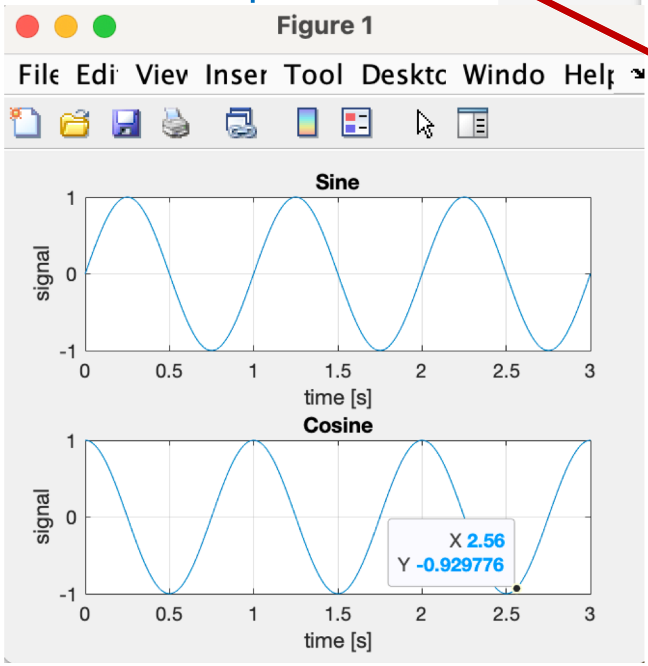
Plot the second

Multiple plots

Multiple windows

subplot(rows,cols,pos)

On a grid of size rows x cols positions the plot at pos



Editor – /Users/tomasoerseghe/Work/Cors

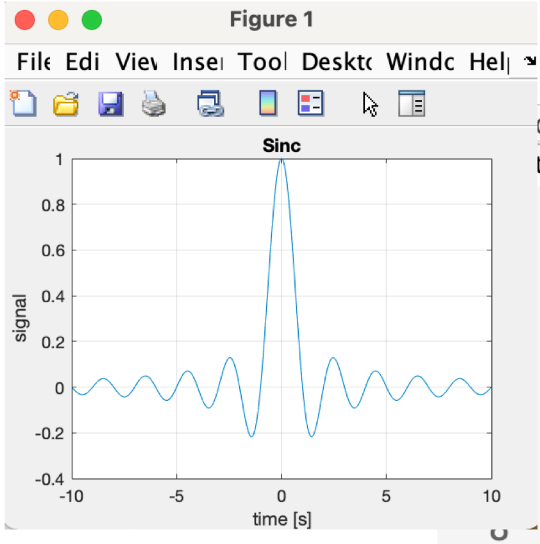
test1.m

```
close all
clear all
clc

t = 0:0.01:3;
s = sin(2*pi*t);
c = cos(2*pi*t);
figure
subplot(2,1,1)
plot(t,s)
grid on
xlabel('time [s]')
ylabel('signal')
title('Sine')
subplot(2,1,2)
plot(t,c)
grid on
xlabel('time [s]')
ylabel('signal')
title('Cosine')
```

Defining a function

entrywise division is needed here!



or – /Users/tomasoerseghe/Work/Corsi/s...

t1.m

```
close all
clear all
clc

t = -10:0.01:10;
s = sinc(t);
figure
plot(t,s)
grid on
xlabel('time [s]')
ylabel('signal')
title('Sinc')

function s = sinc(t)
s = sin(pi*t)./(pi*t);
s(t==0) = 1;
end
```

Function definition must appear at the end of the file (alternatively you put it in file sinc.m)

Correct for t=0 to avoid nan

Choosing the sampling rate

Your sampling spacing must be sufficiently small to capture the function shape!

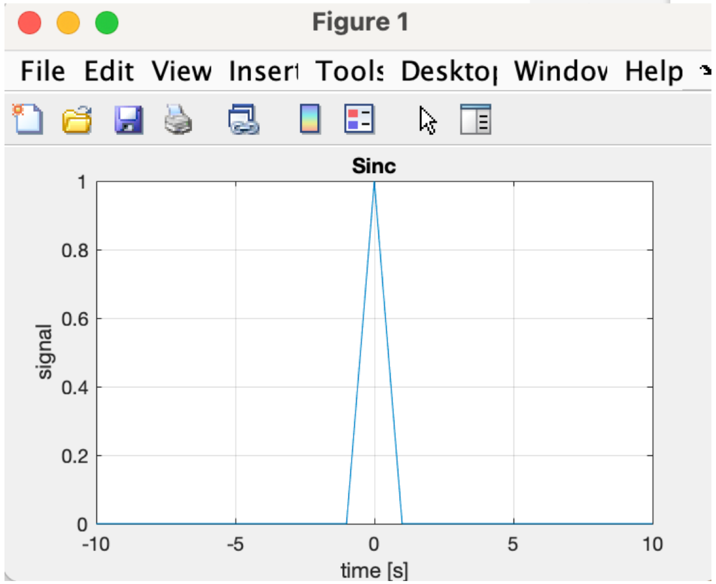
Editor – /Users/tomasoerseghe/Work/Corsi/s...

test1.m

```
close all
clear all
clc

t = -10:1:10;
s = sinc(t);
figure
plot(t,s)
grid on
xlabel('time [s]')
ylabel('signal')
title('Sinc')

function s = sinc(t)
s = sin(pi*t)./(pi*t);
s(t==0) = 1;
end
```



Discrete-time signals

use stem
instead of plot

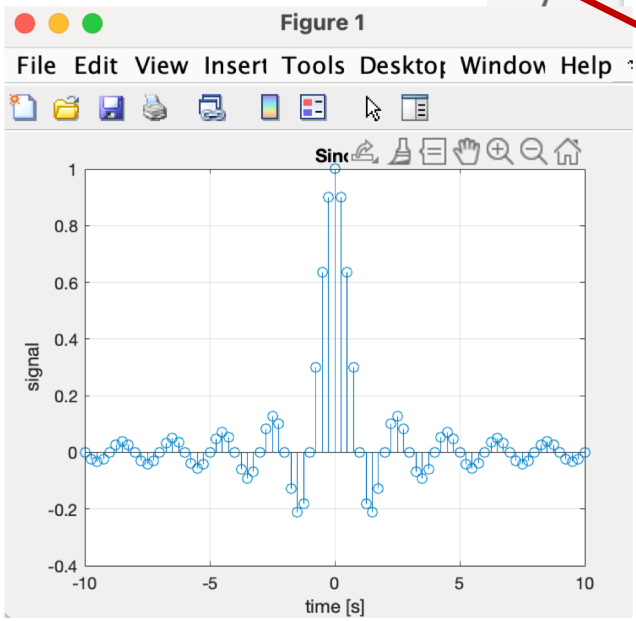
Editor – /Users/tomasoerseghe/Work/Corsi/s...

test1.m

```
close all
clear all
clc

t = -10:.25:10;
s = sinc(t);
figure
stem(t,s)
grid on
xlabel('time [s]')
ylabel('signal')
title('Sinc')

function s = sinc(t)
s = sin(pi*t)./(pi*t);
s(t==0) = 1;
end
```



Exercises

On MatLab plots

Signal plots is a fundamental step that allows you to correctly represent your data.

Practice yourself with plots, multiple plots, and the representation of **complex signals** through their real and imaginary parts, or through absolute value and phase.

Pay particular attention in correctly defining the **sampling rate/spacing** when representing continuous-time signals

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Lecture 10

Fourier series: definition and properties

Tomaso Erseghe

Università degli Studi di Padova

10.1 The Fourier series

for continuous-time periodic signals

- Orthogonal projections in a signal space
- The Fourier series
- Convergence properties
- Useful Fourier pairs

Unveiling filters

A general idea in continuous-time

eigenfunction

$$x(t) = e^{j\omega_0 t} \longrightarrow \boxed{g(t)} \longrightarrow y(t) = \lambda x(t)$$

eigenvalue

$$\lambda = G(j\omega_0) = \int_{-\infty}^{\infty} g(u) e^{-j\omega_0 u} du$$

By linearity we have

$$x(t) = \sum_k A_k \cdot e^{j\omega_k t}$$

$$\Downarrow$$

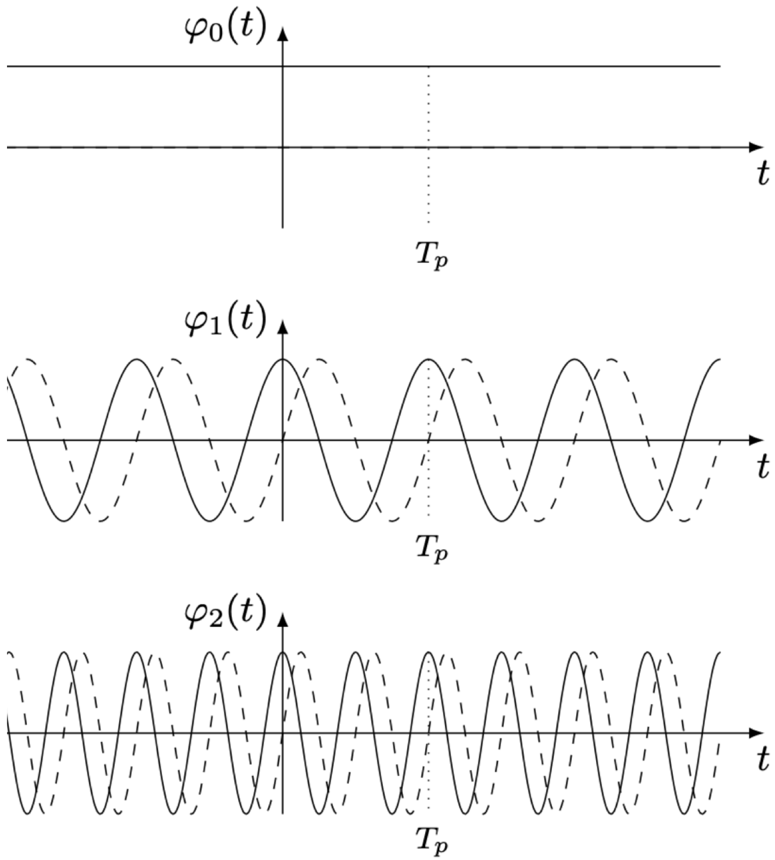
$$y(t) = \sum_k [A_k G(j\omega_k)] \cdot e^{j\omega_k t}$$

We focus on **periodic** signals and linear complex exponentials that are periodic T_p

$\boldsymbol{\omega_k = k\, 2\pi / T_p}$

Complex exponentials

Periodic of period T_p



$$\boxed{\varphi_k(t) = e^{jk\omega_0 t}}$$

$$\boxed{\omega_0 = \frac{2\pi}{T_p}}$$

period

$$T_p(k) = \frac{T_p}{|k|}$$

Main issue

Extracting the signals components

periodic T_p

$$\omega_0 = \frac{2\pi}{T_p}$$

$$x(t) = \sum_k X_k \cdot e^{jk\omega_0 t}$$

signal components

Can be extracted using standard rules on vectors, appropriately rewritten for the periodic signal space

Orthogonal base decomposition

Complex vectors	Periodic signals

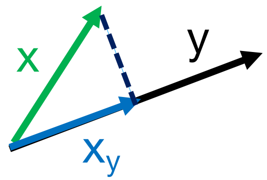
Inner product

$$\langle x, y \rangle = \sum_i x_i y_i^*$$

$$\langle x(t), y(t) \rangle = \int_{t_0}^{t_0+T_p} x(t) y^*(t) dt$$

Projection along y

$$x_y = \frac{y}{\|y\|} \left\langle x, \frac{y}{\|y\|} \right\rangle = y \frac{\langle x, y \rangle}{\|y\|^2}$$

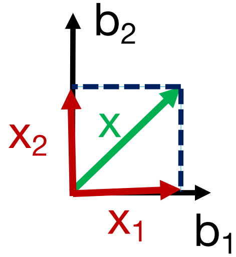


$$x_y(t) = y(t) \frac{\langle x(t), y(t) \rangle}{\|y(t)\|^2}$$

Decomposition wrt. an orthogonal basis

$$\langle b_i, b_{j \neq i} \rangle = 0$$

$$x = \sum_i b_i \frac{\langle x, b_i \rangle}{\|b_i\|^2}$$



$$\langle b_i(t), b_{j \neq i}(t) \rangle = 0$$

$$x(t) = \sum_k \underbrace{b_k(t) \frac{\langle x(t), b_k(t) \rangle}{\|b_k(t)\|^2}}_{\text{signal components}}$$

The orthogonal basis

$$\boxed{b_k(t) = e^{j\omega_0 k t}} \qquad \boxed{\omega_0 = \frac{2\pi}{T_p}}$$

Orthogonality properties

$$\langle b_i(t), b_\ell(t)\rangle = \int_{t_0}^{t_0+T_p} \underbrace{e^{j\omega_0 i t} e^{-j\omega_0 \ell t}}_{e^{j\omega_0 (i-\ell) t}} dt$$

$$= \begin{cases} \boxed{T_p \quad i = \ell} & \text{energy} \\ \boxed{0 \quad i \neq \ell} & \text{orthogonality} \end{cases}$$

The Fourier series

projection onto
an orthogonal
basis

$$\omega_0 = \frac{2\pi}{T_p}$$

Fourier series

$$s(t) = \sum_{k=-\infty}^{\infty} S_k e^{jk\omega_0 t}$$

$$S_k = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} s(t) e^{-jk\omega_0 t} dt$$

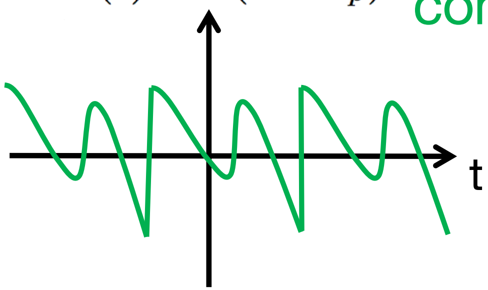
Fourier series
coefficients
signal components

signal
reconstruction

time domain

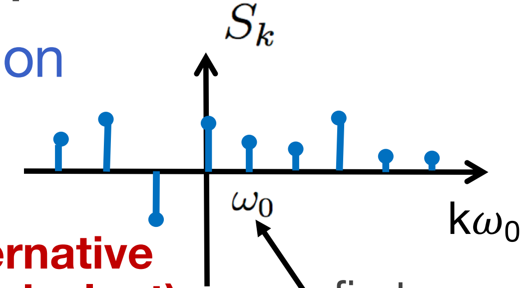
$s(t) = s(t + T_p)$

continuous-time
and periodic



Fourier domain

discrete-pulsation
and aperiodic



**an alternative
(and equivalent)
signal representation**

first
harmonic

Invertibility conditions

From S_k to s(t) and viceversa

absolutely integrable + finite number of finite discontinuities + finite number of max and min (in a period) — **Dirichlet**

finite energy (in a period) — **Riesz-Fisher**

finite number of delta functions (in a period) — **Dirac**

plus any linear combinations of the above

Weak convergence

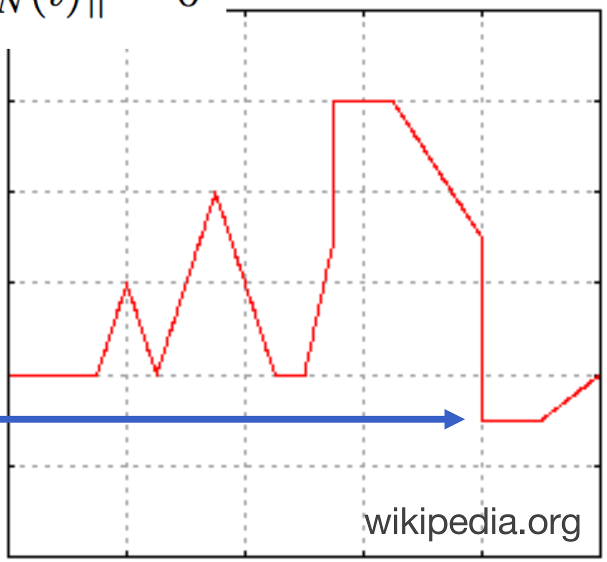
in norm – converges almost everywhere

partial sums

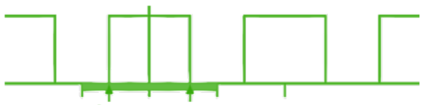
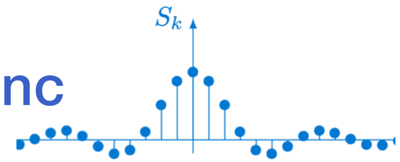
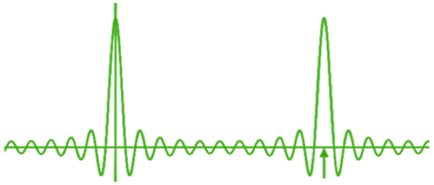
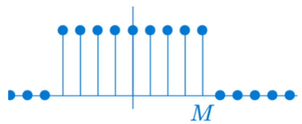
$$s_N(t) = \sum_{k=-N}^{N} S_k e^{jk\omega_0 t}$$

$$\lim_{N\to\infty} \|s(t) - s_N(t)\| = 0$$

Gibbs phenomenon at discontinuities



Useful pairs

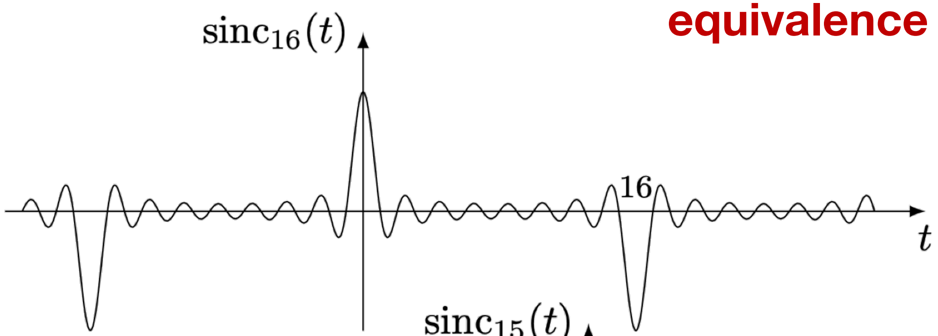
		time domain	Fourier domain		
	constant	1	$\delta(k)$	delta	
	delta dual pair	$\mathrm{rep}_{T_p}\,\delta(t)$	$\dfrac{1}{T_p}$	constant	
	rect	$\mathrm{rep}_{T_p}\,\mathrm{rect}\left(\frac{t}{dT_p}\right)$	$d\ \mathrm{sinc}(kd)$	sinc	
	sinc dual pair	$N\mathrm{sinc}_N\left(\frac{Nt}{T_p}\right)$ $N = 1 + 2M$	$\mathrm{rect}\left(\frac{k}{N}\right)$	rect	
	exponential	$e^{jm\omega_0 t}$	$\delta(k-m)$	delta	
	delta dual pair	$\mathrm{rep}_{T_p}\delta(t-t_1)$	$\frac{1}{T_p}e^{-jk\omega_0 t_1}$	exponential	
	sinusoid	$\cos(n\omega_0 t + \varphi_0)$	$\frac{1}{2}e^{j\varphi_0}\delta(k-n) + \frac{1}{2}e^{-j\varphi_0}\delta(k+n)$		

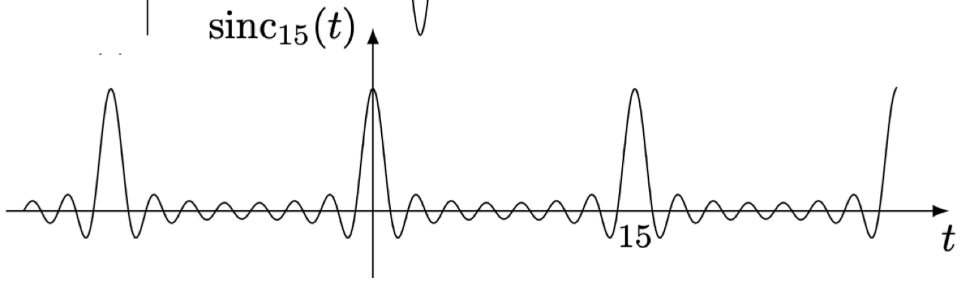
The periodic sinc

$$\operatorname{sinc}_M(x) = \frac{\sin(\pi x)}{M \sin(\pi \frac{x}{M})}$$

$$= \begin{cases} \operatorname{rep}_M \operatorname{sinc}(x) & , M \text{ odd} \\ \operatorname{rep}_{2M} \operatorname{sinc}(x) - \operatorname{sinc}(x - M) & , M \text{ even} \end{cases}$$

will prove this equivalence later on!





Exercises

On the Fourier series

Extracting the **coefficients of the Fourier series** or evaluating the signal shape through its **Fourier series** expression is a fundamental requirement of this course. You need to fully memorize the forward/backward rule and take particular care in evaluating integrals and/or series correctly.

You also need to memorize the **fundamental Fourier pairs** , as these are relevant in what follows.

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Lecture 10

Fourier series: definition and properties

Tomaso Erseghe

Università degli Studi di Padova

10.4 Properties

Of the Fourier series

- Time-reversal, conjugation, and symmetries
- Linearity, time-shift, modulation, convolution, product, and derivative
- Mean value and power
- The minimum period issue (de-periodization)

Symmetries

	time domain	Fourier domain
time-reversal	$x(-t)$	X_{-k}
conjugation	$x^*(t)$	X^*_{-k}
symmetries	**even** $x(t) = x(-t)$	**even** $X_k = X_{-k}$
	odd $x(t) = -x(-t)$	**odd** $X_k = -X_{-k}$
	real $x(t) = x^*(t)$	**Hermitian** $X_k = X^*_{-k}$
	real + even $x(t) = x^*(t) = x(-t)$	real + even $X_k = X^*_{-k} = X_{-k}$
	real + odd $x(t) = x^*(t) = -x(-t)$	imaginary + odd $X_k = X^*_{-k} = -X_{-k}$

Time-reversal + conjugation

$$\frac{1}{T_p}\int_{t_0}^{t_0+T_p} x(-t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T_p}\int_{-t_0-T_p}^{-t_0} x(u)e^{-j(-k)\omega_0 u}du \quad \boxed{= X_{-k}}$$

u = -t

$$\frac{1}{T_p}\int_{t_0}^{t_0+T_p} x^*(t)e^{-jk\omega_0 t}dt$$

$$= \left(\frac{1}{T_p}\int_{t_0}^{t_0+T_p} x(t)e^{jk\omega_0 t}dt\right)^* \quad \boxed{= X^*_{-k}}$$

k = -(-k)

Properties

	time domain	Fourier domain
linearity	$\alpha x(t) + \beta y(t)$	$\alpha X_k + \beta Y_k$
time-shift	$x(t - t_1)$	$X_k e^{-jk\omega_0 t_1}$
modulation	$x(t) e^{jm\omega_0 t}$	X_{k-m}
dual pair		
convolution	$x *_{\text{cir}} y(t)$	$T_p X_k Y_k$
product	$x(t) y(t)$	$X_k * Y_k$
dual pair		
derivative	$\frac{dx(t)}{dt}$	$X_k \cdot jk\omega_0$
mean value	m_x	X_0
power Parseval's theorem	P_x	E_X

Mean + Power

$$X_0 = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x(t)dt = m_x$$

$$P_x = \textbf{power}\left[\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}\right] = \sum_{k=-\infty}^{\infty} |X_k|^2$$

see exercises
Lecture 2

Time-shift + modulation

$$\frac{1}{T_p}\int_{t_0}^{t_0+T_p} x(t-t_1)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T_p}\int_{t_0-t_1}^{t_0-t_1+T_p} x(u)e^{-jk\omega_0(u+t_1)}du$$

one period

u = t-t_1

$$\boxed{= X_k e^{-jk\omega_0 t_1}}$$

$$\frac{1}{T_p}\int_{t_0}^{t_0+T_p} x(t)e^{jm\omega_0 t}e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T_p}\int_{t_0-t_1}^{t_0-t_1+T_p} x(t)e^{-j\omega_0(k-m)t}dt$$

$$\boxed{= X_{k-m}}$$

one period

Convolution + product

$$\frac{1}{T_p}\int_{t_0}^{t_0+T_p}\left(\int_{t_0}^{t_0+T_p} x(t-u)y(u)du\right)e^{-jk\omega_0 t}dt$$

swap

$$= \int_{t_0}^{t_0+T_p} y(u)\left(\frac{1}{T_p}\int_{t_0}^{t_0+T_p} x(t-u)e^{-jk\omega_0 t}dt\right)du$$

$$= \int_{t_0}^{t_0+T_p} y(u)X_k e^{-jk\omega_0 u}du \quad \boxed{= T_pX_kY_k}$$

$$\frac{1}{T_p}\int_{t_0}^{t_0+T_p} x(t)y(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T_p}\int_{t_0}^{t_0+T_p}\left(\sum_{m=-\infty}^{\infty} X_m e^{jm\omega_0 t}\right)y(t)e^{-jk\omega_0 t}dt$$

swap

$$= \sum_{m=-\infty}^{\infty} X_m \underbrace{\left(\frac{1}{T_p}\int_{t_0}^{t_0+T_p} e^{-j(k-m)\omega_0 t}y(t)dt\right)}_{Y_{k-m}}$$

Derivative

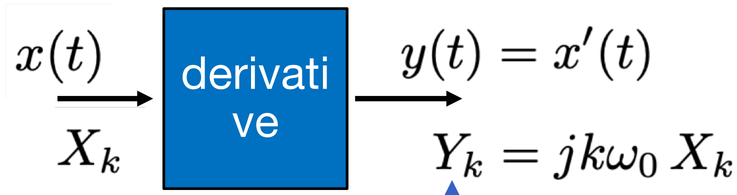
swap

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}\right)$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{X_k \cdot jk\omega_0}\, e^{jk\omega_0 t}$$

Fourier coefficients

Inversion rule



the value at k=0 is lost since Y_0=0, the others are multiplied by a known factor

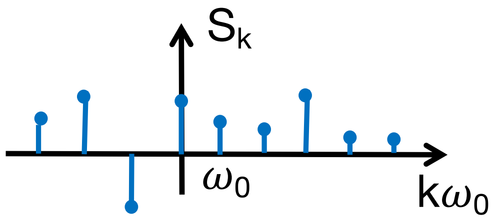
$$X_k = \begin{cases} \dfrac{Y_k}{jk\omega_0} & , k \neq 0 \\ \boxed{m_x} & , k = 0 \end{cases}$$

Incorrect period case

De-periodization in the time domain

true period T_p

$$s(t) = \sum_{k=-\infty}^{\infty} S_k e^{jk\omega_0 t}$$



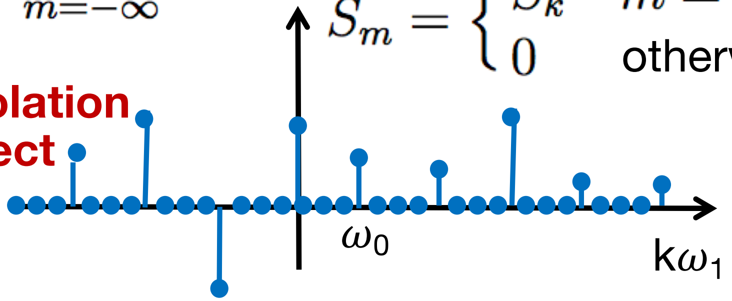
must be equal

period $T'_p = NT_p$ and $\boxed{\omega_1 = \omega_0/N}$

$$s(t) = \sum_{m=-\infty}^{\infty} \tilde{S}_m e^{jm\omega_0/Nt}$$

$$\tilde{S}_m = \begin{cases} S_k & m = Nk \\ 0 & \text{otherwise} \end{cases}$$

interpolation effect



Exercises

On the properties of the Fourier series

By correctly exploiting the **properties of the Fourier series** we can ease the calculation for a large variety of signals, but we first need to properly familiarize with their use.

Remember to always give a **graphical representation** to your signals, as this might very often become useful in correctly interpreting them.

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Lecture 11

Convolution and Fourier series in MatLab

Tomaso Erseghe

Università degli Studi di Padova

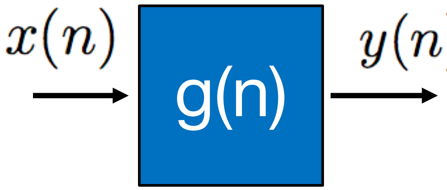
11.1 Convolution in MatLab

An overview

- Discrete-time convolution
- Approximate convolution in continuous-time

Discrete-time convolution

For limited-time signals

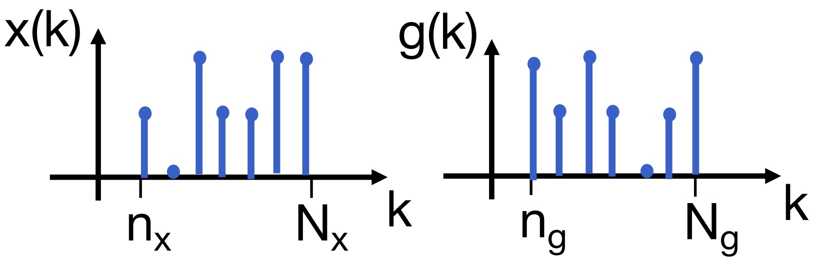


$$y(n) = \sum_{k=-\infty}^{\infty} x(k)g(n-k)$$

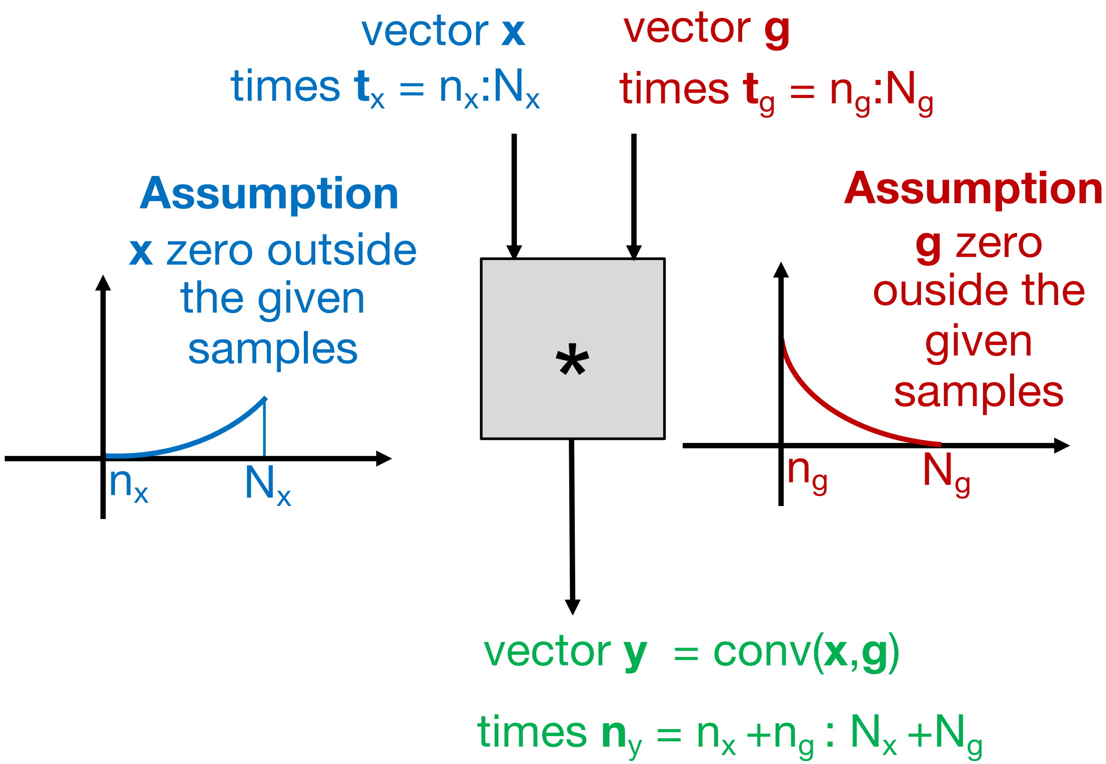
LTI system/filter

Limited only if
limited signals

$e(x*g) = [n_x+n_g, N_x+N_g]$

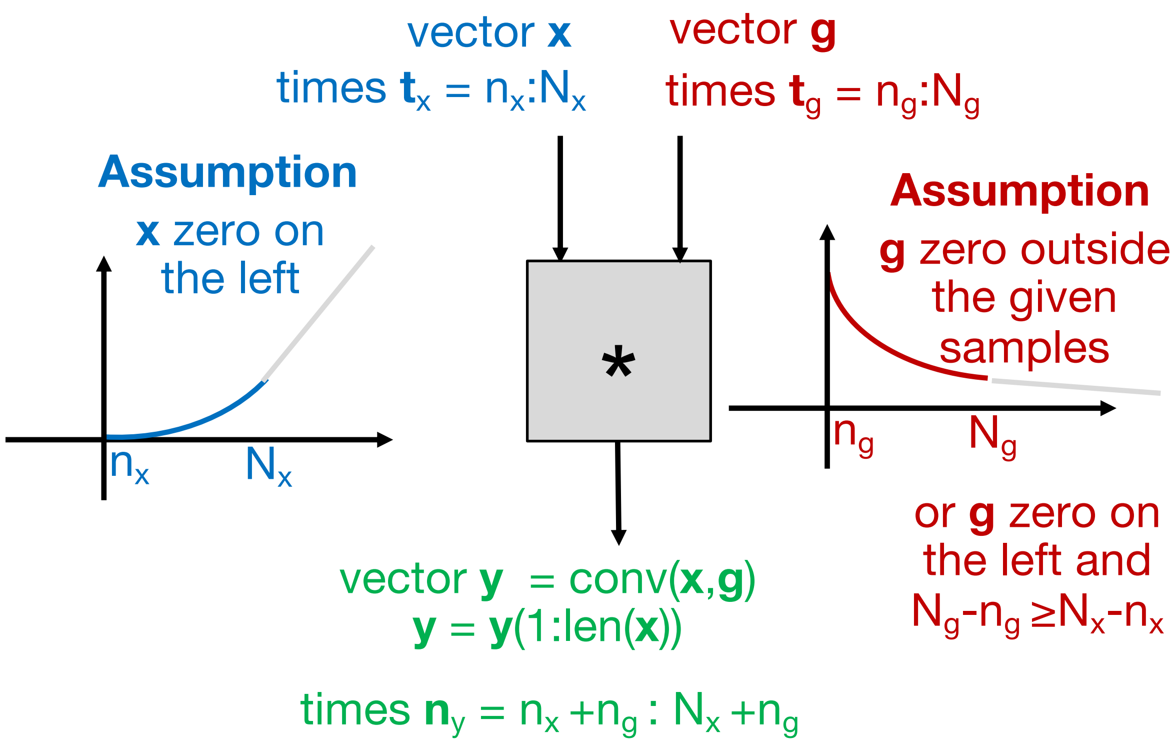


MatLab conv function



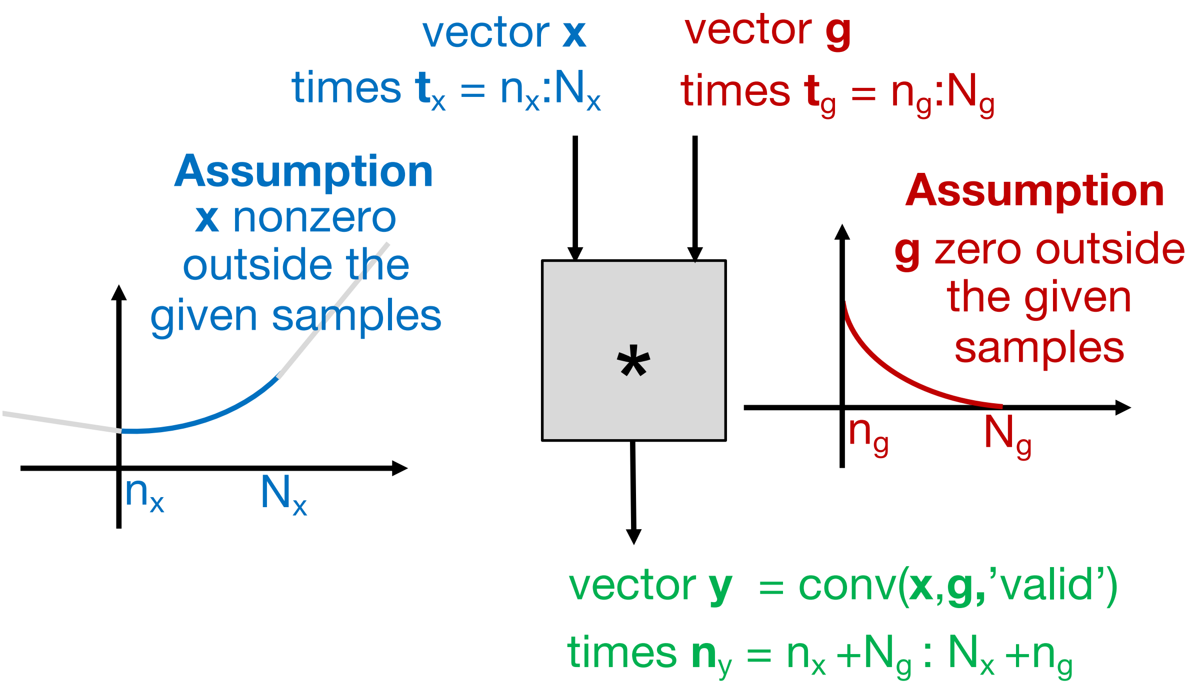
MatLab conv function

cut the result in case nonzero outside



MatLab conv function

‘valid’ = keeps an even smaller part



Continuous-time convolution

An approximation

samples of y

$$x * g(nT) = \int_{-\infty}^{\infty} x(u)g(nT-u)\,du$$

$$\simeq T \sum_{k=-\infty}^{\infty} x(kT)g(nT-kT)$$

sampling spacing: must be sufficiently (very) small

samples of x and g

y = T*conv(x,g)

$\mathbf{t_y} = t_x + t_g : T_x + T_g$

Exercises

On the convolution in MatLab

Get acquainted with MatLab convolution operator **conv** and remind that when approximating a continuous-time convolution you will need to multiply by the sampling spacing T, to have **T*conv(x,g)**

Remember that the output of the convolution is not always valuable everywhere, e.g., in case signals are not zero outside the interval where samples are given

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Lecture 11

Convolution and Fourier series in MatLab

Tomaso Erseghe

Università degli Studi di Padova

11.4 Fourier series in MatLab

Some insights

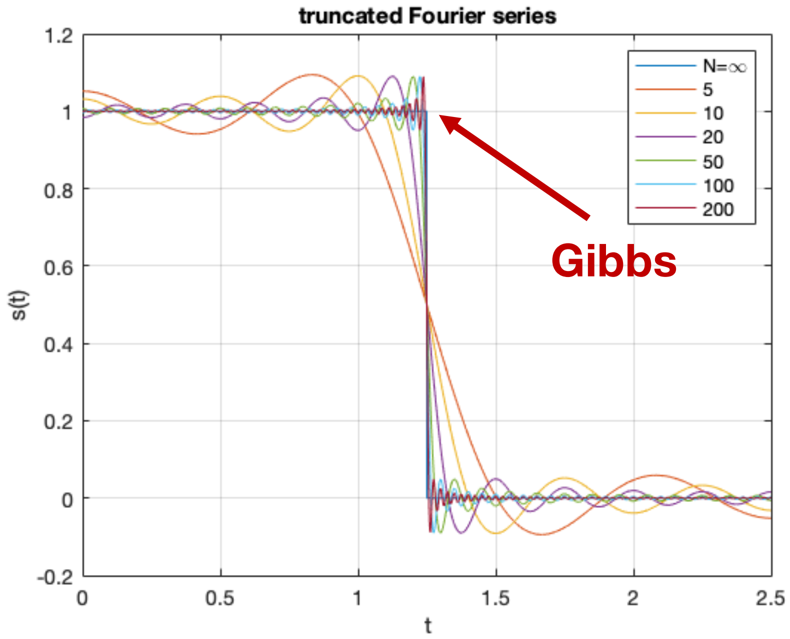
- The Gibbs phenomenon
- Approximating the coefficients via numerical integration

Square wave

And the Gibbs phenomenon

$$s_N(t) = \sum_{k=-N}^{N} a_k \, e^{jk\omega_0 t} \;, \quad \omega_0 = \frac{2\pi}{T_p} \;, \; a_k = d \; \mathrm{sinc}(kd)$$

truncated Fourier series

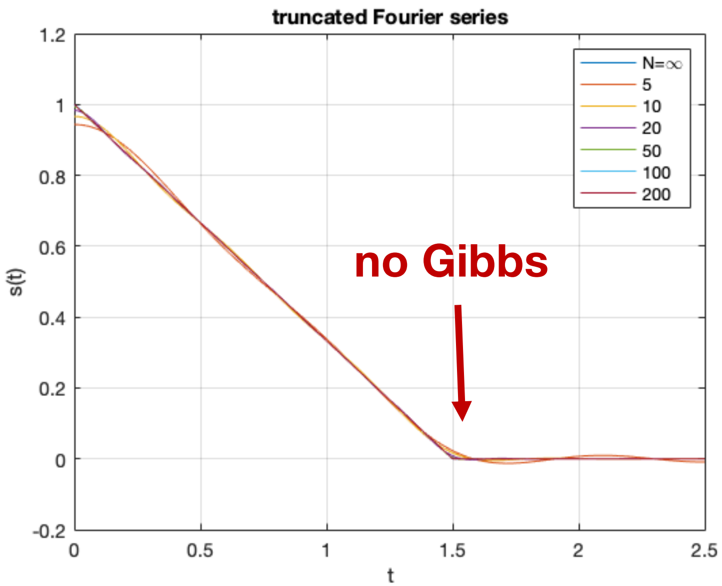


Triangular wave

And the absence of Gibbs phenomenon

$$s_N(t) = \sum_{k=-N}^{N} a_k \, e^{jk\omega_0 t} \; , \quad \omega_0 = \frac{2\pi}{T_p} \; , \; a_k = d \, \mathrm{sinc}^2(kd)$$

truncated Fourier series



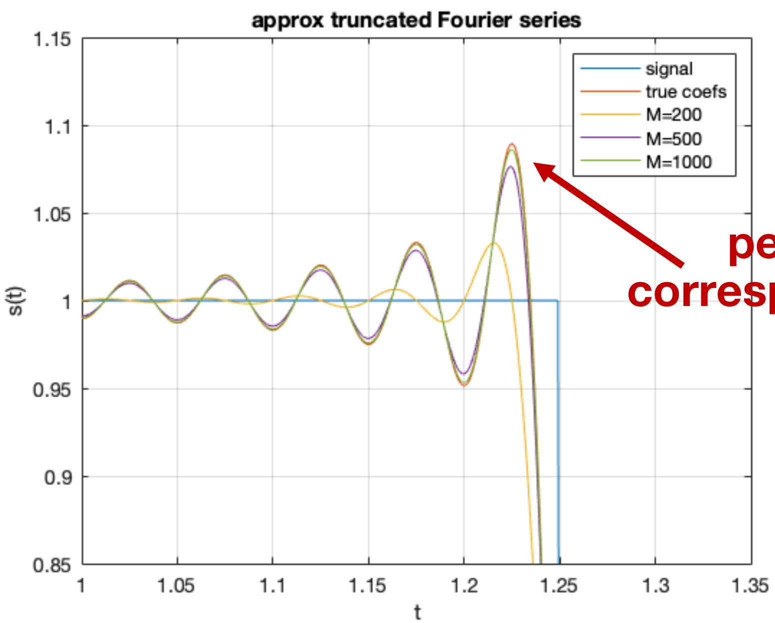
Square wave

Numerically evaluated coefficients

$$a_k = \frac{1}{T_p}\int_0^{T_p} s(t)e^{-jk\omega_0 t}dt \simeq b_k = \frac{1}{T_p}\cdot T\sum_{n=0}^{M-1} s(nT)\,e^{-jk\omega_0 nT}$$

$$T = T_p/M$$

approximated coefficients



perfect correspondence

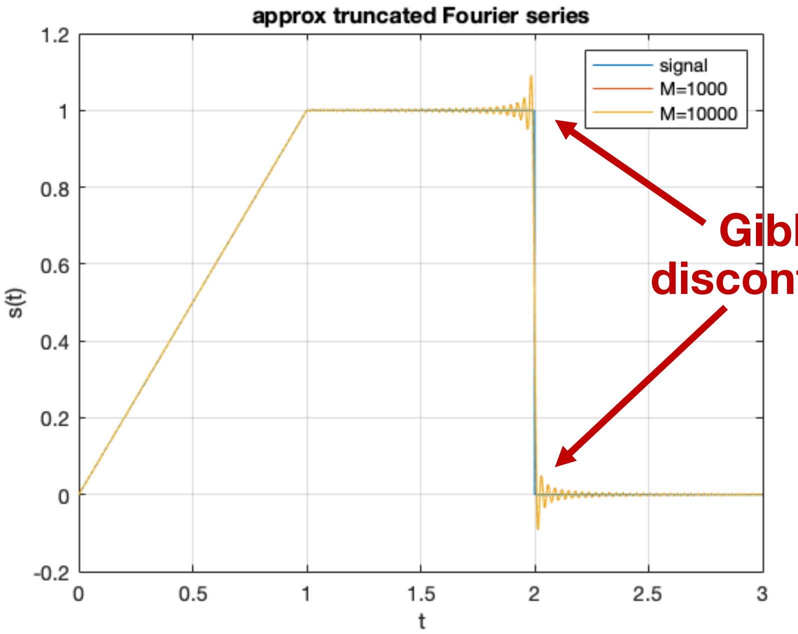
Generic wave

Numerically evaluated coefficients

$$a_k = \frac{1}{T_p}\int_0^{T_p} s(t)e^{-jk\omega_0 t}dt \simeq b_k = \frac{1}{T_p}\cdot T\sum_{n=0}^{M-1} s(nT)\,e^{-jk\omega_0 nT}$$

$$T = T_p/M$$

approximated coefficients



Gibbs at discontinuities

Exercises

On Fourier series

Observe the outcome of truncated Fourier series, and appreciate the presence of the **Gibbs' phenomenon** at discontinuities.

Practice yourself with **numerically evaluated** Fourier coefficients, to be able to represent any periodic signal through its (truncated) Fourier series

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Lecture 12

Discrete Fourier transform: definition and properties

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12.1 The discrete Fourier transform

for discrete-time periodic signals

- Orthogonal projections in a signal space
- The discrete Fourier transform
- The fast Fourier transform
- Useful Fourier pairs and symmetry rule

Unveiling filters

A general idea in discrete-time

eigenfunction

$$x(n) = e^{j\varphi_0 n} \longrightarrow \boxed{g(n)} \longrightarrow y(n) = \lambda x(n)$$

eigenvalue

$$\lambda = G(e^{j\varphi_0}) = \sum_{k=-\infty}^{\infty} g(k) e^{-j\varphi_0 k}$$

- **By linearity we have**

$$x(n) = \sum_k A_k \cdot e^{j\varphi_k n}$$

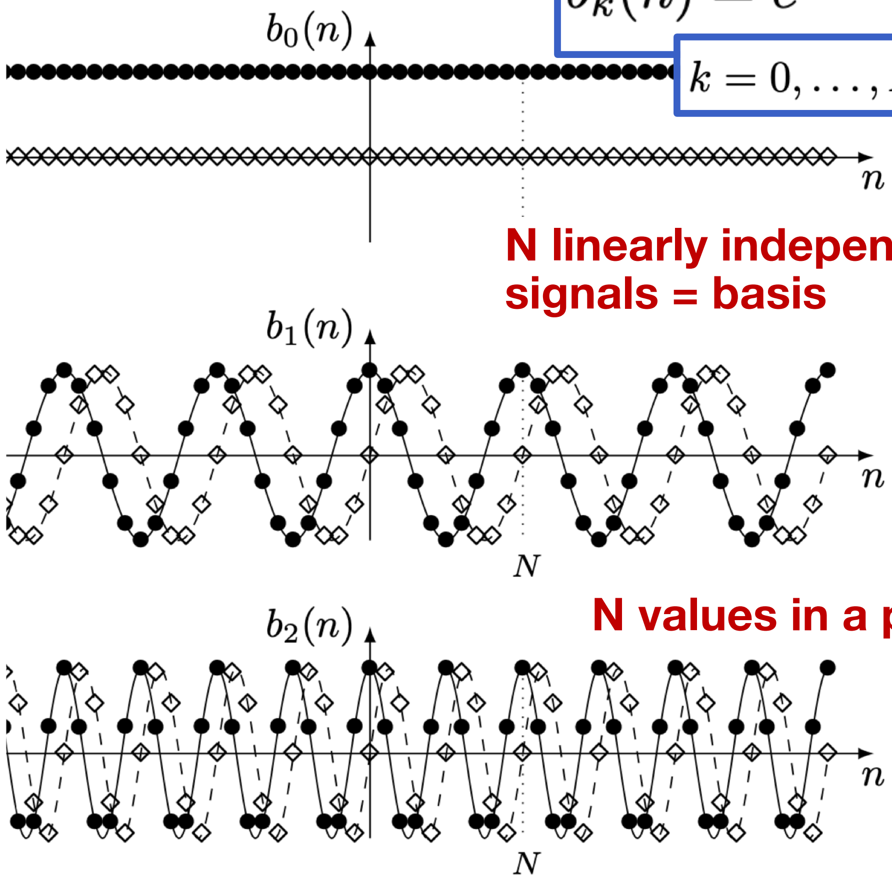
$$y(n) = \sum_k [A_k G(e^{j\varphi_k})] \cdot e^{j\varphi_k n}$$

We focus on **periodic** signals and linear complex exponentials that are periodic N
φ_k = k 2π/N

The orthogonal basis

$$\boxed{b_k(n) = e^{j2\pi\,\frac{kn}{N}}}$$

$$\boxed{k = 0, \ldots, N-1}$$



N linearly independent signals = basis

N values in a period

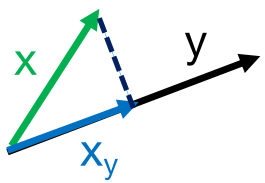
Basis decomposition

— Inner product

$$\langle x(n), y(n)\rangle = \sum_{n=n_0}^{n_0+N-1} x(n)y^*(n)$$

— Projection along y

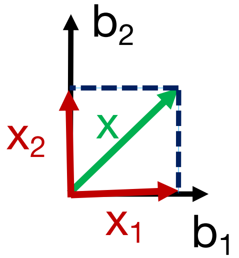
$$x_y(n) = y(n)\frac{\langle x(n), y(n)\rangle}{\|y(n)\|^2}$$



— Decomposition wrt. an orthogonal basis

$$\langle b_k(n), b_\ell(n)\rangle = N\,\delta(k-\ell)$$

$$x(n) = \sum_k b_k(n) \underbrace{\frac{\langle x(n), b_k(n)\rangle}{\|b_k(n)\|^2}}_{\text{signal components}}$$



The Discrete Fourier transform (DFT)

Fourier series

$$s(n) = \sum_{k=k_0}^{k_0+N-1} S_k \, e^{jk\frac{2\pi}{N}n}$$

projection onto an orthogonal basis

$$S_k = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} s(n) \, e^{-jk\frac{2\pi}{N}n}$$

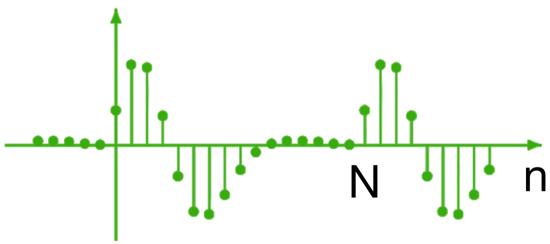
signal reconstruction

Fourier series coefficients

signal components

time domain

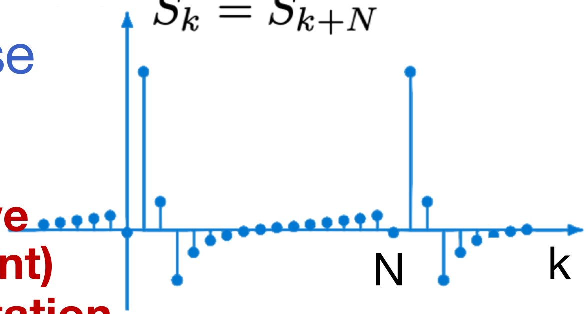
$s(n) = s(n+N)$ discrete-time and periodic



Fourier domain

discrete-phase and periodic

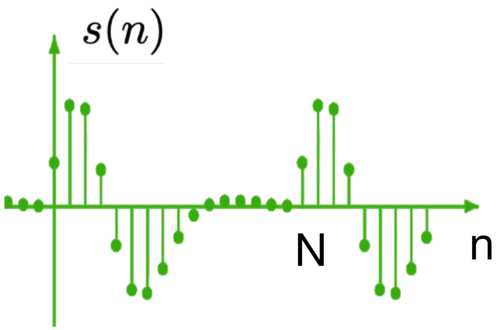
$S_k = S_{k+N}$



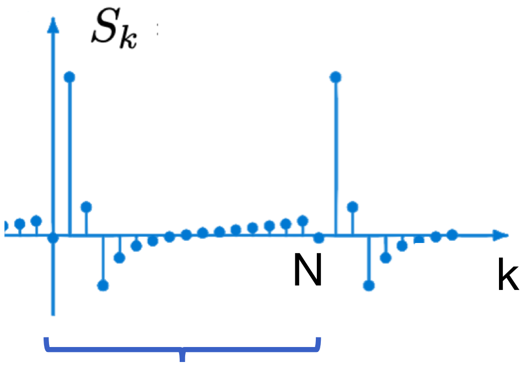
an alternative (and equivalent) signal representation

DFT as a matrix product

Invertibility between S_k and s(n)



collect the N samples of a period into vector
s



collect the N samples of a period into vector
S

S = (1/N) F s

s = F* S

DFT matrix $F = \left[e^{-jk\frac{2\pi}{N}n}\right]_{n,k=1,\ldots,N-1}$

The FFT

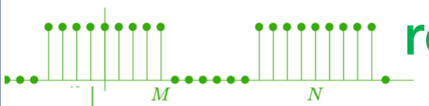
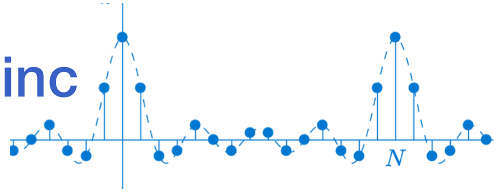
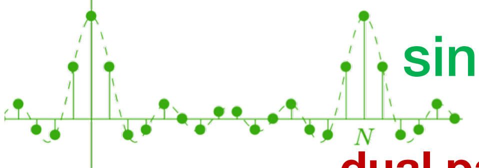
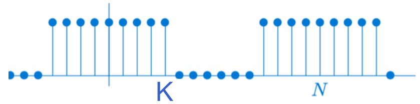
Fast Fourier transform

- The DFT matrix multiplication has complexity N^2
- By exploiting the symmetries of F we can devise an algorithm of complexity $N \log_2 N$ called **FFT**
- For N=1000 we have $N^2=10^6$ and $N \log_2 N=10^4$

  … **100 times faster**
- FFT is among the Top 10 Algorithms of the 20th century

https://en.wikipedia.org/wiki/Fast_Fourier_transform

Useful pairs

	time domain	Fourier domain	
constant	1	$\mathrm{rep}_N \delta(k)$	delta
delta **dual pair**	$\mathrm{rep}_N \delta(n)$	$\frac{1}{N}$	constant
 rect	$\mathrm{rep}_N \, \mathrm{rect}(\frac{n}{M})$	$\frac{M}{N} \, \mathrm{sinc}_M(\frac{M}{N} k)$	sinc 
 sinc **dual pair**	$M \, \mathrm{sinc}_M(\frac{M}{N} n)$ $M = 1 + 2K$	$\mathrm{rep}_N \, \mathrm{rect}(\frac{k}{M})$	rect 
exponential	$e^{jm\frac{2\pi}{N}n}$	$\mathrm{rep}_N \delta(k - m)$	delta
delta **dual pair**	$\mathrm{rep}_N \delta(n - m)$	$\frac{1}{N} e^{-k\frac{2\pi}{N}m}$	exponential
sinusoid	$\cos(m\frac{2\pi}{N}n + \varphi_0)$	$\frac{1}{2} e^{j\varphi_0} \mathrm{comb}_N(k - m) + \frac{1}{2} e^{-j\varphi_0} \mathrm{comb}_N(k + m)$	

DFT symmetry rule

One transform, two couples

$$s(n) \longrightarrow \boxed{\mathcal{F}} \longrightarrow S_k = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} s(n)\, e^{-jk\frac{2\pi}{N}n}$$

$$x(n) = S_n \longrightarrow \boxed{\mathcal{F}} \longrightarrow X_k = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} S_n\, e^{j(-k)\frac{2\pi}{N}n}$$

$$= \frac{1}{N} s(-k)$$

An example

$$s(n) = 1$$

$$S_k = \mathrm{rep}_N \delta(k)$$

$$x(n) = S_n = \mathrm{rep}_N \delta(n)$$

$$X_k = \frac{1}{N} s(-k) = \frac{1}{N}$$

Exercises

On the DFT

Extracting the **coefficients of the DFT** or evaluating the signal shape through its **DFT series** expression is a fundamental requirement of this course. You need to fully memorize the forward/backward rule and take particular care in evaluating series correctly.

You also need to memorize the **fundamental Fourier pairs**, as these are relevant in what follows.

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Lecture 12

Discrete Fourier transform: definition and properties

Tomaso Erseghe

Università degli Studi di Padova

12.4 Properties

Of the DFT

- Time-reversal, conjugation, and symmetries
- Linearity, time-shift, modulation, convolution, product, and increment
- Mean value and power
- Minimum period issue

Symmetries

	time domain	Fourier domain
time-reversal	$x(-n)$	X_{-k}
conjugation	$x^*(n)$	X^*_{-k}
symmetries	**even** $x(n) = x(-n)$	**even** $X_k = X_{-k}$
	odd $x(n) = -x(-n)$	**odd** $X_k = -X_{-k}$
	real $x(n) = x^*(n)$	**Hermitian** $X_k = X^*_{-k}$
	real + even $x(n) = x^*(n) = x(-n)$	real + even $X_k = X^*_{-k} = X_{-k}$
	real + odd $x(n) = x^*(n) = -x(-n)$	imaginary + odd $X_k = X^*_{-k} = -X_{-k}$

Time-reversal + conjugation

$$\frac{1}{N}\sum_{n=0}^{N-1} x(-n)e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{m=-N+1}^{0} x(m)e^{-j(-k)\frac{2\pi}{N}m} \quad \boxed{= X_{-k}}$$

m = -n

$$\frac{1}{N}\sum_{n=0}^{N-1} x^*(n)e^{-jk\frac{2\pi}{N}n}$$

$$= \left(\frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j(-k)\frac{2\pi}{N}n}\right)^* \quad \boxed{= X^*_{-k}}$$

Properties

	time domain	Fourier domain
linearity	$\alpha x(n) + \beta y(n)$	$\alpha X_k + \beta Y_k$
time-shift	$x(n - n_1)$	$X_k e^{-jk\frac{2\pi}{N}n_1}$
modulation	$x(n)e^{jm\frac{2\pi}{N}n}$	X_{k-m}
dual pair		
convolution	$x *_{\text{cir}} y(n)$	$N X_k Y_k$
product	$x(n)y(n)$	$X *_{\text{circ}} Y_k$
dual pair		
increment	$x(n) - x(n-1)$	$X_k \cdot (1 - e^{-jk\frac{2\pi}{N}})$
mean value	$m_x = X_0$	
power Parseval's theorem	$P_x = E_X(N)$	

Mean + Power

$$X_0 = \frac{1}{N} \sum_{n=0}^{N-1} x(n) = m_x$$

$$P_x = \text{power}\left\{ \sum_{k=0}^{N-1} X_k \, e^{jk\frac{2\pi}{N}n} \right\} = \sum_{k=0}^{N-1} |X_k|^2$$

↑

see exercises
Lecture 3

Increment

$$x(n) \xrightarrow{} \boxed{\text{increment}} \xrightarrow{} y(n) = x(n) - x(n-1)$$

$$X_k \qquad Y_k = X_k \cdot \left(1 - e^{-jk\frac{2\pi}{N}}\right)$$

the value at k=0 (mod N) is lost since Y_0=0,
the others are multiplied by a known factor

Inversion rule

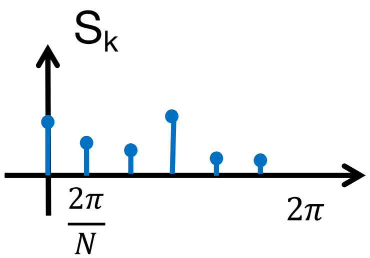
$$X_k = \begin{cases} \dfrac{Y_k}{1 - e^{-jk\frac{2\pi}{N}}} & , k \neq 0 \pmod{N} \\ \boxed{m_x} & , k = 0 \pmod{N} \end{cases}$$

Incorrect period case

De-periodization in the time domain

true period N

$$s(t) = \sum_{k=0}^{N-1} S_k e^{jk\frac{2\pi}{N}n}$$



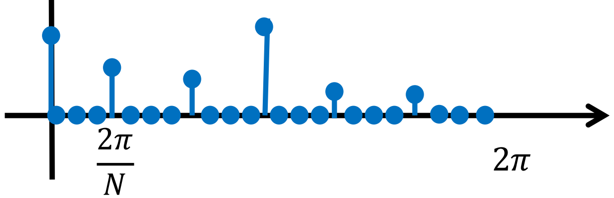
must be equal

period N’ = $\boxed{KN}$

$$s(t) = \sum_{m=0}^{KN-1} \tilde{S}_m e^{jm\frac{2\pi}{KN}n}$$

$$\tilde{S}_m = \begin{cases} S_k & , m = kK \\ 0 & , \text{otherwise} \end{cases}$$

interpolation effect



Exercises

On the properties of the DFT

By correctly exploiting the **properties of the DFT** we can ease the calculation for a large variety of signals, but we first need to properly familiarize with their use.

Please be aware that DFT pairs are in general more cumbersome to deal with, and that only **simple cases** will be considered at the exam.

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Lecture 13

Fourier transform: definition and properties

Tomaso Erseghe

13.1 The Fourier transform

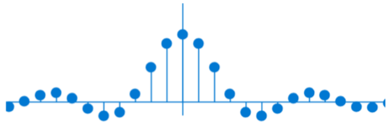
for continuous-time aperiodic signals

- The Fourier transform
- Convergence properties
- Useful Fourier pairs
- Symmetry rule and scale property

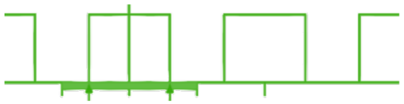
Defining a Fourier transform

From periodic to aperiodic

$1/T_p$



$$s(t) = \frac{\omega_0}{2\pi} \sum_{k=-\infty}^{\infty} \tilde{S}(jk\omega_0) e^{jk\omega_0 t} \qquad \tilde{S}(jk\omega_0) = \int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} s(t) e^{-jk\omega_0 t}\, dt$$

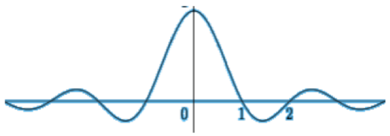


$\omega_0 \rightarrow 0$

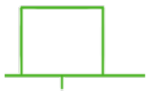
$T_p \rightarrow \infty$

$T_p \rightarrow \infty$

$\omega_0 \rightarrow 0$



$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\omega) e^{j\omega t} d\omega \qquad S(j\omega) = \int_{-\infty}^{\infty} s(t) e^{-j\omega t}\, dt$$



The Fourier transform

projection onto an
orthogonal basis

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\omega)e^{j\omega t}d\omega$$

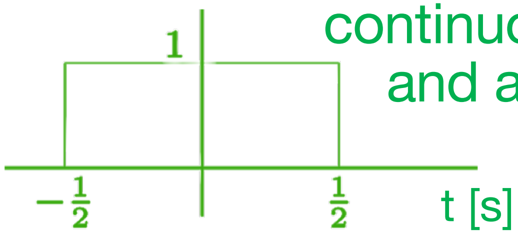
$$S(j\omega) = \int_{-\infty}^{\infty} s(t)e^{-j\omega t}\,dt$$

signal
reconstruction

Fourier transform
signal components
frequency f=$\omega/2\pi$ [Hz]

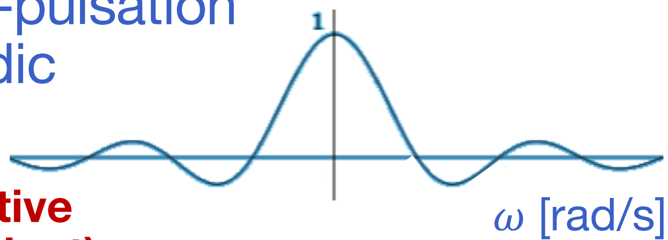
time domain

continuous-time
and aperiodic



Fourier domain

continuous-pulsation
and aperiodic



**an alternative
(and equivalent)
signal representation**

Invertibility conditions

From S(ω) to s(t) and viceversa

absolutely integrable + finite number of finite discontinuities + finite number of max and min — **Dirichlet**

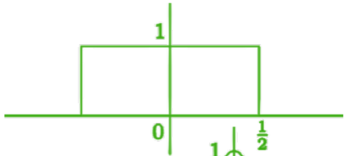
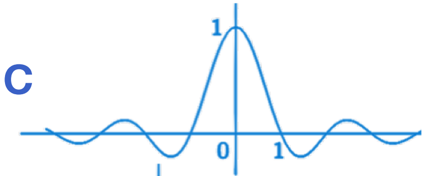
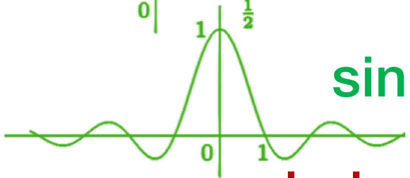
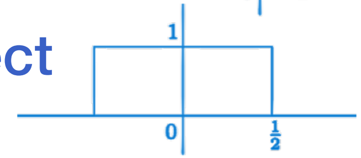
finite energy — **Riesz-Fisher**

finite number of delta functions — **Dirac**

plus any linear combinations of the above

Convergence in norm – converges almost everywhere

Useful pairs

	time domain	Fourier domain	
constant	1	$2\pi\,\delta(\omega)$	delta
delta	$\delta(t)$	1	constant
dual pair			
 rect	$\mathrm{rect}(t)$	$\mathrm{sinc}(\frac{\omega}{2\pi})$	sinc 
 sinc	$\mathrm{sinc}(t)$	$\mathrm{rect}(\frac{\omega}{2\pi})$	rect 
dual pair			
exponential	$e^{j\omega_1 t}$	$2\pi\,\delta(\omega-\omega_1)$	delta
delta	$\delta(t-t_1)$	$e^{-j\omega t_1}$	exponential
dual pair			
sinusoid	$\cos(\omega_1 t+\varphi_1)$	$\pi\, e^{j\varphi_1}\,\delta(\omega-\omega_1)+\pi\, e^{-j\varphi_1}\,\delta(\omega+\omega_1)$	

More pairs

	time domain	Fourier domain	
triangle	$\mathrm{triang}(t)$	$\mathrm{sinc}^2(\frac{\omega}{2\pi})$	squared sinc
squared sinc	$\mathrm{sinc}^2(t)$	$\mathrm{triang}(\frac{\omega}{2\pi})$	triangle
dual pair			
sign	$\mathrm{sgn}(t)$	$\frac{-2j}{\omega}$	hyperbola
hyperbola	$\frac{j}{\pi t}$	$\mathrm{sgn}(\omega)$	sign
dual pair			
unit step	$1(t)$	$\frac{-j}{\omega} + \pi\delta(\omega)$	
	$\frac{j}{2\pi t} + \frac{1}{2}\delta(t)$	$1(\omega)$	unit step
dual pair			

Symmetry rule

One transform, two couples

$$s(t) \longrightarrow \boxed{\mathcal{F}} \longrightarrow S(j\omega) = \int_{-\infty}^{\infty} s(t) e^{-j\omega t}\, dt$$

$$x(t) = S(jt) \longrightarrow \boxed{\mathcal{F}} \longrightarrow X(j\omega) = \int_{-\infty}^{\infty} S(jt)\, e^{-j\omega t}\, dt$$

$$= 2\pi s(-\omega)$$

An example

$$s(t) = \text{rect}(t) \Rightarrow x(t) = S(jt) = \text{sinc}\left(\tfrac{t}{2\pi}\right)$$

$$S(j\omega) = \text{sinc}\left(\tfrac{\omega}{2\pi}\right) \qquad X(j\omega) = 2\pi s(-\omega)$$

$$= 2\pi\, \text{rect}(\omega)$$

Scale property

$$x(t) \xrightarrow{\text{scale}} x(t/a)$$

$$X(j\omega) \xrightarrow{\text{scale}} aX(ja\omega)$$

inverse scale in the pulsation domain

Proof

$$\int_{-\infty}^{\infty} x(t/a)e^{-j\omega t}\, dt = a\int_{-\infty}^{\infty} x(u)e^{-j\omega a u}\, du$$

An example

$$x(t) = \mathrm{sinc}(\tfrac{t}{2\pi})$$

$$X(j\omega) = 2\pi\, \mathrm{rect}(\omega)$$

$$\xrightarrow{a = \frac{1}{2\pi}}$$

$$y(t) = x(t/a) = \mathrm{sinc}(t)$$

$$Y(j\omega) = a\, X(j\omega)$$

$$= \mathrm{rect}(\tfrac{\omega}{2\pi})$$

Exercises

On the Fourier transform

Deriving the **Fourier transform** or evaluating the signal shape through the **inverse Fourier transform** is a fundamental requirement of this course. You need to fully memorize the forward/backward rule and take particular care in evaluating integrals correctly.

You also need to memorize the **fundamental Fourier pairs**, as these are relevant in what follows.

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Lecture 13

Fourier transform: definition and properties

Tomaso Erseghe

Università degli Studi di Padova

13.4 Properties

Of the Fourier transform

- Time-reversal, conjugation, and symmetries
- Linearity, time-shift, modulation, convolution, product, derivative, product by t, integration
- Mean value and power

Symmetries

	time domain	Fourier domain
time-reversal	$x(-t)$	$X(-j\omega)$
conjugation	$x^*(t)$	$X^*(-j\omega)$
symmetries	**even** $x(t) = x(-t)$	**even** $X(j\omega) = X(-j\omega)$
	odd $x(t) = -x(-t)$	**odd** $X(j\omega) = -X(-j\omega)$
	real $x(t) = x^*(t)$	**Hermitian** $X(j\omega) = X^*(-j\omega)$
	real + even $x(t) = x^*(t) = x(-t)$	real + even $X(j\omega) = X^*(-j\omega) = X(-j\omega)$
	real + odd $x(t) = x^*(t) = -x(-t)$	imaginary + odd $X(j\omega) = X^*(-j\omega) = -X(-j\omega)$

Time-reversal + conjugation

$$\int_{-\infty}^{\infty} x(-t)e^{-j\omega t}\, dt \quad = \int_{-\infty}^{\infty} x(u)e^{j\omega u}\, du$$

u = -t

$$\boxed{= X(-j\omega)}$$

$$\int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\, dt \quad = \left(\int_{-\infty}^{\infty} x(t)e^{j\omega t}\, dt\right)^*$$

$$\boxed{= X^*(-j\omega)}$$

Properties

	time domain	Fourier domain
linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(j\omega) + \beta Y(j\omega)$
time-shift	$x(t - t_0)$	$X(j\omega)e^{-j\omega t_0}$
modulation	$x(t)e^{j\omega_0 t}$	$X(j\omega - j\omega_0)$
dual pair		
convolution	$x * y(t)$	$X(j\omega)Y(j\omega)$
product	$x(t)y(t)$	$\frac{1}{2\pi} X * Y(j\omega)$
dual pair		
derivation	$x'(t)$	$j\omega \cdot X(j\omega)$
product by t	$tx(t)$	$jX'(j\omega)$
dual pair		
integration	$\int_{-\infty}^{t} x(u)\, du$	$\frac{X(j\omega)}{j\omega} + \pi\, A_x\, \delta(\omega)$
area	$A_x =$	$S(j0)$
energy	$E_x =$	$\frac{1}{2\pi} E_X$
Parseval's theorem		

Convolution + product

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(u)y(t-u)du\right)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(u)\left(\int_{-\infty}^{\infty} y(t-u)e^{-j\omega t}dt\right)\,du$$

swap

$$= \int_{-\infty}^{\infty} x(u)Y(j\omega)e^{-j\omega u}\,du \quad \boxed{= X(j\omega)Y(j\omega)}$$

$$\int_{-\infty}^{\infty} x(t)y(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(jv)e^{jvt}dv\right)e^{-j\omega t}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(jv)\underbrace{\left(\int_{-\infty}^{\infty} x(t)e^{-j(\omega-v)t}dt\right)}_{X(j(\omega-v))}dv$$

swap

Area + Energy

$$S(j0) = \int_{-\infty}^{\infty} s(t) \underbrace{e^{-j0t}}_{1} \, dt = A_s$$

$$z(t) = x(t) y^*(t)$$

$$Z(j\omega) = \frac{1}{2\pi} X * Y_-^*(j\omega)$$

$$A_z = \int_{-\infty}^{\infty} x(t) y^*(t) \, dt = Z(j0)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(jv) \underbrace{Y_-^*(-jv)}_{Y^*(jv)} \, dv$$

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt = \frac{1}{2\pi} E_S = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(j\omega)|^2 d\omega$$

Derivative

swap

$$x'(t) = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d}{dt}\left(e^{j\omega t}\right)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(j\omega)j\omega}_{\text{Fourier transform}} \cdot e^{j\omega t}\, d\omega$$

Inversion rule

$x(t)$ → derivative → $y(t) = x'(t)$

$X(j\omega)$ → derivative → $Y(j\omega) = j\omega \cdot X(j\omega)$

the value at ω=0 is lost since Y(j0)=0, the others are multiplied by a known factor

$$X(j\omega) = \frac{Y(j\omega)}{j\omega} + \boxed{m_x} 2\pi\,\delta(\omega)$$

No need to correct for a finite value at ω=0

Integration

$x(t) \longrightarrow$ integration $\longrightarrow$

$$y(t) = \int_{-\infty}^{t} x(u)\, du$$
$$= x * 1(t)$$

$$Y(j\omega) = X(j\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$
$$= \frac{X(j\omega)}{j\omega} + \pi X(j\omega)\delta(\omega)$$
$$= \frac{X(j\omega)}{j\omega} + \pi \underbrace{\boxed{X(0)}}_{\text{area}}\delta(\omega)$$

Exercises

On the properties of the Fourier transform

Through the **properties of the Fourier transform** we can ease the calculation for a large variety of signals, but we first need to properly familiarize with their use.

The Fourier transform is by far the case where we can deal with the most complex waveforms, also thanks to the fact that managing integrals is easier than managing series.

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Lecture 14

Relations among the transforms and filters in continuous-time

Tomaso Erseghe

14.1 Relations among Fourier transforms

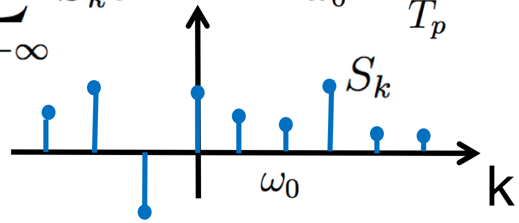
Link between Fourier transform, Fourier series, and DFT

- De-periodization property
- Periodic repetition and sampling properties

De-periodisation

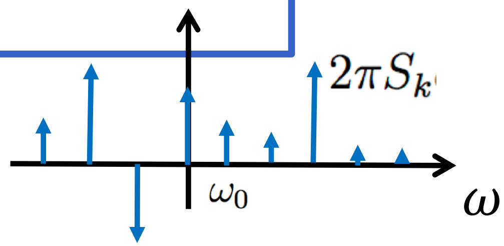
From the Fourier series to the transform

$$s(t) = s(t + T_p) = \sum_{k=-\infty}^{\infty} S_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_p}$$



The Fourier transform of a periodic signal

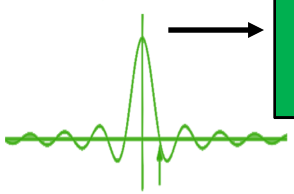
$$\boxed{\tilde{S}(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi S_k \delta(\omega - k\omega_0)}$$



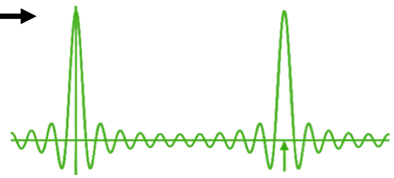
Periodic repetition

From the Fourier transform to the series

$$x(t) \longrightarrow \boxed{\mathrm{rep}_{T_p}} \longrightarrow y(t) = \mathrm{rep}_{T_p} x(t)$$

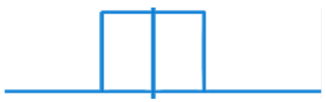


continuous-time
aperiodic



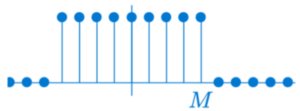
continuous-time
periodic T_p

$$X(j\omega) \longrightarrow \boxed{\downarrow} \longrightarrow Y_k = \frac{1}{T_p} X(jk\omega_0)$$



continuous-pulsation
aperiodic

$$\omega_0 = \frac{2\pi}{T_p}$$



discrete-pulsation
aperiodic

A proof

From the Fourier transform to the series

$$Y_k = \frac{1}{T_p}\int_0^{T_p}\left(\sum_{m=-\infty}^{\infty} x(t-mT_p)\right) e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T_p}\sum_{m=-\infty}^{\infty}\left(\int_0^{T_p} x(t-mT_p)\, e^{-jk\omega_0 t}\,dt\right)$$

swap

$$= \frac{1}{T_p}\sum_{m=-\infty}^{\infty}\int_{-mT_p}^{(1-m)T_p} x(u)\,\underbrace{e^{-jk\omega_0(u+mT_p)}}_{e^{-jk\omega_0 u}}\,du$$

$\boxed{u = t - mT_p}$

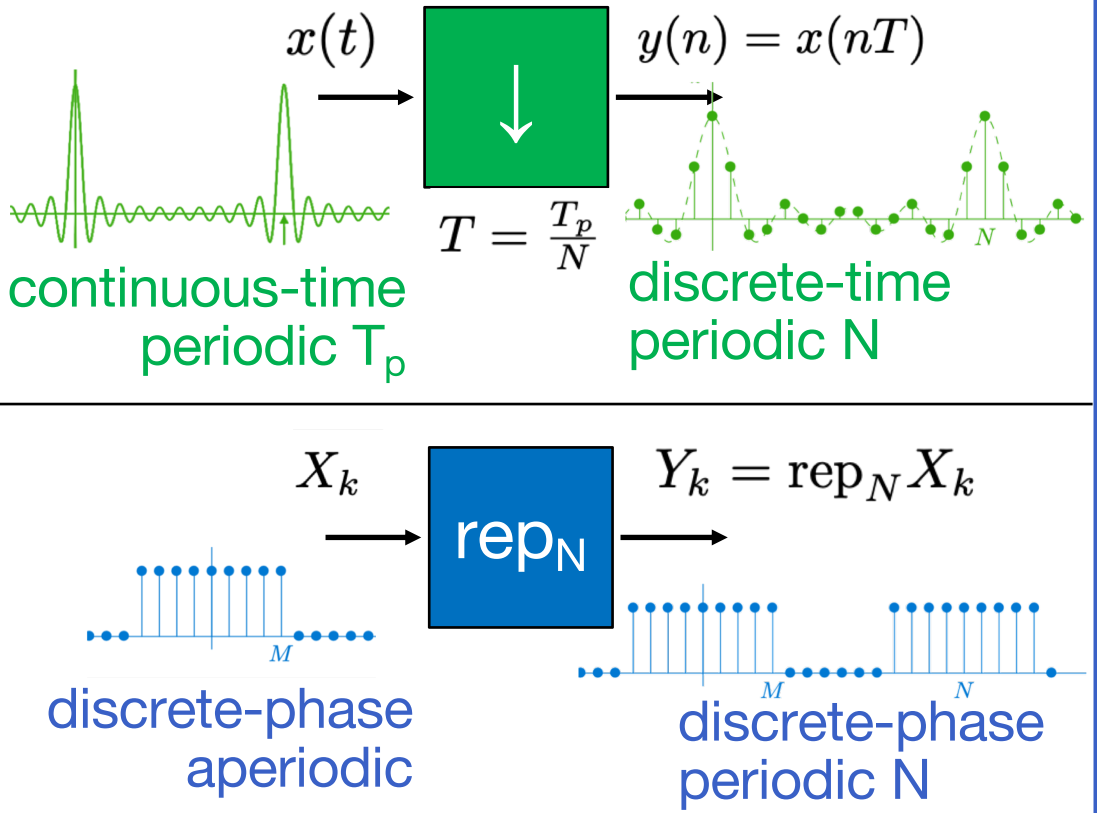
$\boxed{k\omega_0 \cdot mT_p = 2\pi km}$

$$= \frac{1}{T_p}\int_{-\infty}^{\infty} x(u)\, e^{-jk\omega_0 u}\,du$$

$$= \frac{1}{T_p} X(jk\omega_0)$$

Sampling

From the Fourier series to the DFT



A proof

From the Fourier series to the DFT

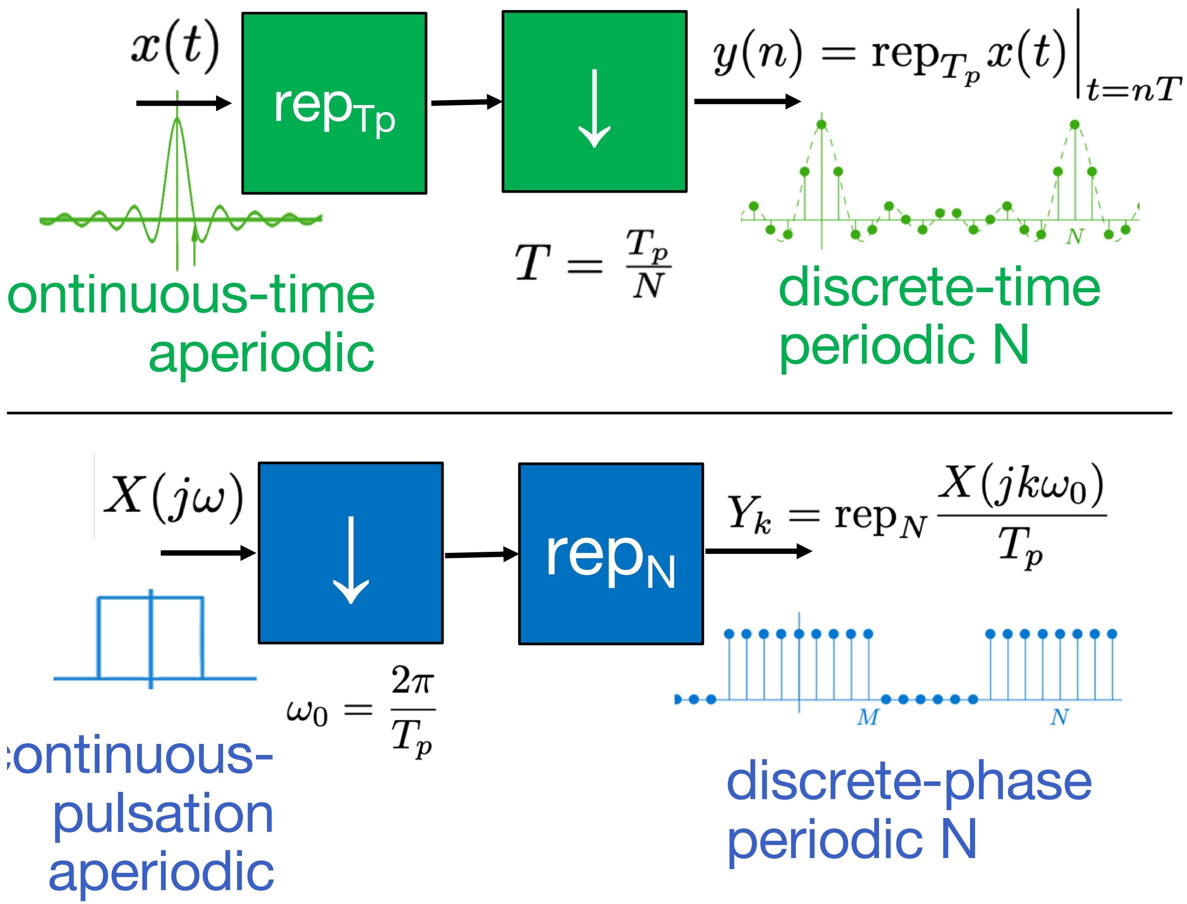
$$y(n) = \sum_{k=-\infty}^{\infty} X_k \, e^{jk\frac{2\pi}{T_p}t}\Big|_{t=n\frac{T_p}{N}} = \sum_{k=-\infty}^{\infty} X_k \, e^{jk\frac{2\pi}{N}n}$$

$$= \sum_{\ell=0}^{N-1} \underbrace{\sum_{m=-\infty}^{\infty} X_{\ell-mN}}_{\mathrm{rep}_N X_\ell} \underbrace{e^{j(\ell-mN)\frac{2\pi}{N}n}}_{e^{j\ell\frac{2\pi}{N}n}}$$

$k = \ell - mN$

Combining the two

From the Fourier transform to the DFT



Exercises

On the relations among Fourier transforms

Evaluating the **Fourier series** or the **DFT coefficients** from a **Fourier transform pair** can be a much easier way of calculating the transforms.

You need to get acquainted with this powerful rule, as it might turn out very useful.

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Lecture 14

Relations among the transforms and filters in continuous-time

Tomaso Erseghe

Università degli Studi di Padova

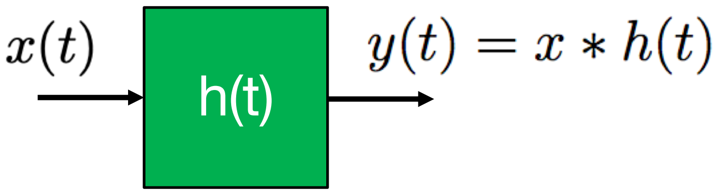
14.4 Filters in continuous-time

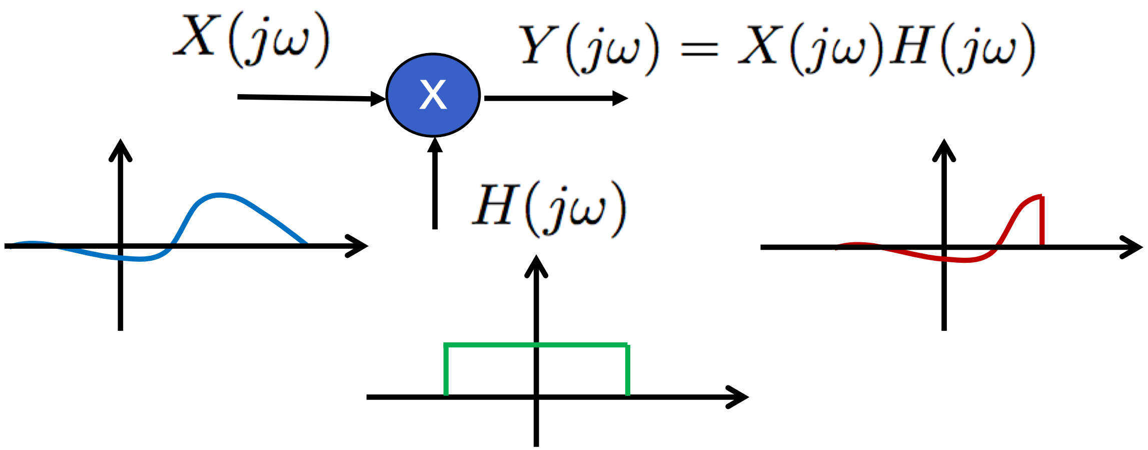
A Fourier transform perspective

- Filters in the Fourier domain
- Selective filters
- Response to sinusoids and exponentials

Filters in the Fourier domain

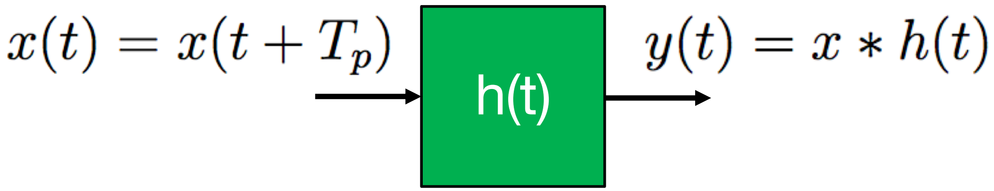
Continuous-time aperiodic

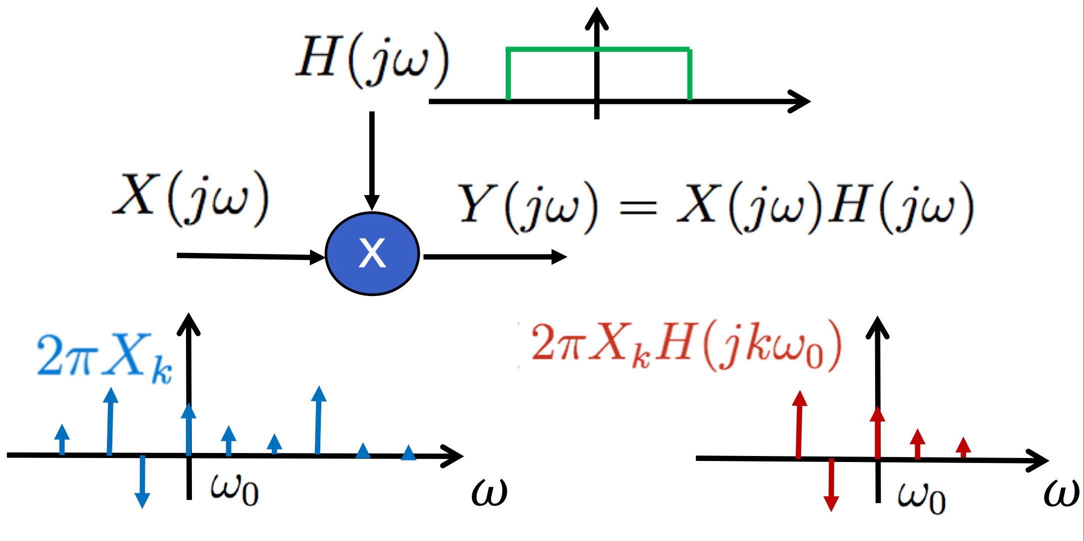




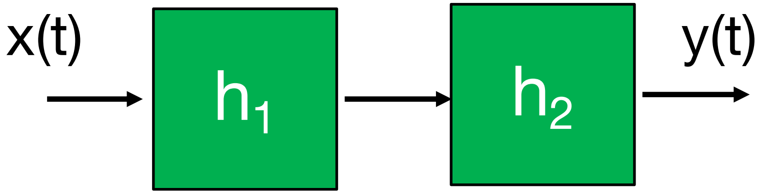
Filters in the Fourier domain

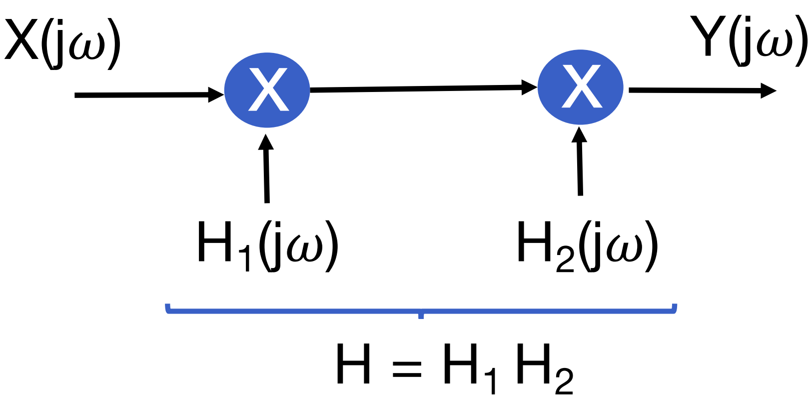
Continuous-time periodic



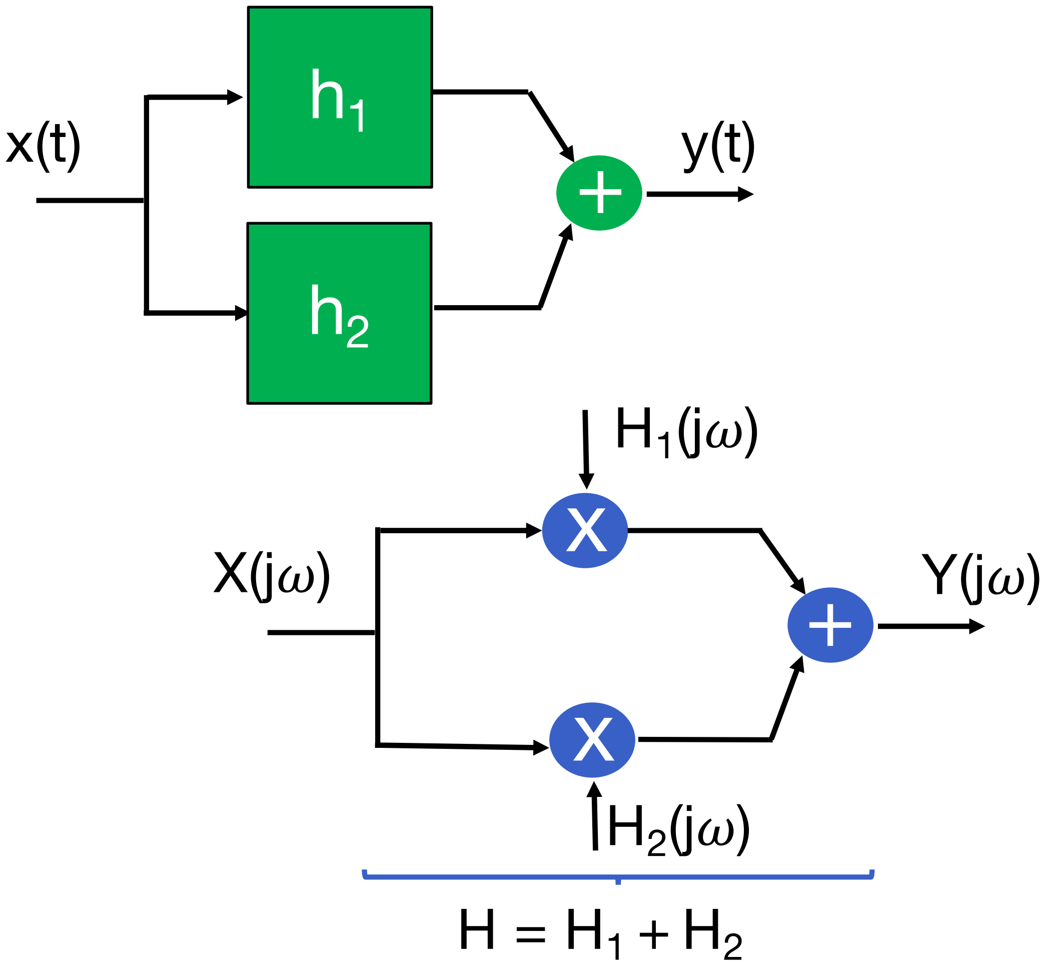


Series

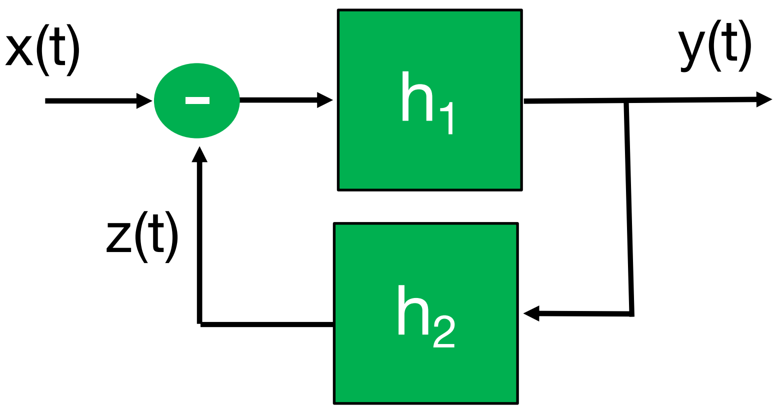


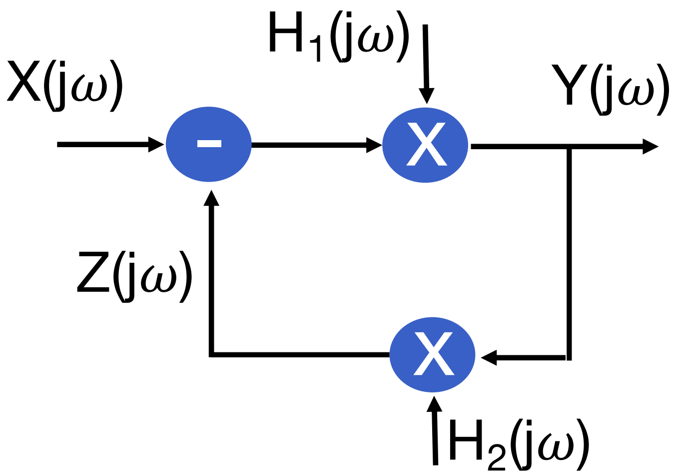


Parallel



Feedback



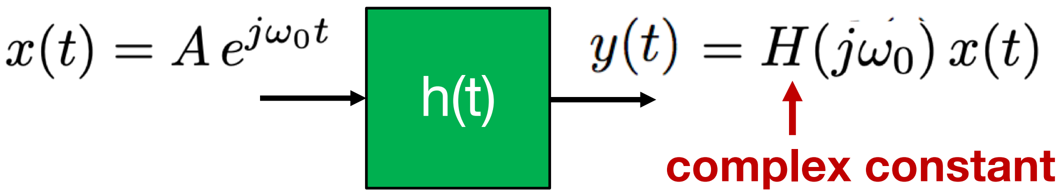


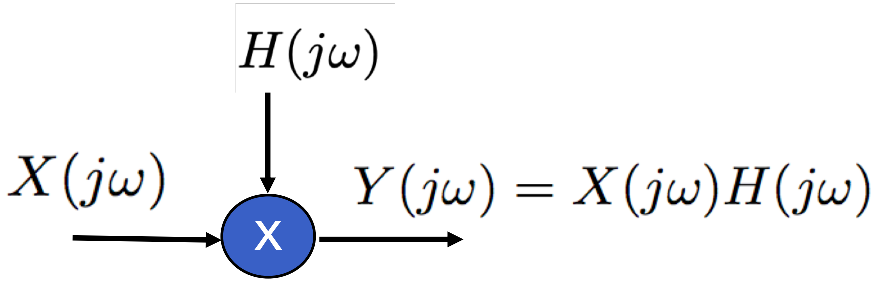
$$Z = H_2\,Y$$

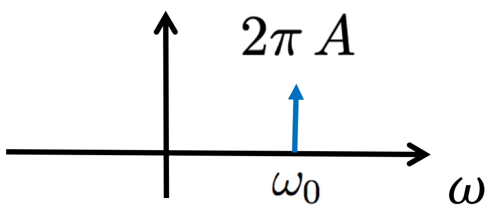
$$\begin{aligned} Y &= H_1\,(X - Z) \\ &= H_1\,(X - H_2 Y) \\ &= H\,X \end{aligned}$$

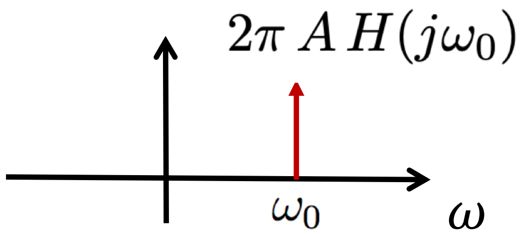
$$\boxed{H = \frac{H_1}{1 + H_1 H_2}}$$

Complex exponentials

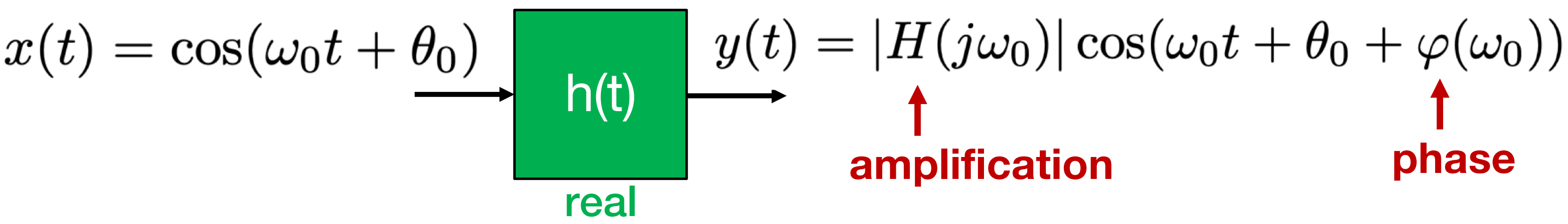


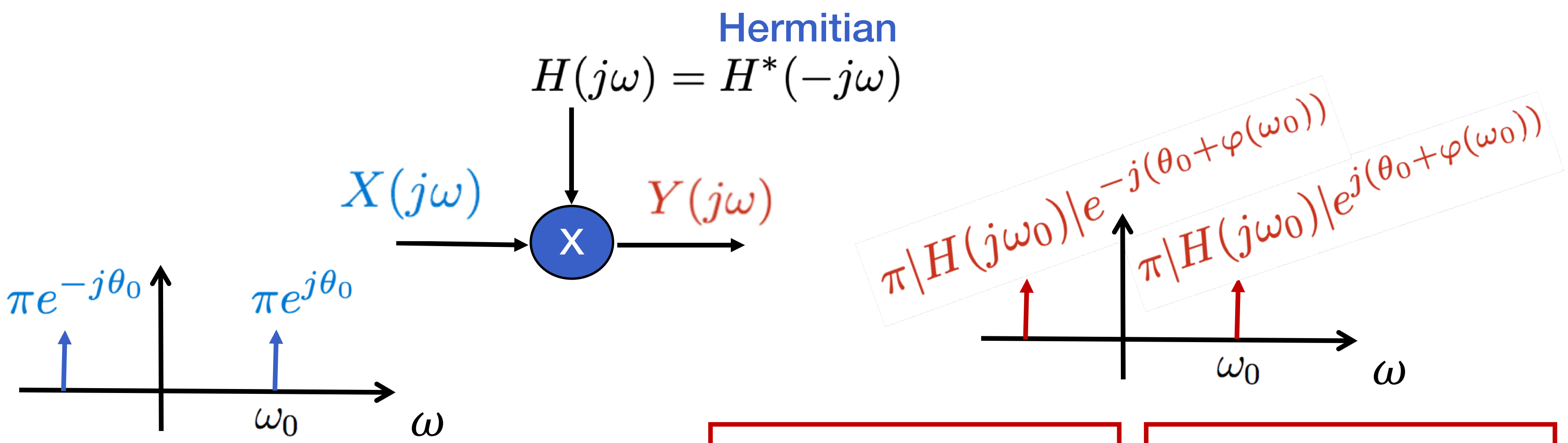






Sinusoids through a real filter





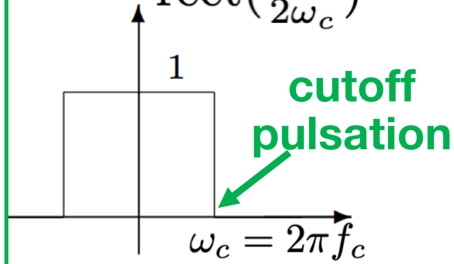
$$\boxed{\begin{aligned} H(-j\omega) &= H^*(\omega) \\ &= |H(j\omega)|e^{-j\varphi(\omega)} \end{aligned}}$$

$$\boxed{\begin{aligned} &H(j\omega) \\ &\quad = |H(j\omega)|e^{j\varphi(\omega)} \end{aligned}}$$

Selective (ideal) filters

low-pass

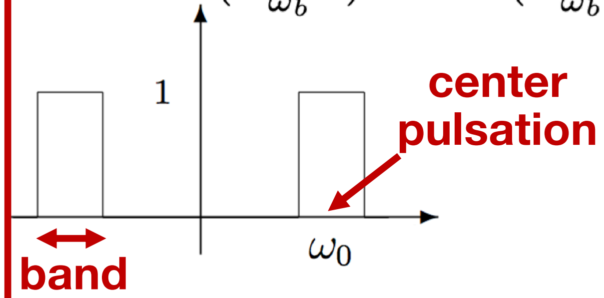
$\mathrm{rect}(\frac{\omega}{2\omega_c})$



$2f_c\ \mathrm{sinc}(2f_c t)$

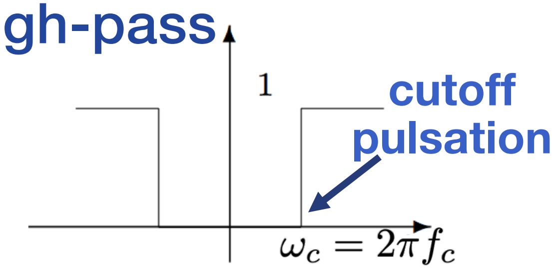
band-pass

$\mathrm{rect}(\frac{\omega-\omega_0}{\omega_b}) + \mathrm{rect}(\frac{\omega+\omega_0}{\omega_b})$



$2f_b\ \mathrm{sinc}(f_b t)\ \cos(\omega_0 t)$

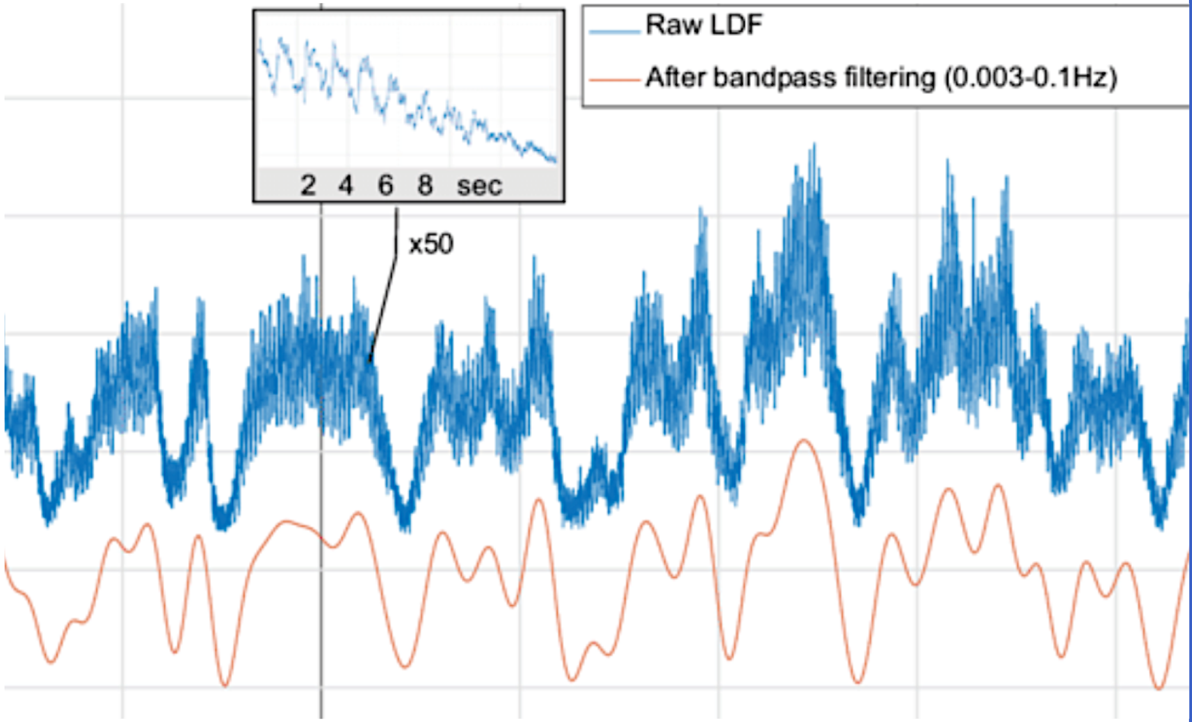
high-pass



all non
BIBO
stable

Polishing action of a filter

With a band-pass filter

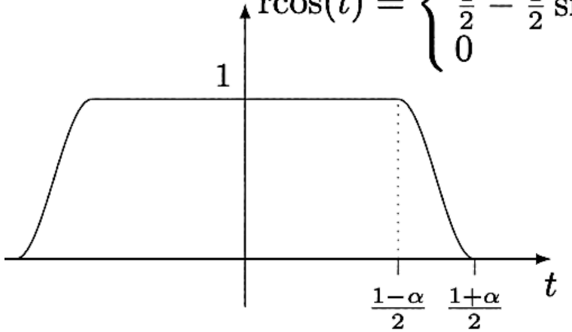


Real filters

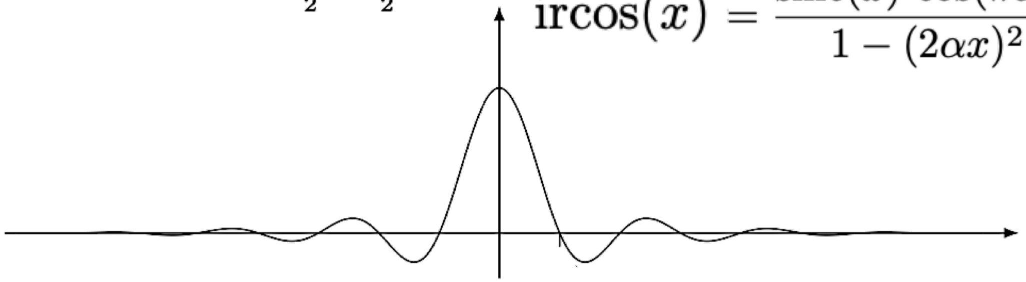
With BIBO stability properties

low-pass $$H(j\omega) = \mathrm{rcos}\left(\frac{\omega}{2\omega_c}\right)$$
$$h(t) = 2f_c\,\mathrm{ircos}(2f_c t)$$

$$\mathrm{rcos}(t) = \begin{cases} 1 & , |t| < \frac{1-\alpha}{2} \\ \frac{1}{2} - \frac{1}{2}\sin\left(\frac{\pi}{\alpha}\left(|t| - \frac{1}{2}\right)\right) & , \frac{1-\alpha}{2} \le |t| \le \frac{1+\alpha}{2} \\ 0 & , \text{otherwise} \end{cases}$$



$$\mathrm{ircos}(x) = \frac{\mathrm{sinc}(x)\,\cos(\pi\alpha x)}{1 - (2\alpha x)^2}$$



Exercises

On continuous-time filters

Through the **Fourier transform approach** we can ease the understanding and calculation of a filtering operation.

Carry in mind the rule on complex exponentials and sinusoids, as it might turn out to be very useful in many practical cases.

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Lecture 15

The discrete-time Fourier transform: definition, properties, relation with other transforms

Tomaso Erseghe

Università degli Studi di Padova

15.1 The discrete-time Fourier transform

for discrete-time aperiodic signals

- The discrete-time Fourier transform
- Useful Fourier pairs
- Time-reversal, conjugation, and symmetries
- Linearity, time-shift, modulation, convolution, product, increment, product by n, and current sum
- Mean value and power

Swapping the domains

From periodic to discrete-time

$T_p = 2\pi$

Fourier series

$$x(-t) = \sum_{n=-\infty}^{\infty} X_n \, e^{jnt}$$

Fourier series coefficients

$$X_n = \frac{1}{2\pi} \int_0^{2\pi} x(-t) \, e^{-jnt} \, dt$$

$\omega_0 = \frac{2\pi}{T_p} = 1$

swap

θ = -t

Fourier series coefficients

$$X_n = \frac{1}{2\pi} \int_0^{2\pi} x(\theta) \, e^{jn\theta} \, d\theta$$

Fourier series

$$x(\theta) = \sum_{n=-\infty}^{\infty} X_n \, e^{-jn\theta}$$

The discrete-time Fourier transform (DTFT)

Fourier series coefficients

$$s(n) = \frac{1}{2\pi}\int_0^{2\pi} S(e^{j\theta})e^{j\theta n}d\theta$$

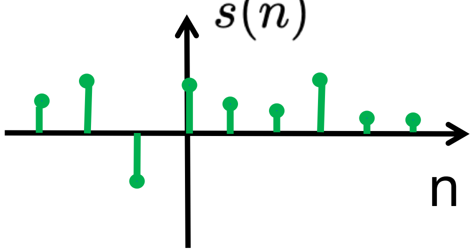
projection onto an orthogonal basis

Fourier series

$$S(e^{j\theta}) = \sum_{n=-\infty}^{\infty} s(n)e^{-j\theta n}$$

time domain

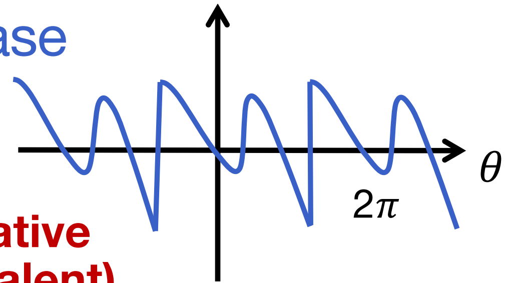
discrete-time and aperiodic



Fourier domain

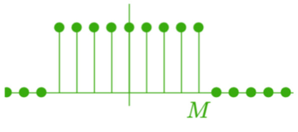
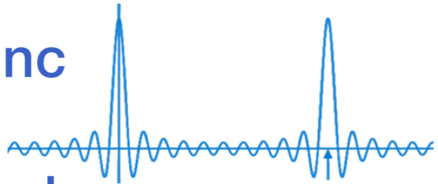
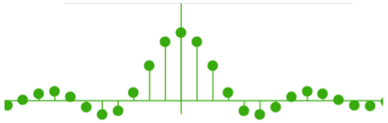
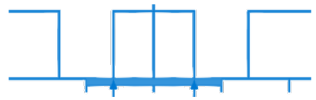
continuous-phase and periodic

$$S(e^{j(\theta+2\pi)}) = S(e^{j\theta})$$



an alternative (and equivalent) signal representation

Useful pairs

	time domain	Fourier domain	
constant	1	$2\pi\,\mathrm{comb}_{2\pi}(\theta)$	delta
delta	$\delta(n)$	1	constant
dual pair			
 rect	$\mathrm{rect}(\frac{n}{N})$ $N = 1 + 2M$	$N\,\mathrm{sinc}_N(\frac{\theta N}{2\pi})$	sinc 
 sinc dual pair	$d\,\mathrm{sinc}(nd)$	$\mathrm{rep}_{2\pi}\,\mathrm{rect}(\frac{\theta}{2\pi d})$	rect 
exponential	$e^{j\theta_0 n}$	$2\pi\,\mathrm{comb}_{2\pi}(\theta - \theta_0)$	delta
delta dual pair	$\delta(n - m)$	$e^{-j\theta m}$	exponential
sinusoid	$\cos(n\theta_0 + \vartheta_0)$	$\pi e^{j\vartheta_0}\mathrm{comb}_{2\pi}(\theta - \theta_0) + \pi e^{-j\vartheta_0}\mathrm{comb}_{2\pi}(\theta + \theta_0)$	

Symmetries

	time domain	Fourier domain
time-reversal	$x(-n)$	$X(e^{-j\theta})$
conjugation	$x^*(n)$	$X^*(e^{-j\theta})$
symmetries	**even** $x(n) = x(-n)$	**even** $X(e^{j\theta}) = X(e^{-j\theta})$
	odd $x(n) = -x(-n)$	**odd** $X(e^{j\theta}) = -X(e^{-j\theta})$
	real $x(n) = x^*(n)$	**Hermitian** $X(e^{j\theta}) = X^*(e^{-j\theta})$
	real + even $x(n) = x^*(n) = x(-n)$	real + even $X(e^{j\theta}) = X^*(e^{-j\theta}) = X(e^{-j\theta})$
	real + odd $x(n) = x^*(n) = -x(-n)$	imaginary + odd $X(e^{j\theta}) = X^*(e^{-j\theta}) = -X(e^{-j\theta})$

Properties

	time domain	Fourier domain
linearity	$\alpha x(n) + \beta y(n)$	$\alpha X(j\omega) + \beta Y(j\omega)$
time-shift	$x(n - n_0)$	$X(e^{j\theta})e^{-j\theta n_0}$
modulation	$x(n)e^{j\theta_0 n}$	$X(e^{j(\theta-\theta_0)})$
dual pair		
convolution	$x * y(n)$	$X(e^{j\theta})Y(e^{j\theta})$
product	$x(n)y(n)$	$\frac{1}{2\pi}X *_{\text{cir}} Y(e^{j\theta})$
dual pair		
increment	$x(n) - x(n-1)$	$X(e^{j\theta})\left(1 - e^{-j\theta}\right)$
product by n	$nx(n)$	$jX'(e^{j\theta})$
dual pair		
current sum	$\sum_{k=-\infty}^{n} x(k)$	$\frac{X(e^{j\theta})}{1 - e^{-j\theta}} + \pi A_x \text{comb}_{2\pi}(\theta)$
area	A_s	$= S(e^{j0})$
energy	E_s	$= P_S$

Parseval's theorem

Area + Energy

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x(n) \underbrace{e^{-jn0}}_{1} = A_x$$

$$P_X = \text{power}\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\theta}\right] = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

see exercises
Lecture 3

Increment

$x(n)$ → increme nt → $y(n) = x(n) - x(n-1)$

$X(e^{j\theta})$ → $Y(e^{j\theta}) = X(e^{j\theta})\left(1 - e^{-j\theta}\right)$

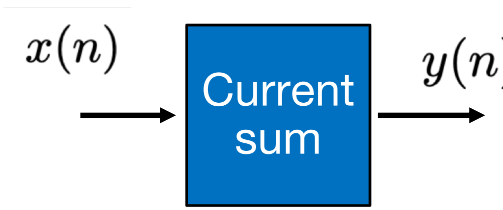
the value at θ=0 (mod 2π) is lost since Y(e^{j0})=0,
the others are multiplied by a known factor

Inversion rule

$$X(e^{j\theta}) = \frac{Y(e^{j\theta})}{1 - e^{-j\theta}} + 2\pi \boxed{m_x} \mathrm{comb}_{2\pi}(\theta)$$

No need to correct for a
finite value at θ=0 (mod 2π)

Current sum



$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

$$= x * 1_0(n)$$

$$Y(e^{j\theta}) = X(e^{j\theta}) \left[\frac{1}{1 - e^{-j\theta}} + \pi \text{comb}_{2\pi}(\theta) \right]$$

$$= \frac{X(e^{j\theta})}{1 - e^{-j\theta}} + \pi X(e^{j\theta}) \text{comb}_{2\pi}(\theta)$$

$$= \frac{X(e^{j\theta})}{1 - e^{-j\theta}} + \pi \boxed{X(e^{j0})} \text{comb}_{2\pi}(\theta)$$

area

Exercises

On the discrete-time Fourier transform

The **discrete-time Fourier transform** is evidently closely linked to the Fourier series, with similar properties and signal pairs. However, these slightly differ in their expression, and, moreover, **new properties** are available for the DTFT. Memorize the forward/backward rule and take particular care in evaluating integrals and/or series correctly.

You also need to memorize the **fundamental Fourier pairs**, as these are relevant in what follows.

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Lecture 15

The discrete-time Fourier transform: definition, properties, relation with other transforms

Tomaso Erseghe

Università degli Studi di Padova

15.4 Relations among Fourier transforms

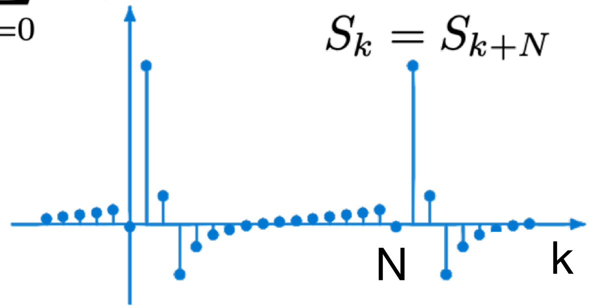
Link between Fourier transform, DTFT, and DFT

- De-periodization property of the DTFT
- Periodic repetition and sampling properties

De-periodisation

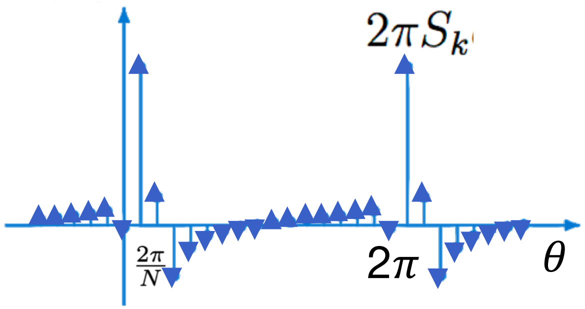
From the DFT to the DTFT

$$s(n) = s(n+N) = \sum_{k=0}^{N-1} S_k e^{jk\frac{2\pi}{N}n}$$



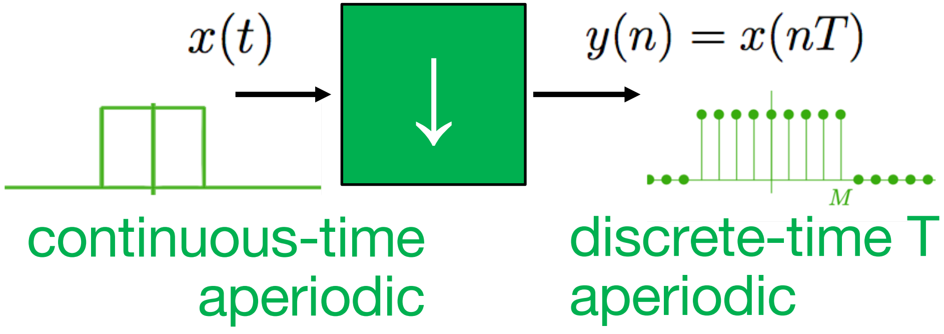
The DTFT of a periodic signal

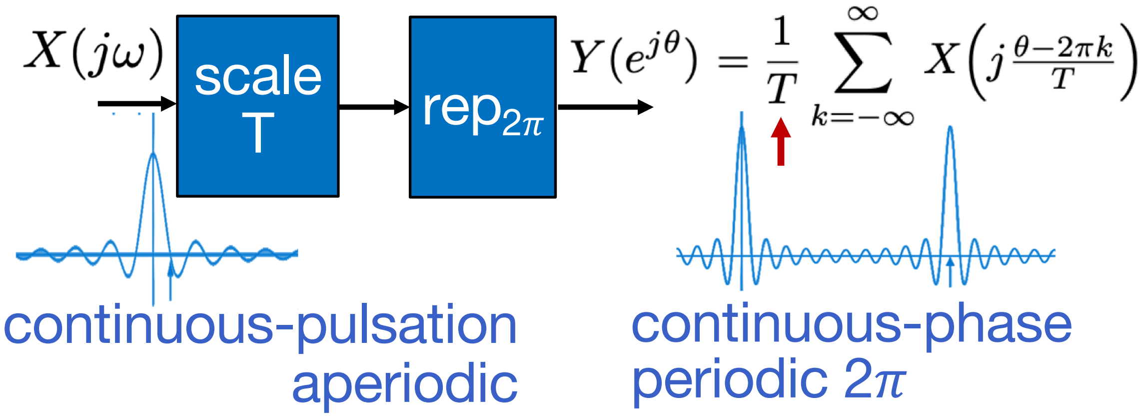
$$\tilde{S}(e^{j\theta}) = \sum_{k=0}^{N-1} 2\pi\, S_k\, \mathrm{comb}_{2\pi}(\theta - k\tfrac{2\pi}{N})$$



Sampling

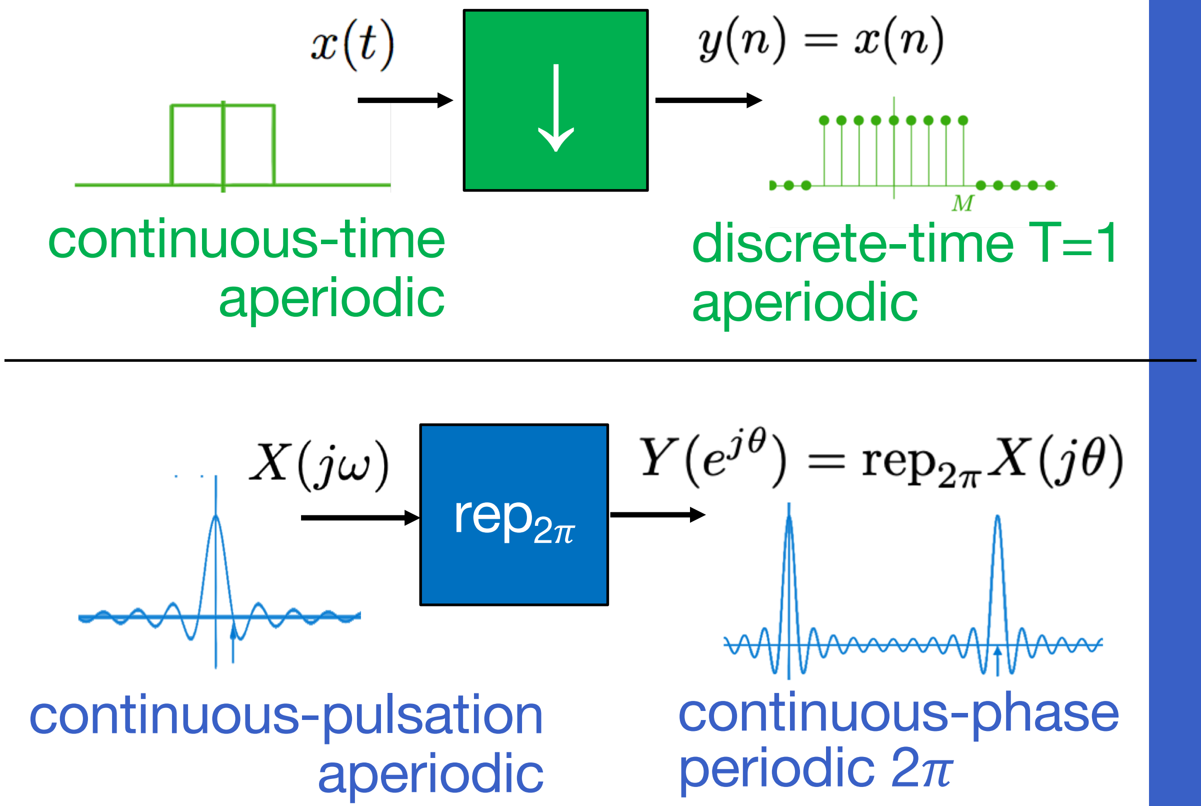
From the Fourier transform to the DTFT





Sampling for T=1

From the Fourier transform to the DTFT



A proof

From the Fourier transform to the DTFT

$$Y(e^{j\theta}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega n T} d\omega \right) e^{-j\theta n}$$

$$= \int_{-\infty}^{\infty} X(j\omega) \left(\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{j(\omega T)n} e^{-j\theta n} \right) d\omega$$

swap

$$= \int_{-\infty}^{\infty} X(j\omega)\, \mathrm{comb}_{2\pi}(\theta - \omega T)\, d\omega$$

$$= \int_{-\infty}^{\infty} X(j\omega) \sum_{k=-\infty}^{\infty} \delta(\theta - \omega T - 2\pi k)\, d\omega$$

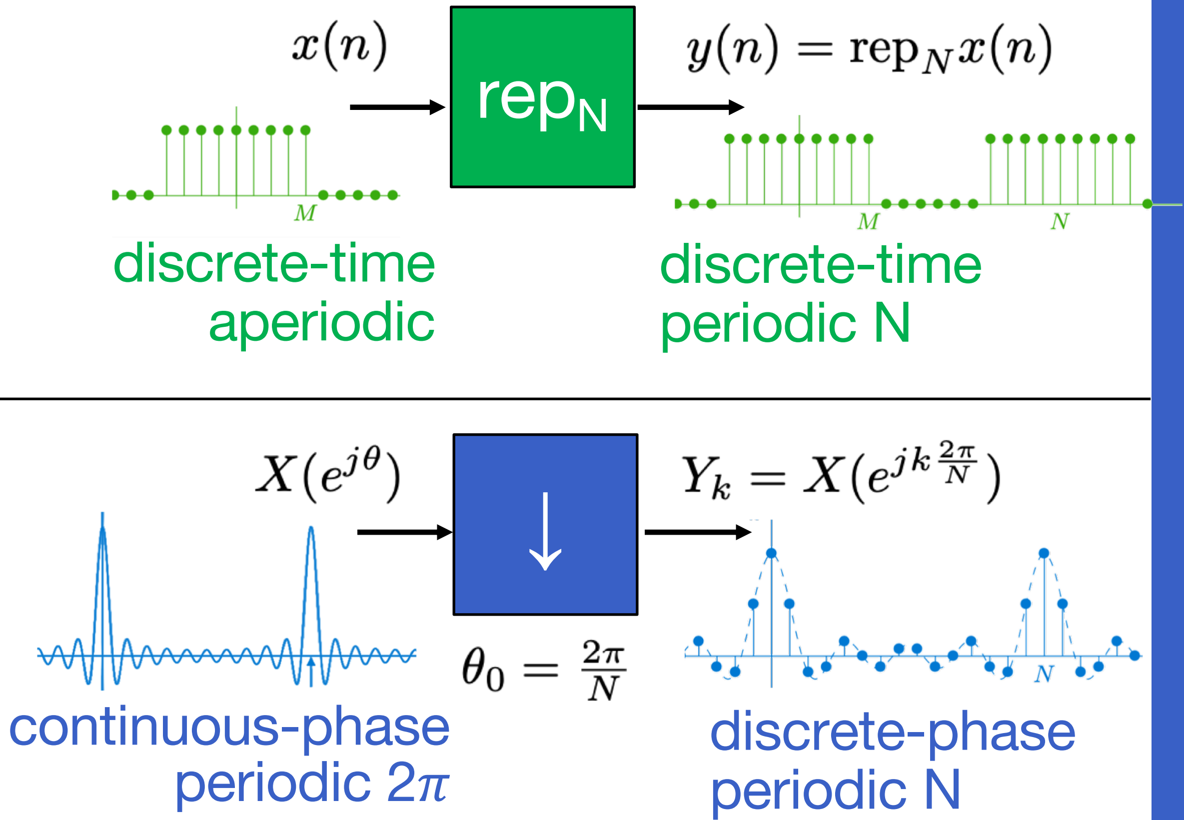
$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X(j\omega)\, \tfrac{1}{T}\, \delta(\tfrac{\theta - 2\pi k}{T} - \omega)\, d\omega$$

swap

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(j\, \tfrac{\theta - 2\pi k}{T}\right)$$

Periodic repetition

From the DTFT to the DFT



A proof

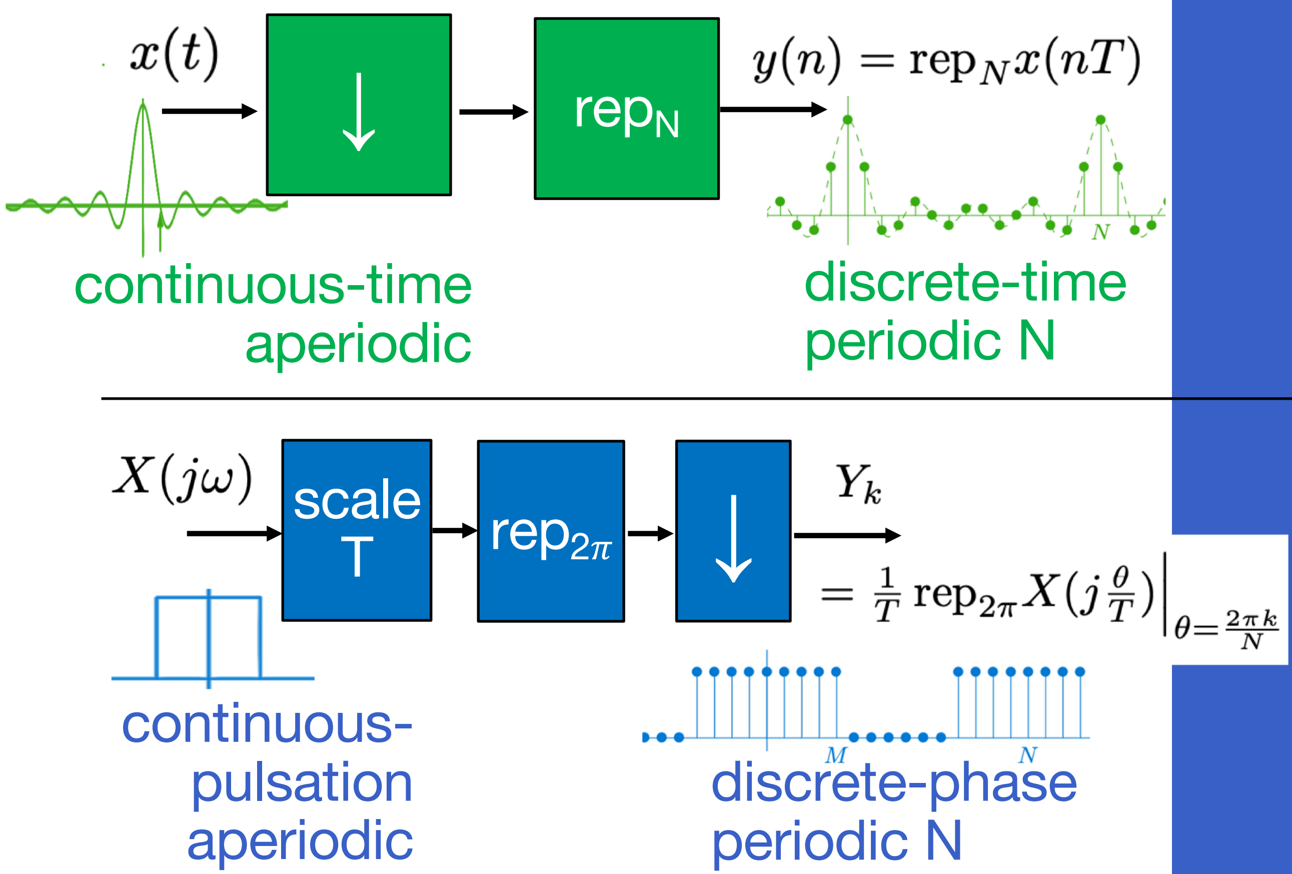
From the DTFT to the DFT

$$
\begin{aligned}
Y_k &= \sum_{n=0}^{N-1} \left(\sum_{m=-\infty}^{\infty} x(n-mN) \right) e^{-jk\frac{2\pi}{N}n} \\
&= \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} x(n-mN)\, e^{-jk\frac{2\pi}{N}(n-mN)} \\
&= \sum_{\ell=-\infty}^{\infty} x(\ell)\, e^{-jk\frac{2\pi}{N}\ell} \\
&= X(e^{jk\frac{2\pi}{N}})
\end{aligned}
$$

$\ell = n - mN$

Combining the two

From the Fourier transform to the DFT



Exercises

On the relations among Fourier transforms

Evaluating the **discrete-time Fourier transform** from a **Fourier transform pair** can be a much easier way of calculating the transforms.

You need to get acquainted with this powerful rule, as it might turn out very useful.

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Lecture 16

Filters in discrete-time and wrap-up un Fourier transforms

Tomaso Erseghe

Università degli Studi di Padova

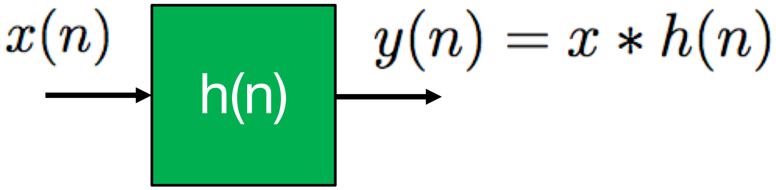
16.1 Filters in discrete-time

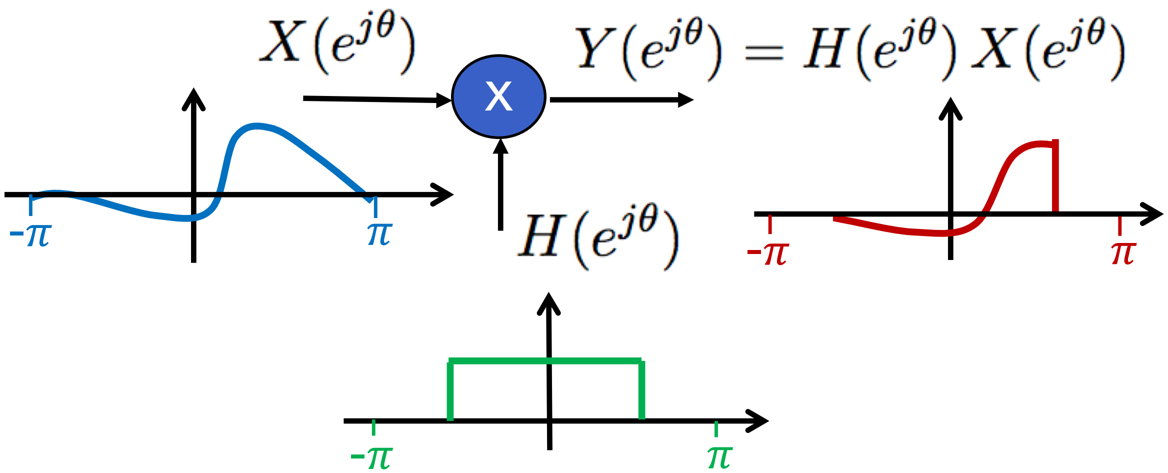
A Fourier transform perspective

- Filters in the Fourier domain
- Selective filters
- Response to sinusoids and exponentials

Filters in the Fourier domain

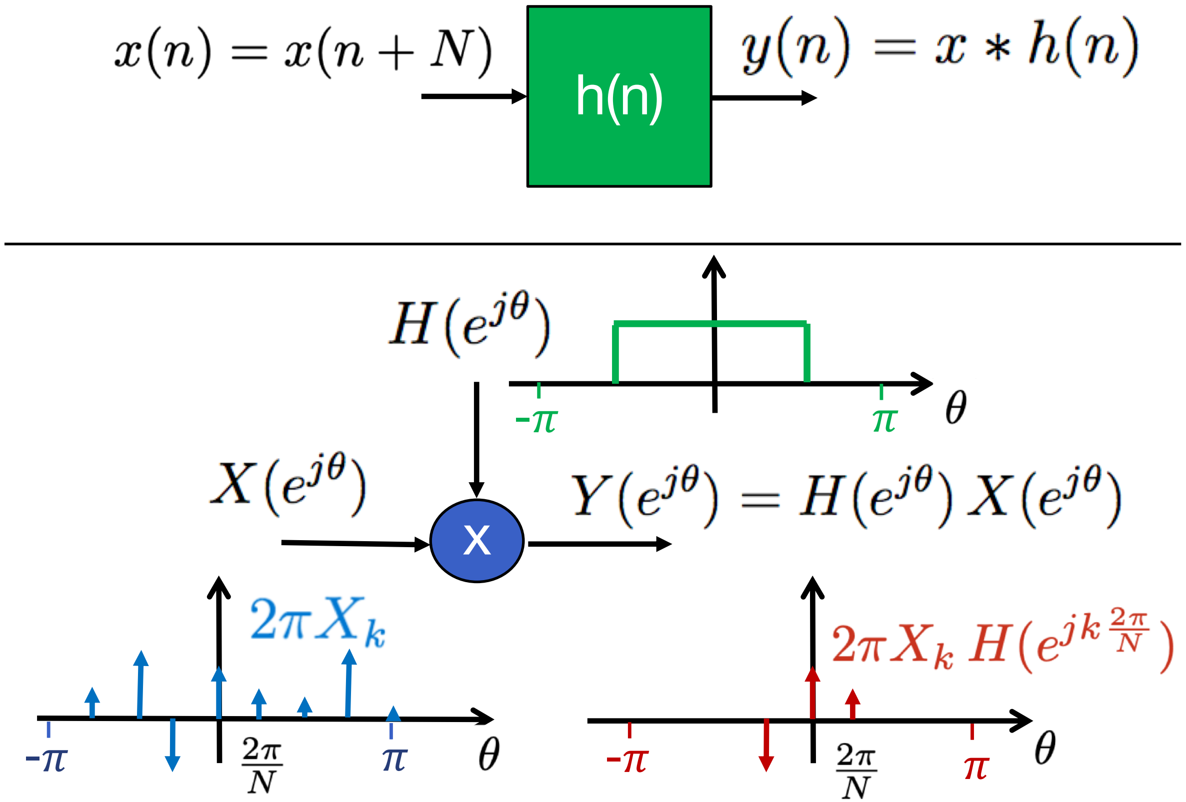
Discrete-time aperiodic



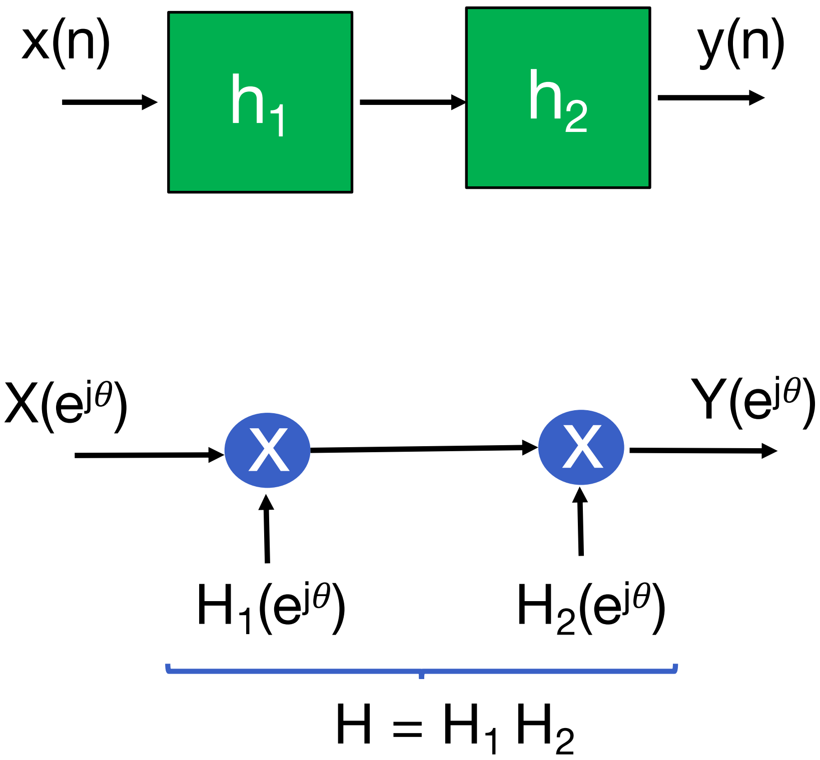


Filters in the Fourier domain

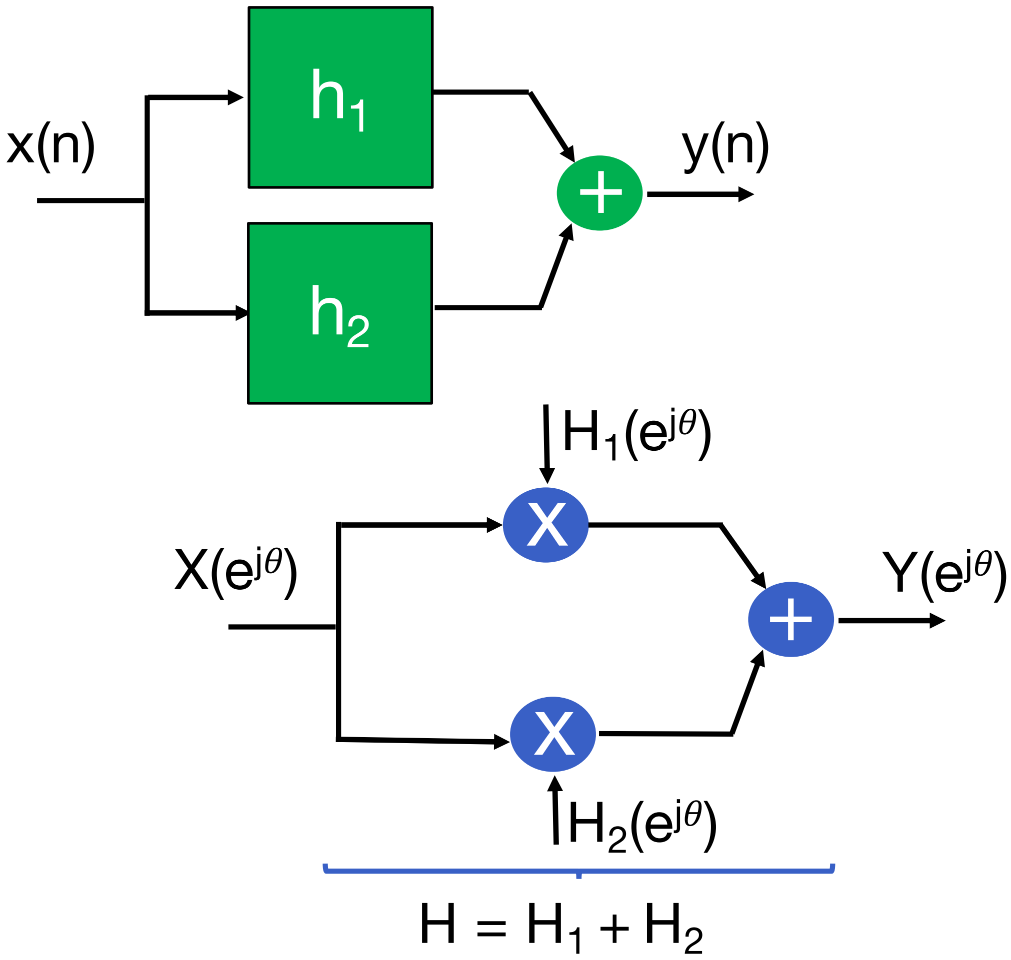
Discrete-time periodic



Series



Parallel

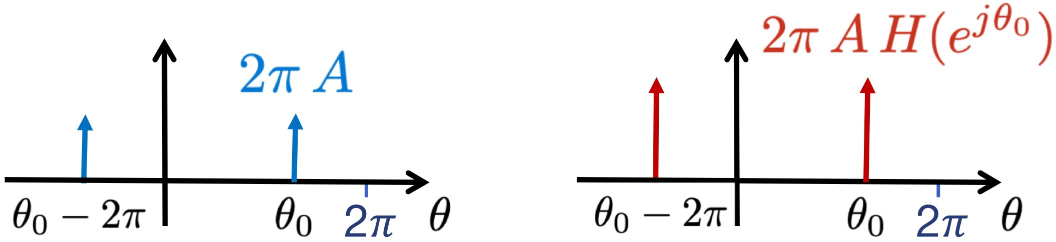


Complex exponentials

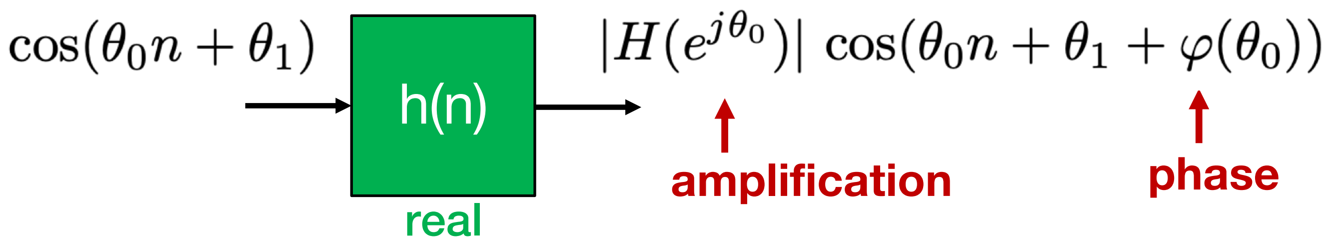
$$x(n) = A\, e^{jn\theta_0} \longrightarrow \boxed{h(n)} \longrightarrow y(n) = A\, H(e^{j\theta_0})\, e^{jn\theta_0}$$

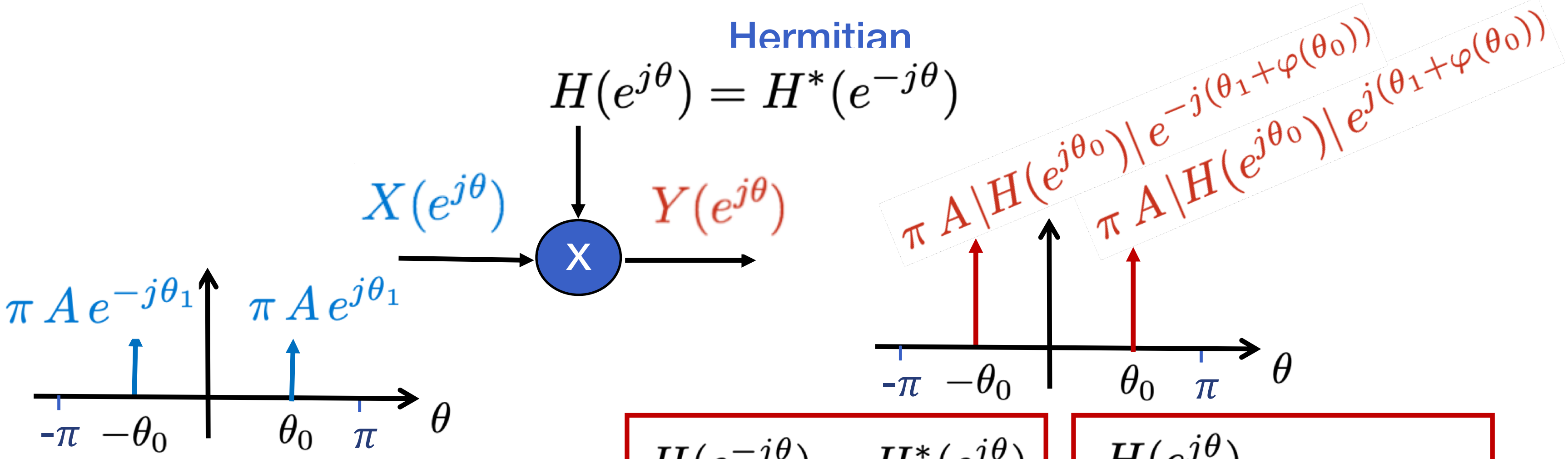
complex constant

$$X(e^{j\theta}) \longrightarrow \underset{\uparrow H(e^{j\theta})}{\otimes} \longrightarrow Y(e^{j\theta}) = H(e^{j\theta})\, X(e^{j\theta})$$



Sinusoids through a real filter





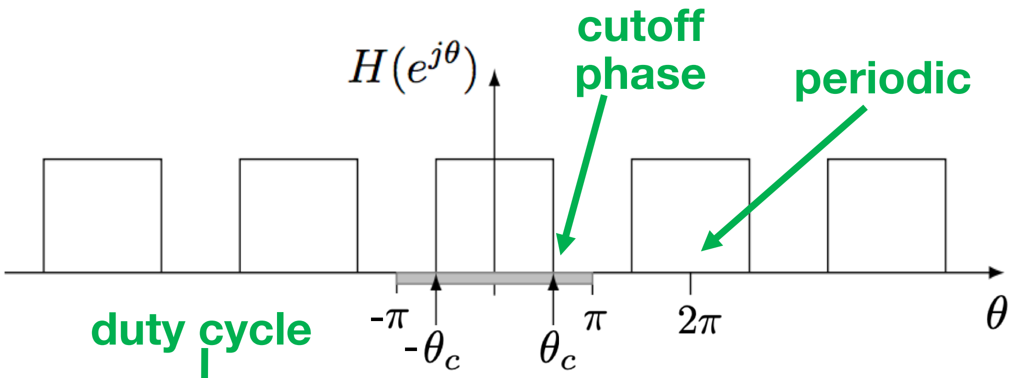
$$\boxed{\begin{aligned} H(e^{-j\theta}) &= H^*(e^{j\theta}) \\ &= |H(e^{j\theta})|\, e^{-j\varphi(\theta)} \end{aligned}}$$

$$\boxed{\begin{aligned} H(e^{j\theta}) & \\ &= |H(e^{j\theta})|\, e^{j\varphi(\theta)} \end{aligned}}$$

Selective (ideal) filters

low-pass

$$H(e^{j\theta}) = \text{rep}_{2\pi}\, \text{rect}\left(\frac{\theta}{2\theta_c}\right)$$



duty cycle

$$h(n) = \frac{\theta_c}{\pi}\, \text{sinc}\left(\frac{\theta_c}{\pi} n\right)$$

non BIBO stable

BIBO stable low-pass

$$H(e^{j\theta}) = \text{rep}_{2\pi}\text{rcos}\left(\frac{\theta}{2\theta_c}\right) \qquad h(n) = \frac{\theta_c}{\pi}\, \text{ircos}\left(\frac{\theta_c}{\pi} n\right)$$

Exercises

On discrete-time filters

Through the **Fourier transform approach** we can ease the understanding and calculation of a filtering operation.

Carry in mind the rule on sampled complex exponentials and sinusoids, as it might turn out to be very useful in many practical cases. Be aware of the fact that now the Fourier transform is periodic, and that a sampled complex exponential with phase θ_0 is equivalent to one with phase $\boldsymbol{\theta_0 + 2\pi k}$

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Lecture 16

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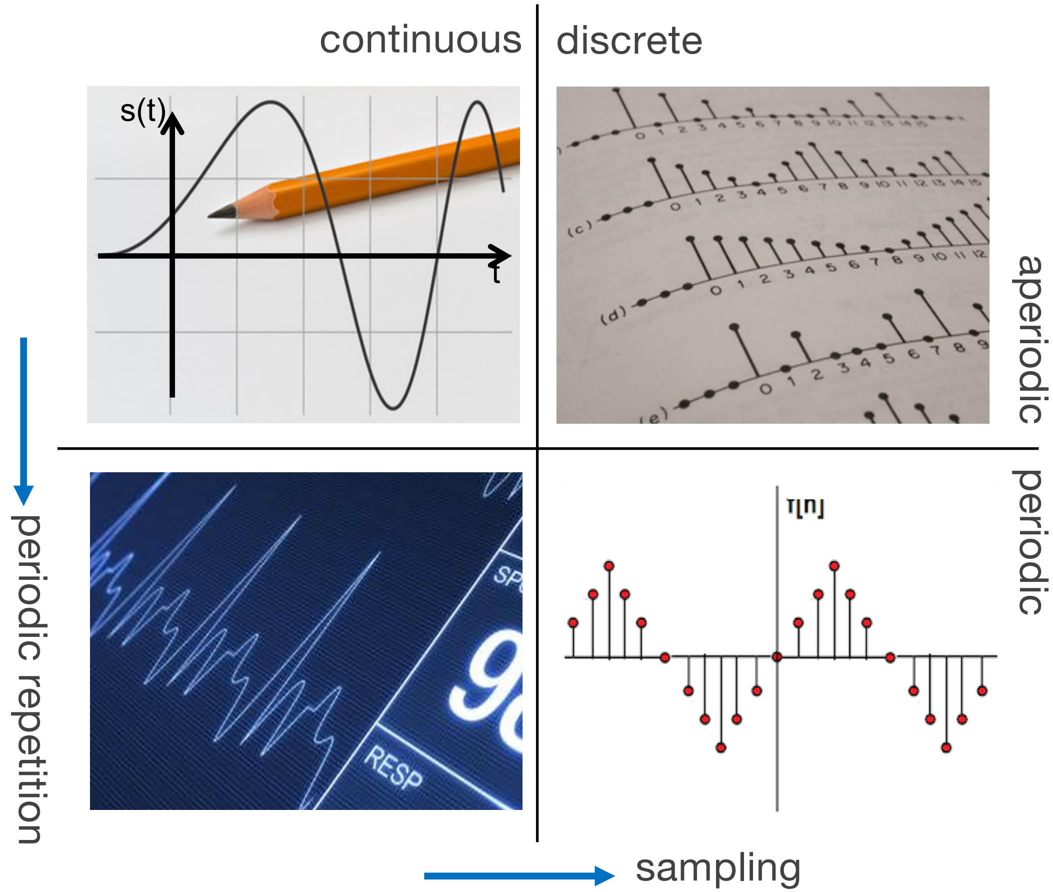
16.4 Wrap-up on Fourier transforms

On their similarity

- Four signal domains, four transforms
- Four versions of each property or Fourier pair

Four classes of signals



Four transforms

continuous-time | **discrete-time**

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\omega)e^{j\omega t}d\omega$$

$$S(j\omega) = \int_{-\infty}^{\infty} \boxed{s(t)e^{-j\omega t}} dt$$

ω t

Fourier transform

$$s(n) = \frac{1}{2\pi}\int_{0}^{2\pi} S(e^{j\theta})e^{j\theta n}d\theta$$

$$S(e^{j\theta}) = \sum_{n=-\infty}^{\infty} \boxed{s(n)e^{-j\theta n}}$$

θ n

DTFT

time aperiodic

$$\omega_0 = \frac{2\pi}{T_p}$$

Fourier series

$$s(t) = \sum_{k=-\infty}^{\infty} S_k e^{jk\omega_0 t}$$

$$S_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} \boxed{s(t)e^{-jk\omega_0 t}} dt$$

$k\omega_0$ t

DFT

$$s(n) = \sum_{k=k_0}^{k_0+N-1} S_k\, e^{jk\frac{2\pi}{N}n}$$

$$S_k = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} \boxed{s(n)\, e^{-jk\frac{2\pi}{N}n}}$$

$k\,\frac{2\pi}{N}$ n

time periodic

Four modulation properties

	continuous-time	discrete-time	
	$x(t-t_0) \quad X(j\omega)e^{-j\omega t_0}$	$x(n-n_0) \quad X(e^{j\theta})e^{-j\theta n_0}$	time aperiodic
	$x(t)\boxed{e^{j\omega_0 t}} \quad X(j\omega - j\omega_0)$ ω_0 t	$x(n)\boxed{e^{j\theta_0 n}} \quad X(e^{j(\theta-\theta_0)})$ θ_0 n	
	Fourier transform	DTFT	
$\omega_0 = \frac{2\pi}{T_p}$	Fourier series	DFT	time periodic
	$x(t-t_1) \quad X_k e^{-jk\omega_0 t_1}$	$x(n-n_1) \quad X_k e^{-jk\frac{2\pi}{N}n_1}$	
	$x(t)\boxed{e^{jm\omega_0 t}} \quad X_{k-m}$ mω_0 t	$x(n)\boxed{e^{jm\frac{2\pi}{N}n}} \quad X_{k-m}$ m $\frac{2\pi}{N}$ n	

Four convolution properties

	continuous-time	discrete-time	
	$x * y(t)$ → $X(j\omega)Y(j\omega)$ $x(t)y(t)$ → $\frac{1}{2\pi} X * Y(j\omega)$ Fourier transform	$x * y(n)$ → $X(e^{j\theta})Y(e^{j\theta})$ $x(n)y(n)$ → $\frac{1}{2\pi} X *_{\text{cir}} Y(e^{j\theta})$ DTFT	time aperiodic
	Fourier series $\omega_0 = \frac{2\pi}{T_p}$ $x *_{\text{cir}} y(t)$ → $T_p X_k Y_k$ $x(t)y(t)$ → $X_k * Y_k$	DFT $x *_{\text{cir}} y(n)$ → $N X_k Y_k$ $x(n)y(n)$ → $X *_{\text{circ}} Y_k$	time periodic

Four derivative properties

	continuous-time	discrete-time	
Fourier transform / **DTFT**	$x'(t)$ $j\omega \cdot X(j\omega)$ $tx(t)$ $jX'(j\omega)$	$x(n) - x(n-1)$ $X(e^{j\theta})\,(1 - e^{-j\theta})$ $nx(n)$ $jX'(e^{j\theta})$	time aperiodic
Fourier series / **DFT**	$\omega_0 = \frac{2\pi}{T_p}$ $x'(t)$ $X_k \cdot jk\omega_0$	$x(n) - x(n-1)$ $X_k \cdot (1 - e^{-jk\frac{2\pi}{N}})$	time periodic

Four area/energy properties

continuous-time	discrete-time
$A_x = S(j0)$ $E_x = \frac{1}{2\pi} E_X$	$A_s = S(e^{j0})$ $E_s = P_S$
Fourier transform	DTFT
$\omega_0 = \frac{2\pi}{T_p}$ Fourier series	DFT
$m_x = X_0$ $P_x = E_X$	$m_x = X_0$ $P_x = E_X(N)$

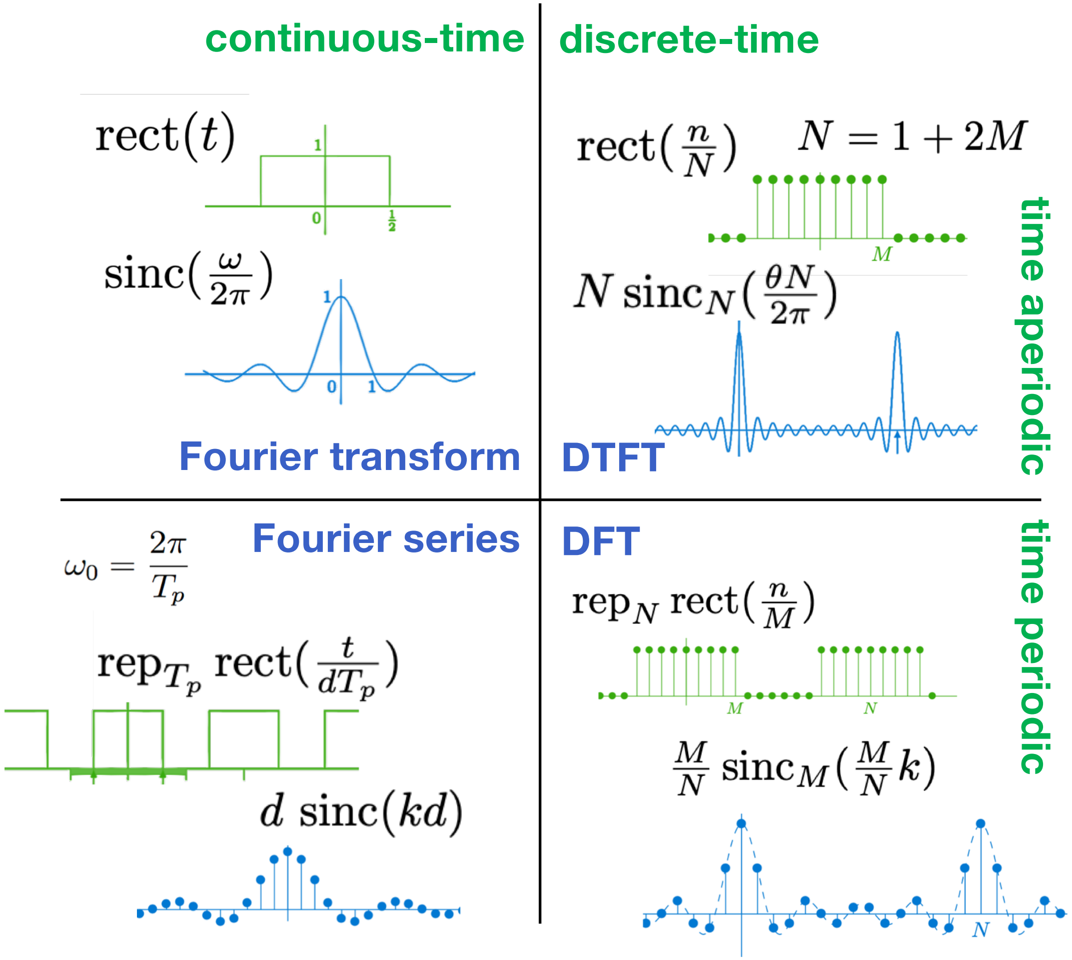
time aperiodic

time periodic

Four delta transforms

	continuous-time	discrete-time	
time aperiodic	$e^{j\omega_1 t}$ $\quad 2\pi\,\delta(\omega-\omega_1)$ $\delta(t-t_1)$ $\quad \boxed{e^{-j\omega t_1}}$ **Fourier transform**	$e^{j\theta_0 n}$ $\quad 2\pi\,\mathrm{comb}_{2\pi}(\theta-\theta_0)$ $\delta(n-m)$ $\quad \boxed{e^{-j\theta m}}$ **DTFT**	time aperiodic
time periodic	**Fourier series** $\omega_0 = \frac{2\pi}{T_p}$ $e^{jm\omega_0 t}$ $\quad \delta(k-m)$ $\mathrm{rep}_{T_p}\delta(t-t_1)$ $\quad \frac{1}{T_p}\boxed{e^{-jk\omega_0 t_1}}$	**DFT** $e^{jm\frac{2\pi}{N}n}$ $\quad \mathrm{rep}_N\delta(k-m)$ $\mathrm{rep}_N\delta(n-m)$ $\quad \frac{1}{N}\boxed{e^{-k\frac{2\pi}{N}m}}$	time periodic

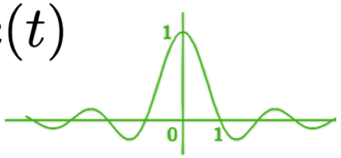
Four rect transforms



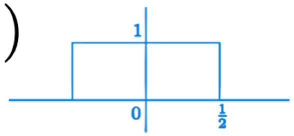
Four sinc transforms

continuous-time | **discrete-time**

$\operatorname{sinc}(t)$

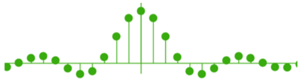


$\operatorname{rect}\left(\frac{\omega}{2\pi}\right)$

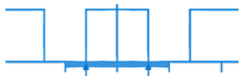


Fourier transform

$d \operatorname{sinc}(nd)$



$\operatorname{rep}_{2\pi} \operatorname{rect}\left(\frac{\theta}{2\pi d}\right)$



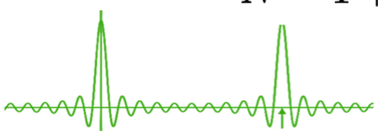
DTFT

time aperiodic

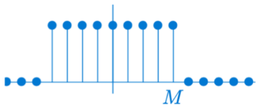
Fourier series

$\omega_0 = \frac{2\pi}{T_p}$

$N \operatorname{sinc}_N\left(\frac{Nt}{T_p}\right) \quad N = 1 + 2M$

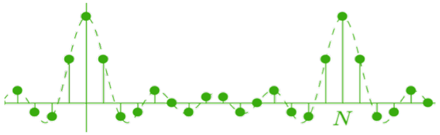


$\operatorname{rect}\left(\frac{k}{N}\right)$



DFT

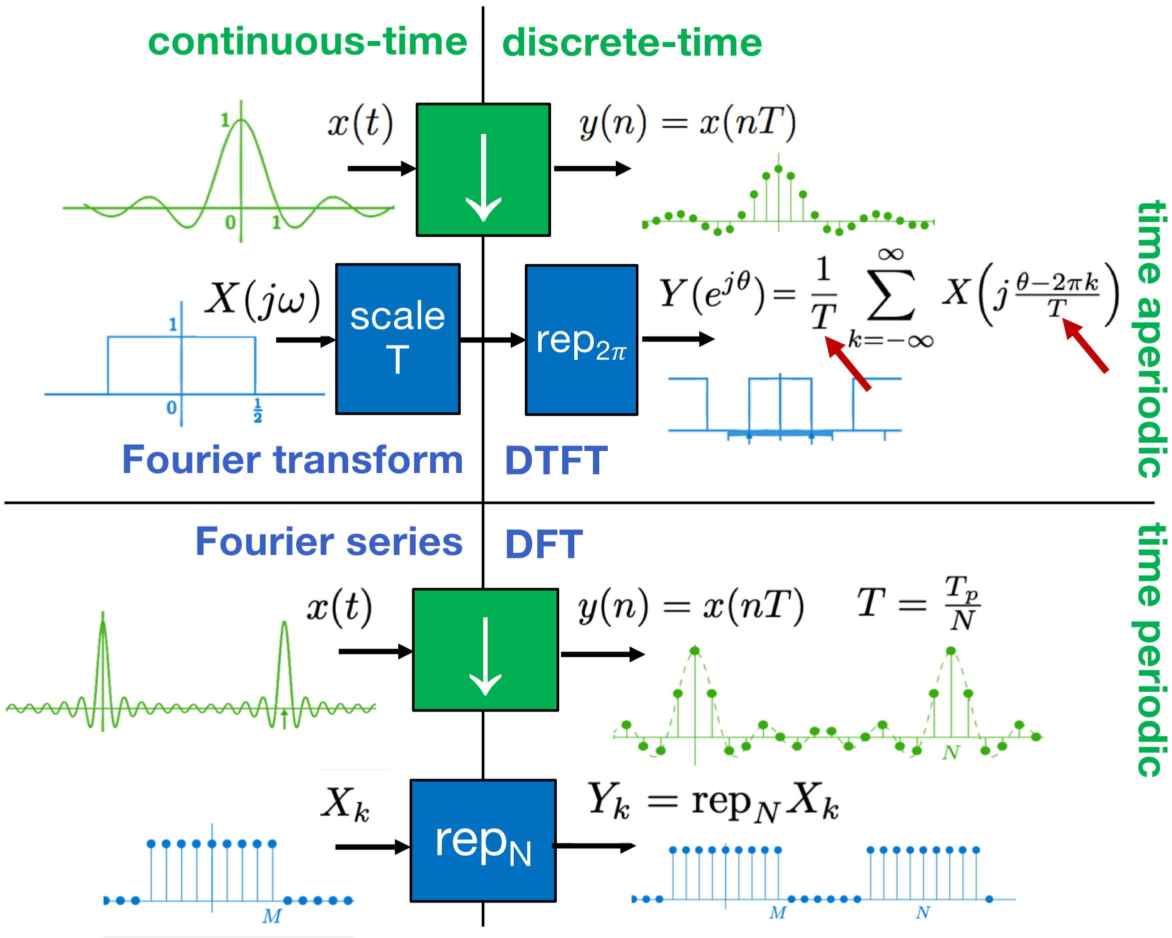
$M \operatorname{sinc}_M\left(\frac{M}{N} n\right)$



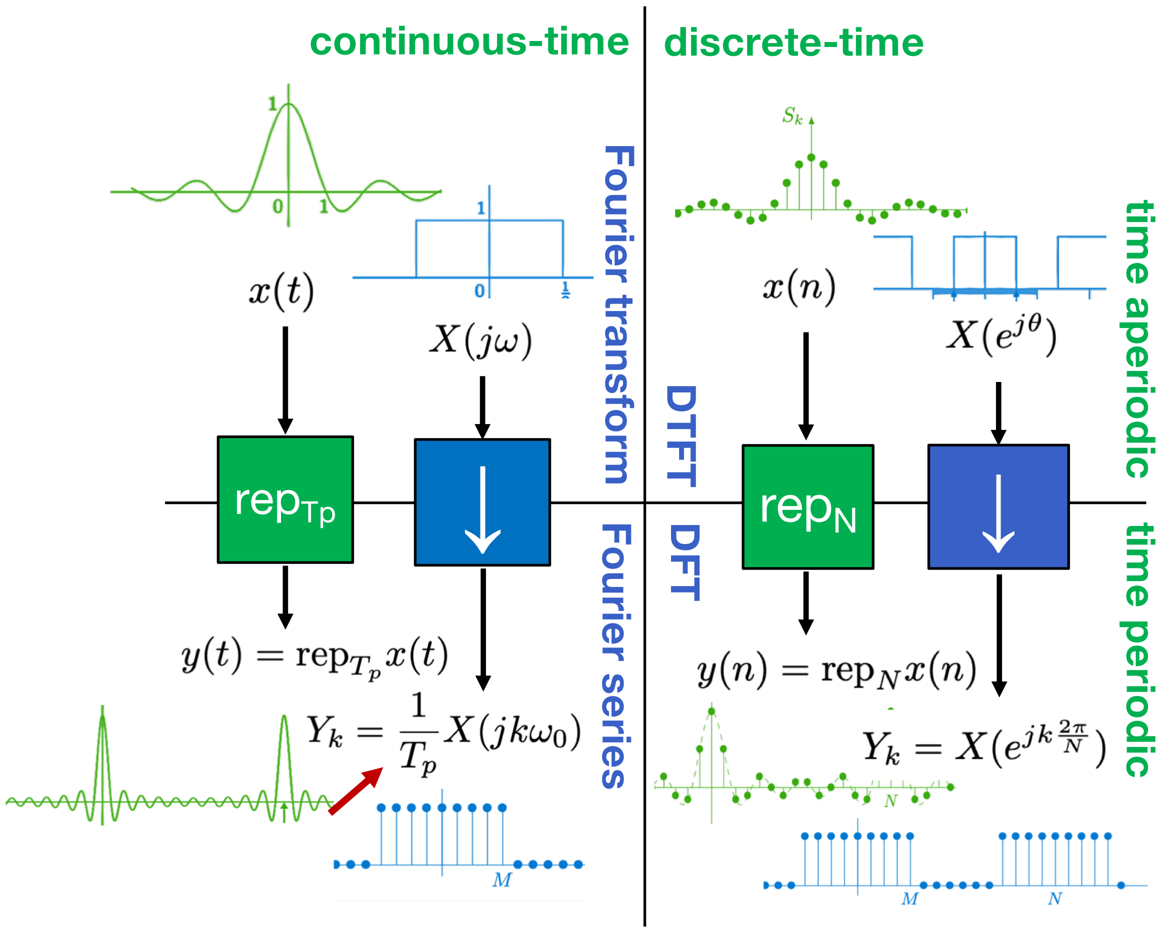
$\operatorname{rep}_N \operatorname{rect}\left(\frac{k}{M}\right)$

time periodic

Four links: 1) sampling



Four links: 2) periodic rep



Exercises

On all the Fourier transforms

Try memorizing the similarities among **Fourier transforms** at your best, since this can ease your understanding.

Solve some review exercises to check your comprehension level.

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Lecture 17

Interpolation and Shannon's Sampling Theorem

Tomaso Erseghe

Università degli Studi di Padova

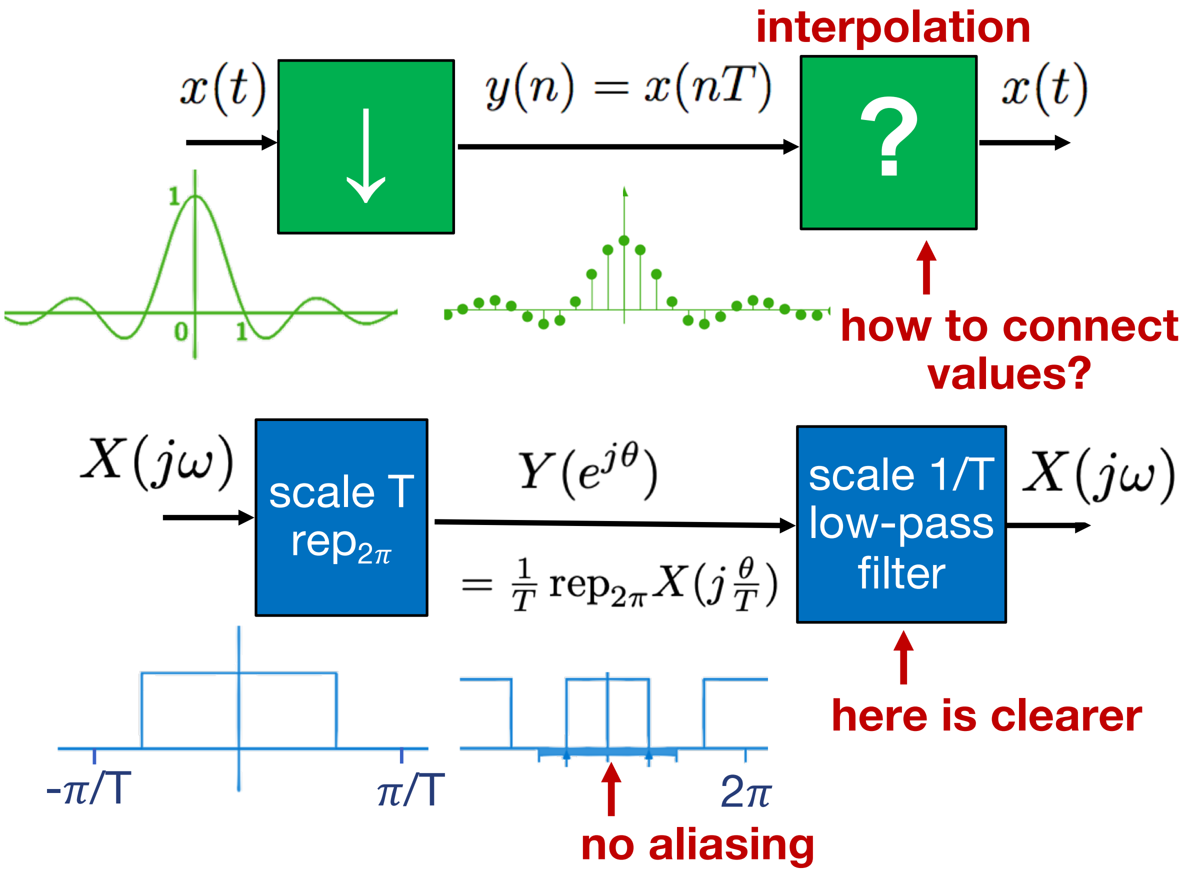
17.1 Interpolation

A time-domain perspective

- The need for interpolation
- Interpolation in the time-domain
- Correct interpolation property

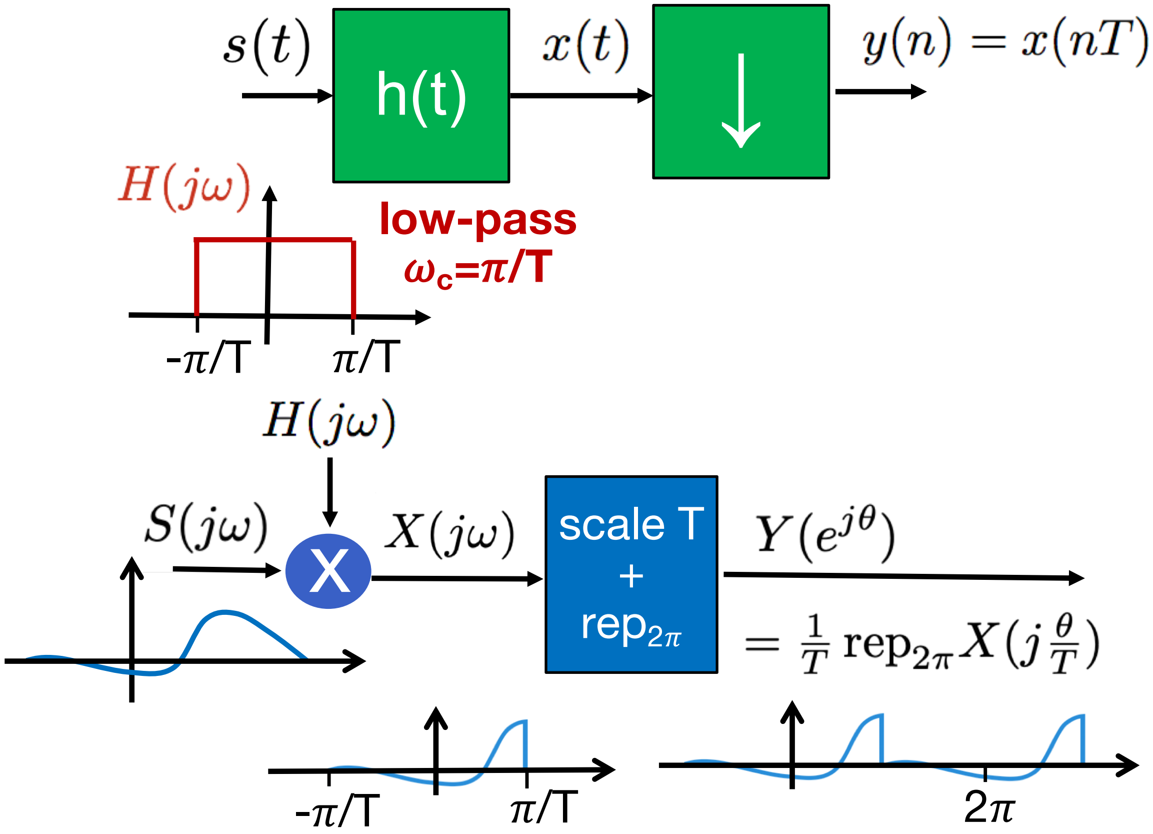
The need for interpolation

How to reconstruct a signal?



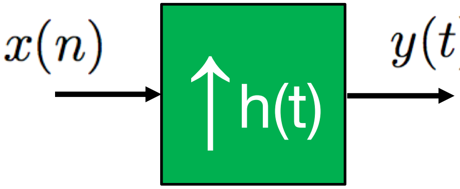
Pre-filtering

Real-world samplers avoid aliasing

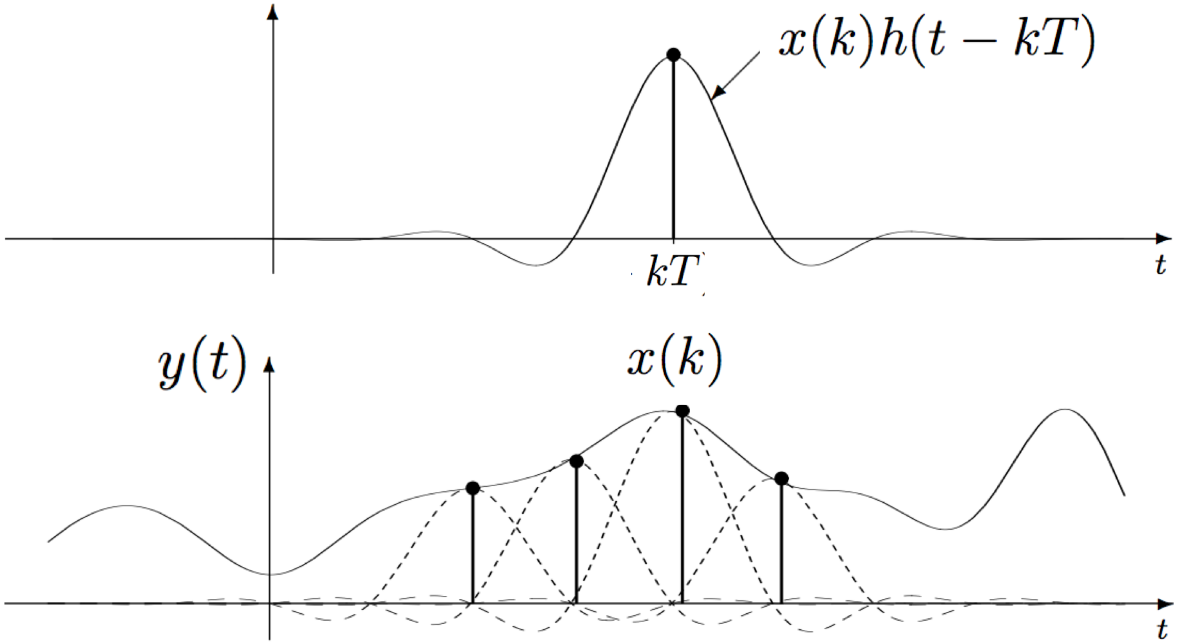


Interpolation filter

To connect samples



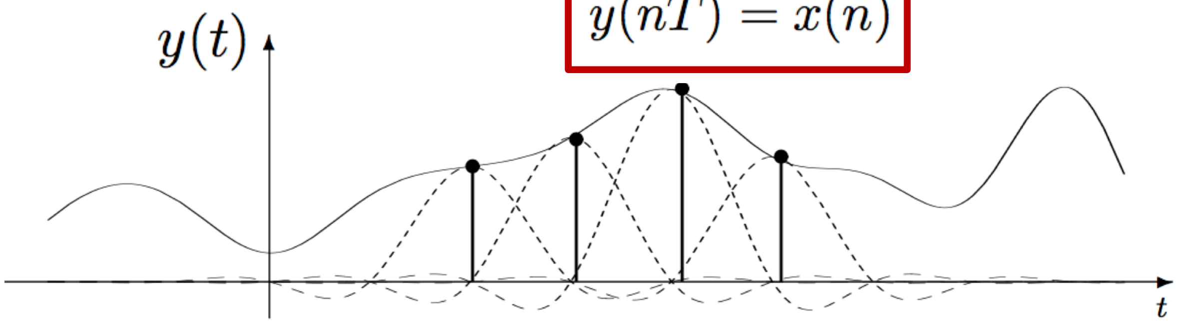
$$y(t) = \sum_{k=-\infty}^{\infty} x(k)h(t-kT)$$



Correct interpolation

To really connect samples

$$\boxed{y(nT) = x(n)}$$



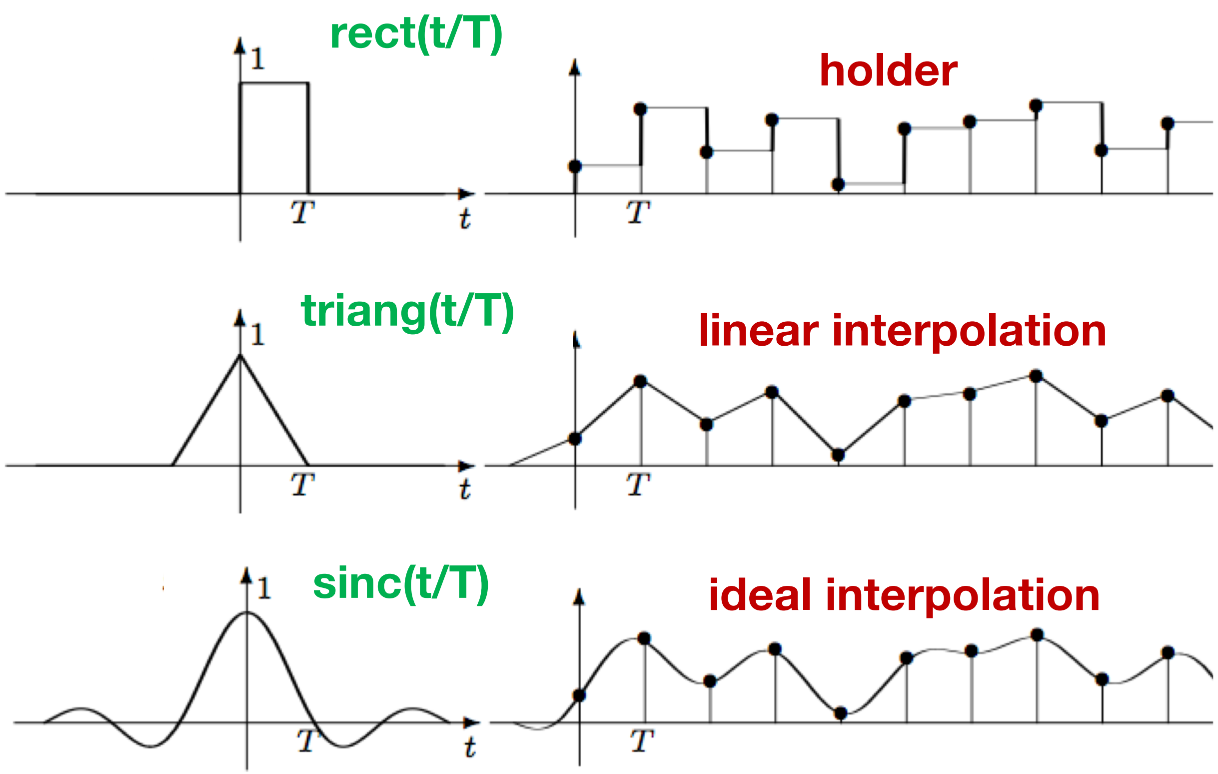
$$y(nT) = \sum_k x(k)h(nT - kT)$$

$$= x(n) * h(nT) = x(n)$$

$$\boxed{h(nT) = \delta(n)}$$

Filter examples

Satisfying the correct interpolation



Correct interpolation

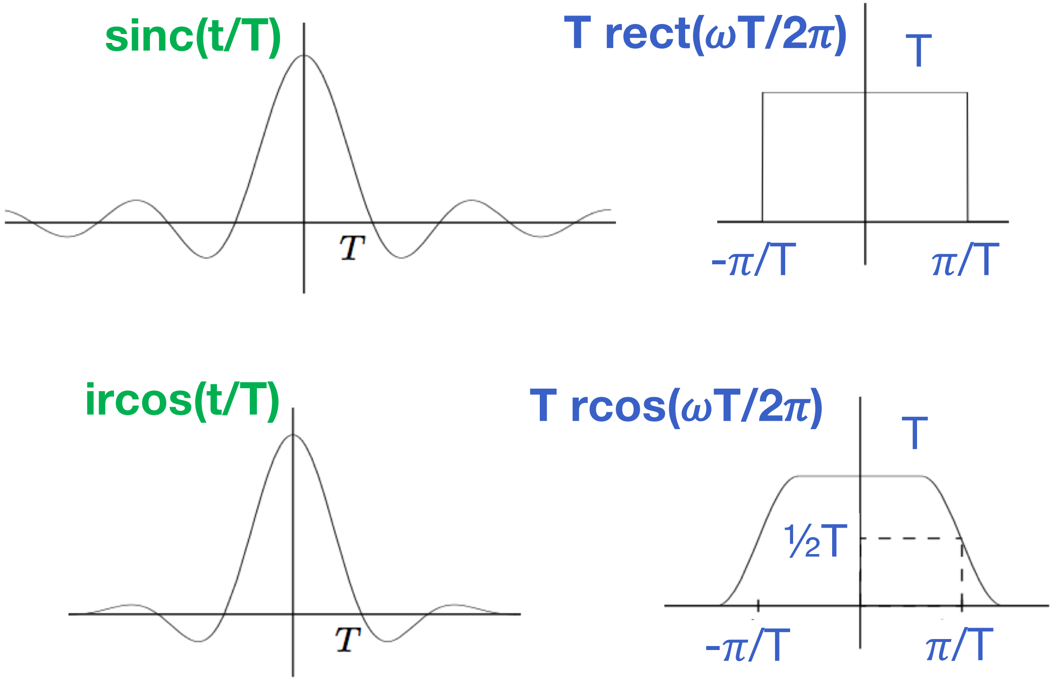
In the Fourier domain

$$
\begin{aligned}
\delta(n) = h(nT) \quad \implies \quad 1 &= \tfrac{1}{T}\,\mathrm{rep}_{2\pi} H(j\tfrac{\theta}{T}) \\
&= \tfrac{1}{T} \sum_{k=-\infty}^{\infty} H(j\tfrac{\theta - 2\pi k}{T}) \\
&= \tfrac{1}{T} \sum_{k=-\infty}^{\infty} H(j(\tfrac{\theta}{T} - \tfrac{2\pi}{T}k)) \\
&= \tfrac{1}{T}\,\mathrm{rep}_{2\pi/T} H(j\omega)\Big|_{\omega = \frac{\theta}{T}}
\end{aligned}
$$

$$\boxed{\mathrm{rep}_{2\pi/T} H(j\omega) = T}$$

Filter examples

Satisfying the correct interpolation



No exercises ☺

On interpolation in the time-domain

We need to introduce further concepts in the Fourier domain before being able to fully manage interpolation

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Lecture 17

Interpolation and Shannon's Sampling Theorem

Tomaso Erseghe

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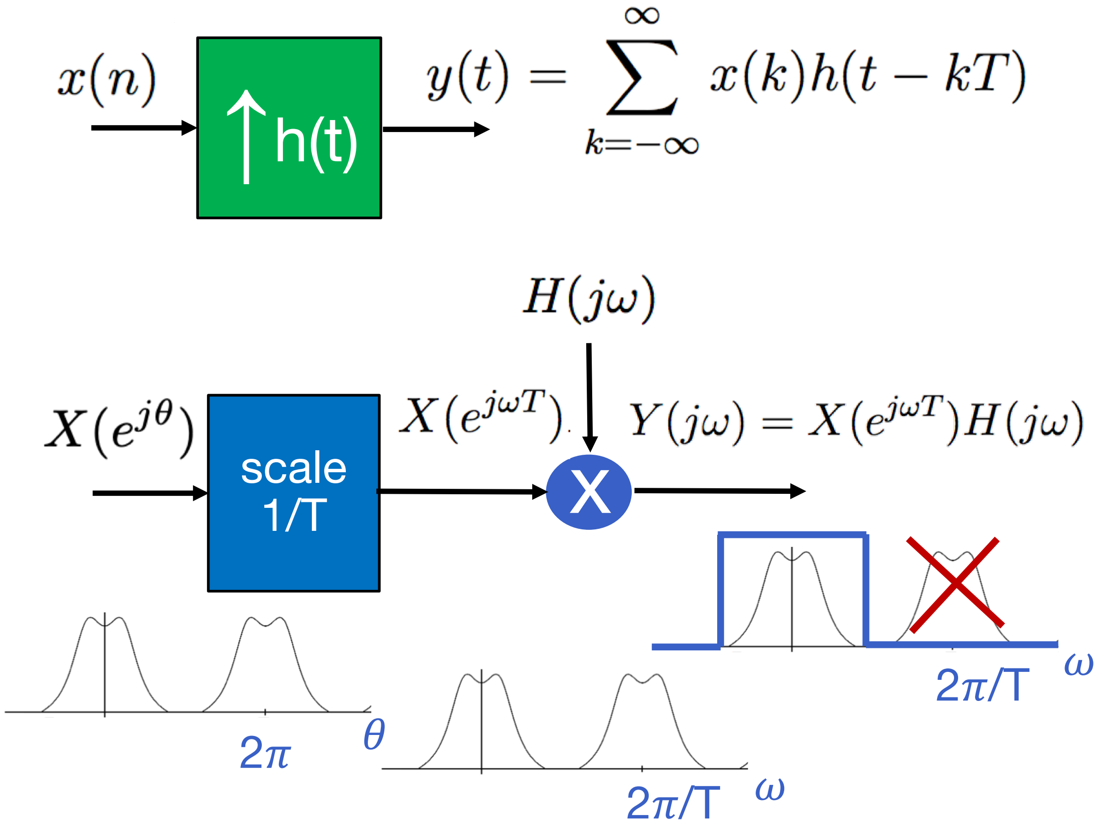
17.2 Shannon's Sampling Theorem

An application of interpolation

- Interpolation in the Fourier domain
- Series of sampling and interpolation
- Shannon's Sampling Theorem

Interpolation

In the Fourier domain



A proof

$$Y(j\omega) = \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x(k)h(t-kT) \right) e^{-j\omega t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} x(k) \left(\int_{-\infty}^{\infty} h(t-kT)e^{-j\omega t}\, dt \right)$$

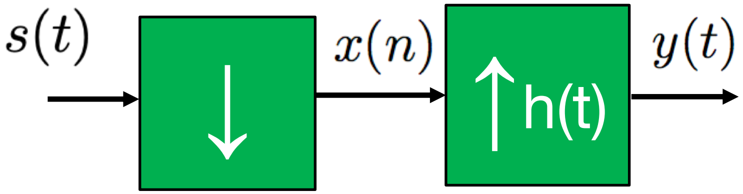
swap

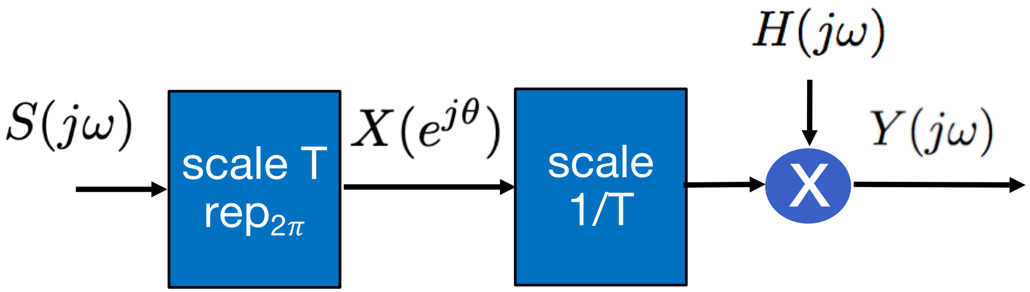
$$= \sum_{k=-\infty}^{\infty} x(k)H(j\omega)e^{-j\omega kT}$$

$$= X(e^{j\omega T})H(j\omega)$$

Sampling + interpolation

In the Fourier domain

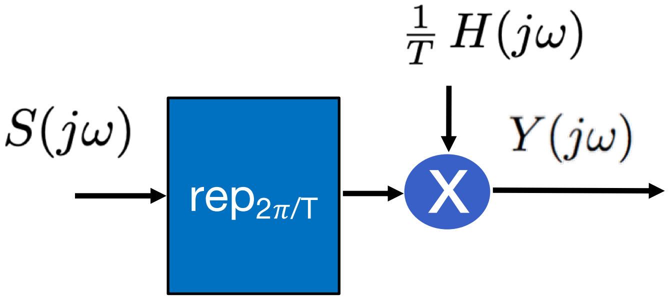




$$X(e^{j\theta}) = \frac{1}{T}\,\mathrm{rep}_{2\pi} S(j\tfrac{\theta}{T})$$

$$Y(j\omega) = X(e^{j\omega T})H(j\omega)$$

Unveiling the series



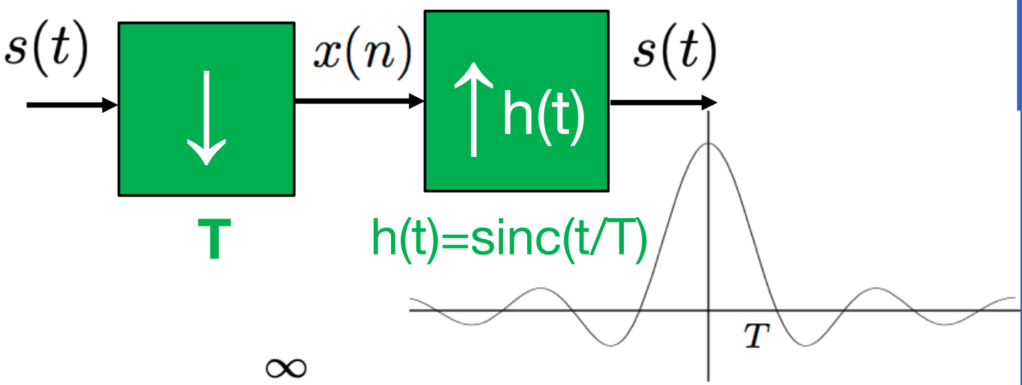
$$
\begin{aligned}
X(e^{j\theta}) &= \tfrac{1}{T}\,\text{rep}_{2\pi} S(j\tfrac{\theta}{T}) \\
&= \tfrac{1}{T} \sum_{k=-\infty}^{\infty} S(j(\tfrac{\theta}{T} - \tfrac{2\pi}{T}k))
\end{aligned}
$$

$$
\begin{aligned}
Y(j\omega) &= X(e^{j\omega T})\,H(j\omega) \\
&= \tfrac{1}{T}\,H(j\omega) \sum_{k=-\infty}^{\infty} S(j(\omega - \tfrac{2\pi}{T}k))
\end{aligned}
$$

Sampling Theorem

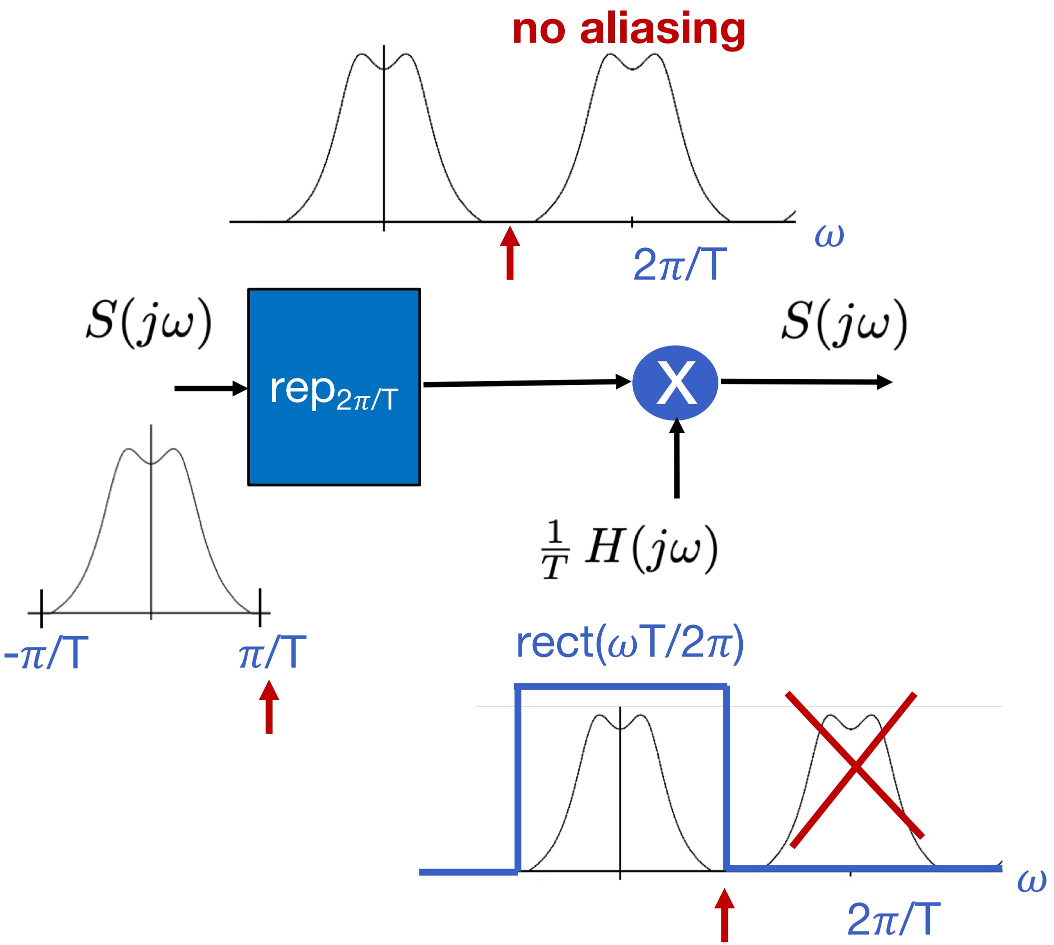
Shannon's base-band version

A continuous-time signal s(t) can be **perfectly recovered** from its samples x(n) = s(nT), with **sampling spacing T**, under the condition that its Fourier transform S(jω) has an **extension e(S) contained in the interval [-π/T, π/T]**



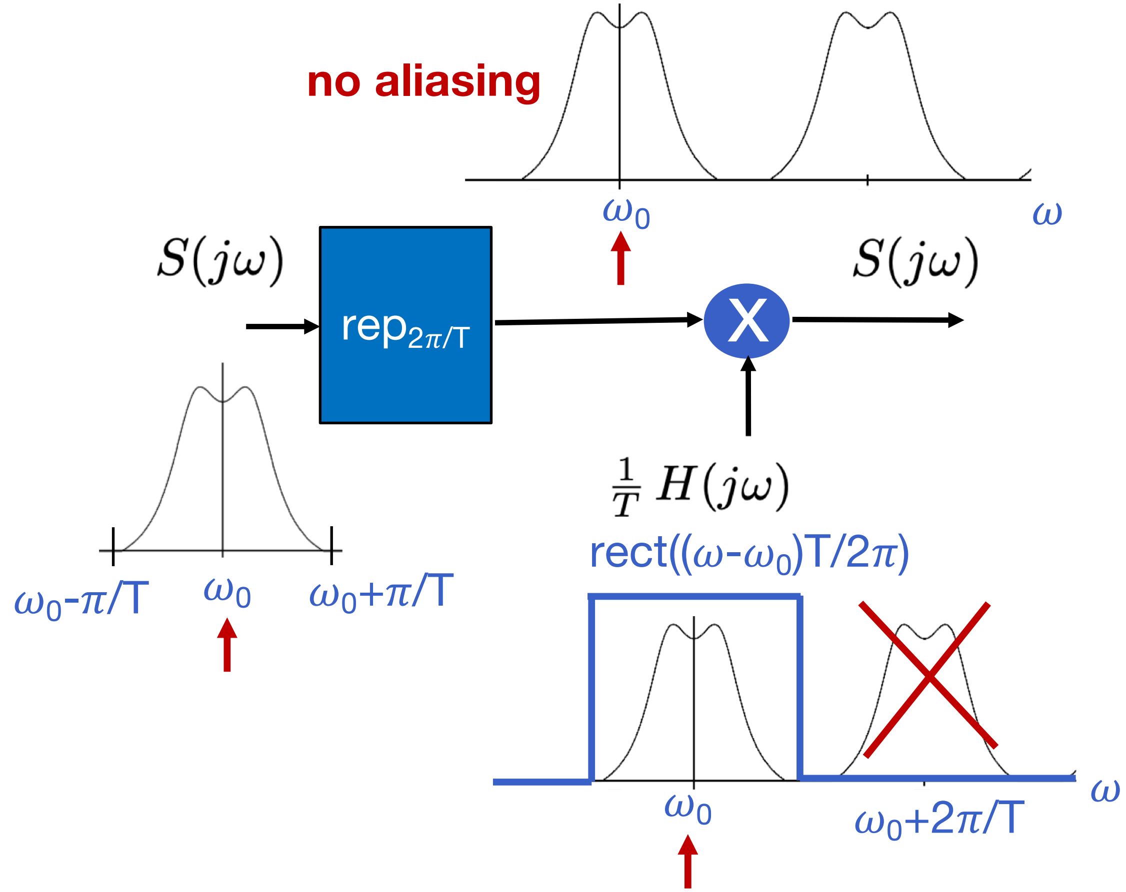
$$s(t) = \sum_{n=-\infty}^{\infty} s(nT)\ \mathrm{sinc}(\tfrac{t-nT}{T})$$

A proof



Generalisation

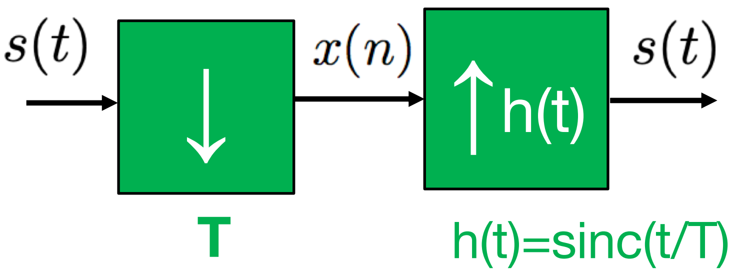
To pass-band signals



Sampling Theorem

Its pass-band version

A continuous-time signal s(t) can be **perfectly recovered** from its samples $x(n) = s(nT)$, with **sampling spacing T**, under the condition that its Fourier transform $S(j\omega)$ has an **extension e(S) contained in the interval $[\omega_0 - \pi/T,\ \omega_0 + \pi/T]$**



$h(t) = \mathrm{sinc}(t/T)\exp(j\omega_0 t)$

Exercises

On sampling and interpolation

Shannon's Sampling Theorem and the general concept of **sampling and interpolation** are key tools in signal processing that must be well understood.

Test your comprehension level with the exercises proposed.

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Lecture 18

Fourier transforms in MatLab

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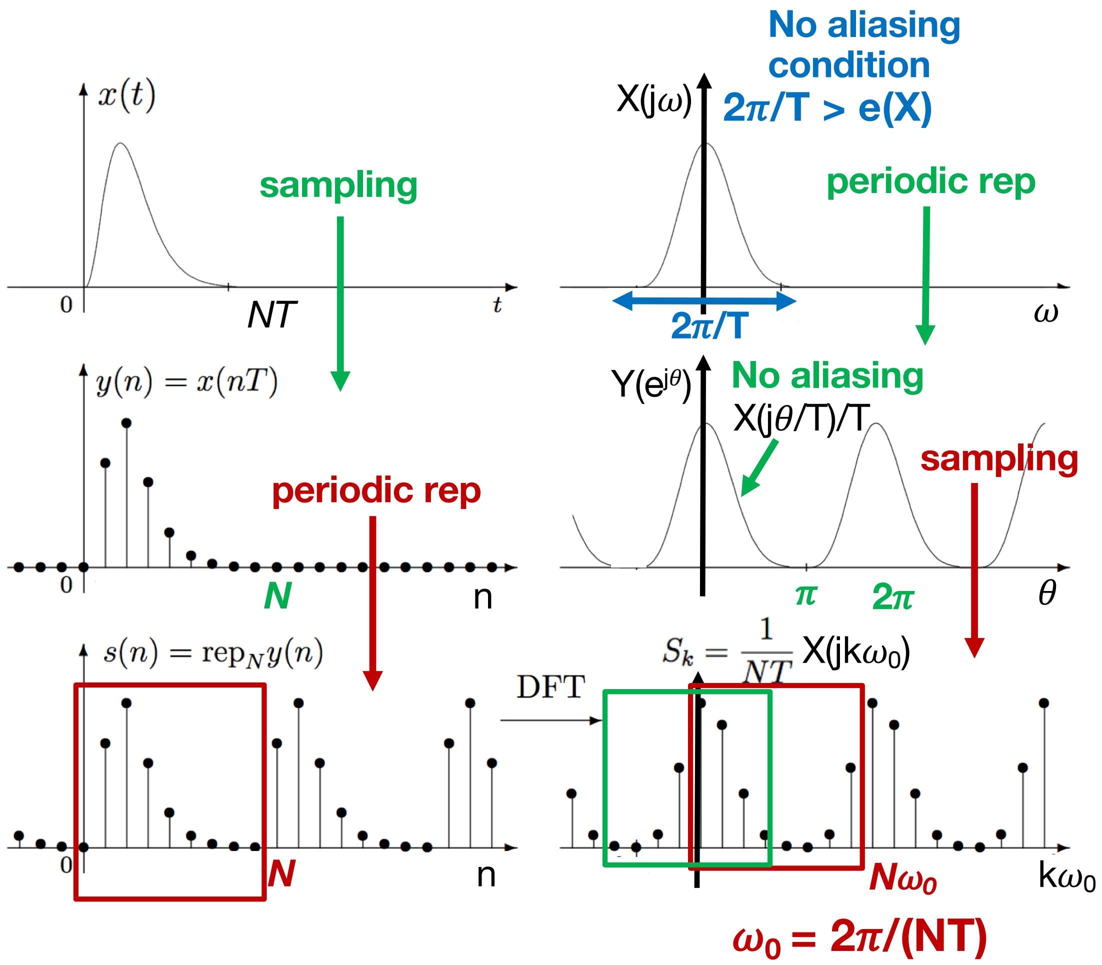
18.1 The Fourier transform in MatLab

An overview

- Sampled signals and DFT
- The fft and fftshift functions

The Fourier transform



MatLab fft function

From time to Fourier samples

```
x % signal samples

T % sampling spacing

N = length(x); % samples length

t = (0:N-1)*T; % time samples

X = ifftshift(T*fft(x)); % Fourier samples

ω = (-round((N-1)/2):round(N/2)-1) *2*pi/(N*T);
                                    % pulsations (in a period)
```

-N/2: N/2-1
N even

-(N-1)/2: (N-1)/2
N odd

Corrections for time-samples

Using the time-shift dual

```
x % signal samples

T % sampling spacing

N = length(x); % samples length

t = (0:N-1)*T + t0; % time samples starting at t0

X = ifftshift(T*fft(x)); % Fourier samples

ω = (-round((N-1)/2):round(N/2)-1) *2*pi/(N*T);
                                    % pulsations (in a period)

X = X .* exp(-1j*ω*t(1)); % Modulation effect
```

Exercises

On the Fourier transform in MatLab

Get acquainted with MatLab Fourier operators **fft** and **fftshift** and learn how to correctly calculate Fourier transforms.

Check that you get the analytical expression of the Fourier transform for known signal couples.

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Lecture 18

Fourier transforms in MatLab

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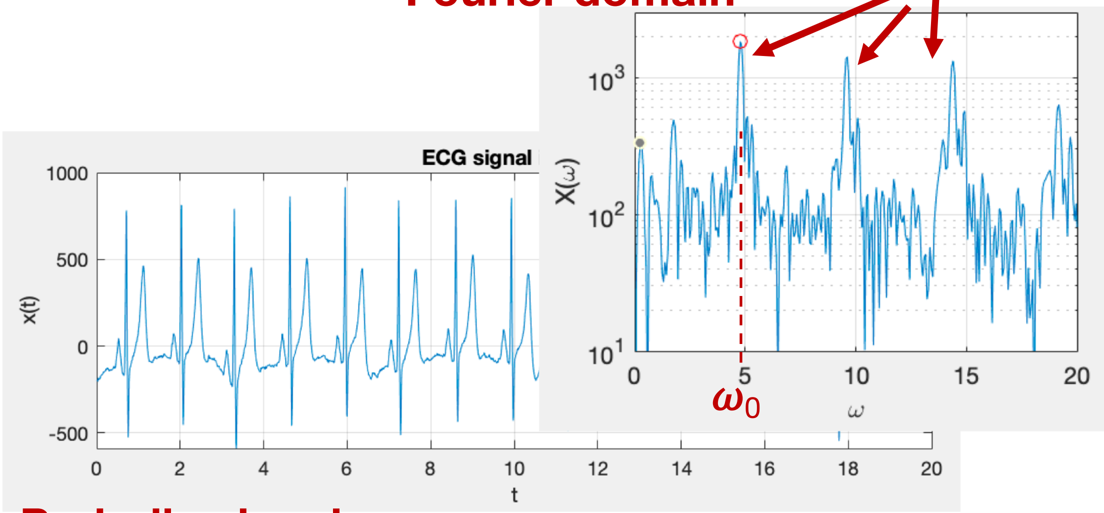
18.4 Periodic signals in MatLab

Some insights on their Fourier transform

- Presence of spectral lines
- Estimating the period from the Fourier domain
- Filtering sinusoidal noises

Periodic signals

A view in the Fourier domain



Periodic signal

$$s(t) \quad \Longrightarrow \quad S(j\omega) = \sum_k 2\pi S_k \delta(\omega - k\omega_0)\ , \quad \omega_0 = \frac{2\pi}{T_p}$$

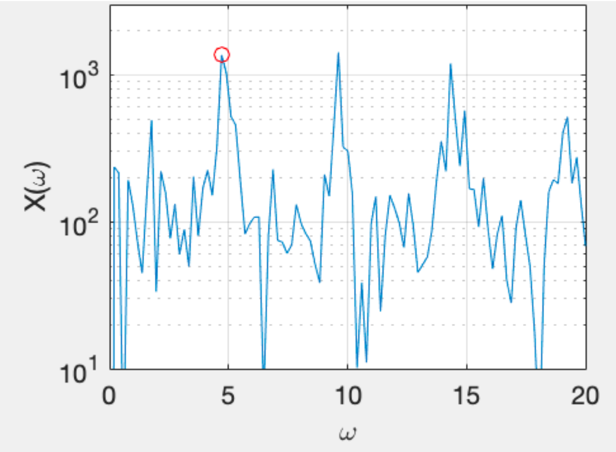
Windowed signal

$$s(t)\,\mathrm{rect}(\tfrac{t}{T_w}) \quad \Longrightarrow \quad S(j\omega) = \sum_k T_w S_k\, \mathrm{sinc}(\tfrac{\omega - k\omega_0}{2\pi/T_w})$$

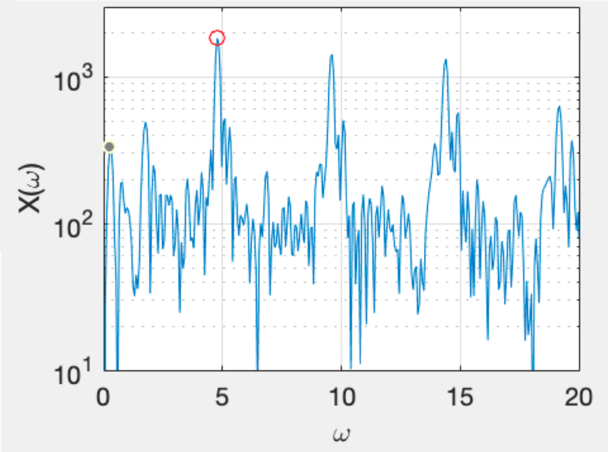
Increasing precision

In the Fourier domain

Natural Fourier sampling



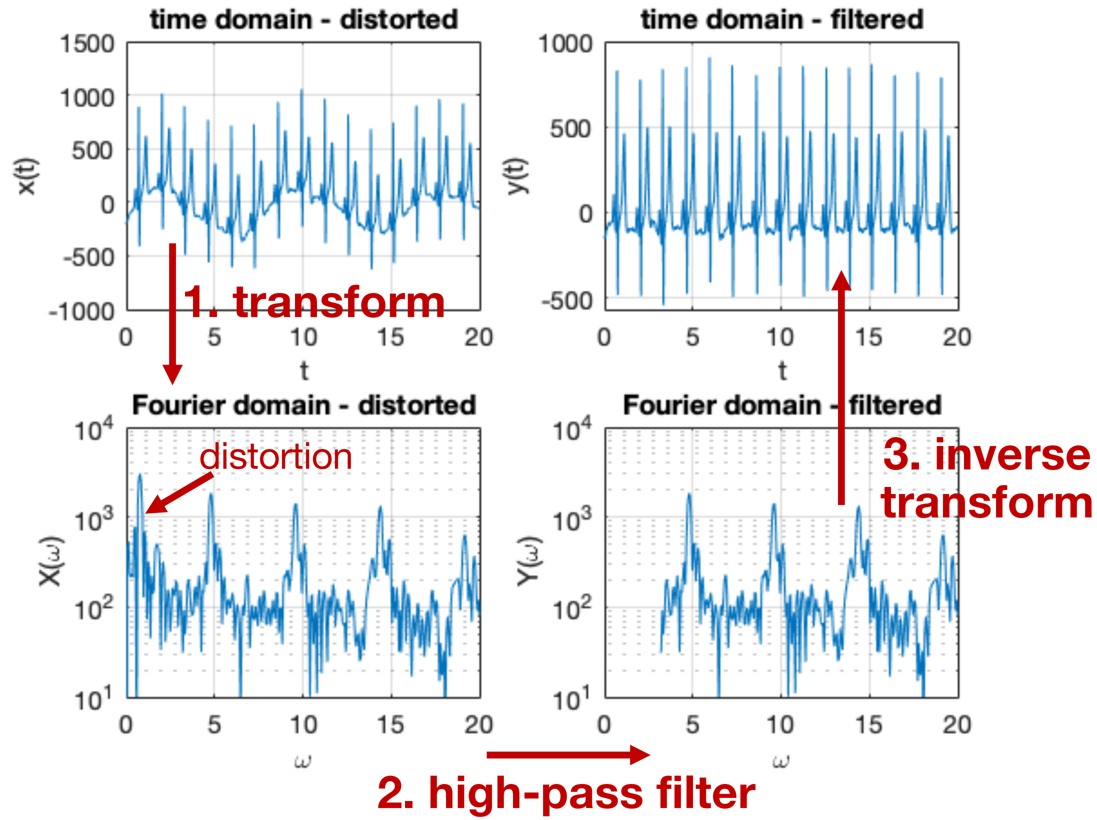
Increased precision by zero padding




```
x = [x, zeros(1,2*length(x))]
```

Filtering

In the Fourier domain



Exercises

On ECG signal processing

Observe the outcome of **quasi-periodic signals** like the ECG signal displaying spectral lines! Try estimating its period from the Fourier domain.

Practice yourself with **filters in the Fourier domain** by removing a sinusoidal distortion applied to an ECG signal

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Lecture 19

Laplace transform: definition and properties

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19.1 The Laplace transform

for continuous-time aperiodic signals

- The Laplace transform
- Region of convergence
- Useful Laplace pairs
- Properties of the Laplace transform

The Laplace transform

projection onto
an orthogonal
basis

Laplace transform

complex valued

$s = \sigma + j\omega$

analytic in the ROC

$$x(t) = \frac{1}{j2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \quad \text{+ ROC}$$

signal
reconstruction

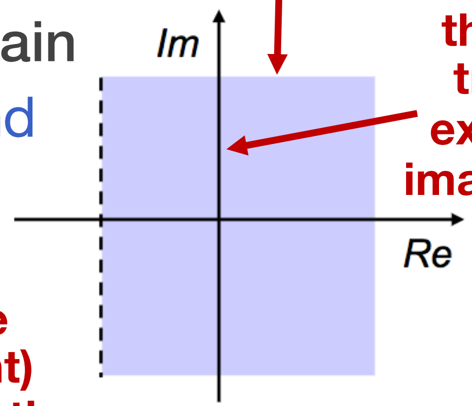
ROC = region of convergence

time domain

continuous-time
and aperiodic

Laplace domain

continuous and
complex s



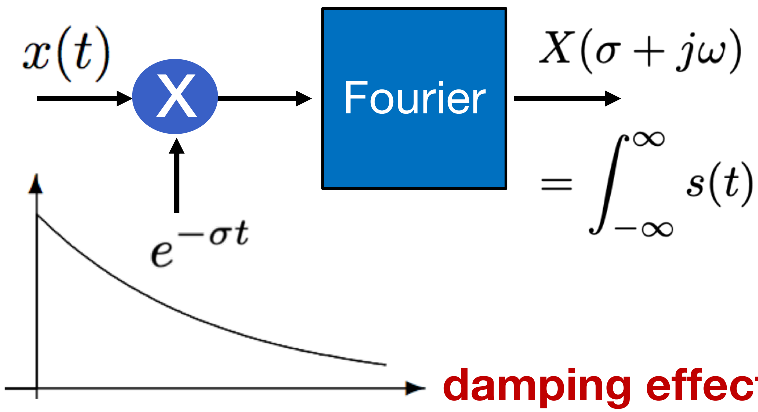
the Fourier transform exists in the imaginary axis

$X(j\omega)$

**an alternative
(and equivalent)
signal representation**

Interpretation

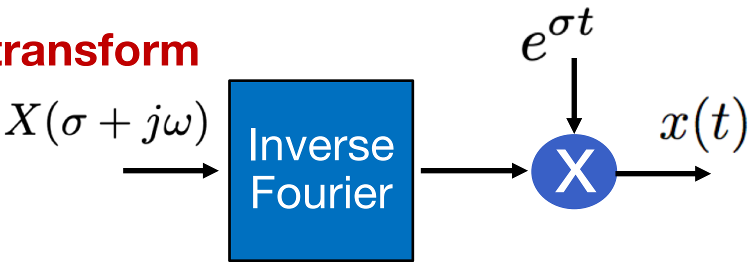
For a causal signal



$$= \int_{-\infty}^{\infty} s(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$$

damping effect
(the signal can also increase in t)

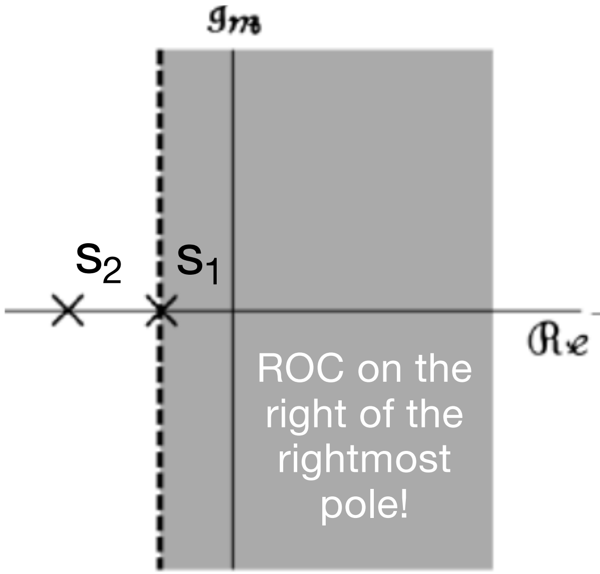
Inverse transform



ROC shapes

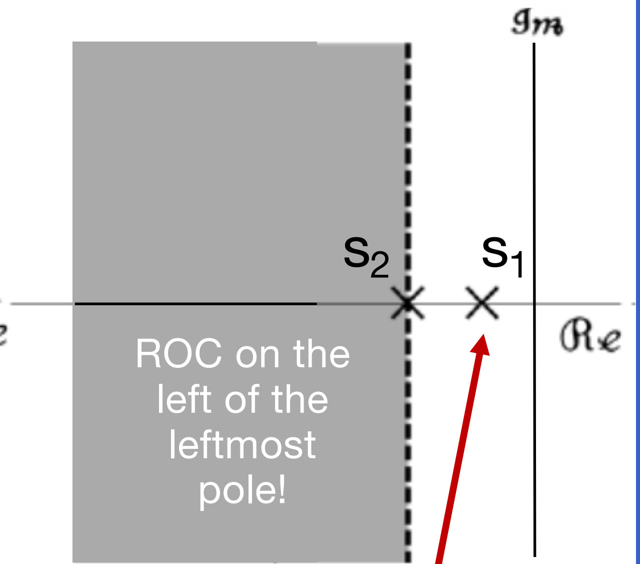
causal signals

$$x(t) = (e^{s_1 t} + e^{s_2 t})\,1(t)$$



anti-causal signals

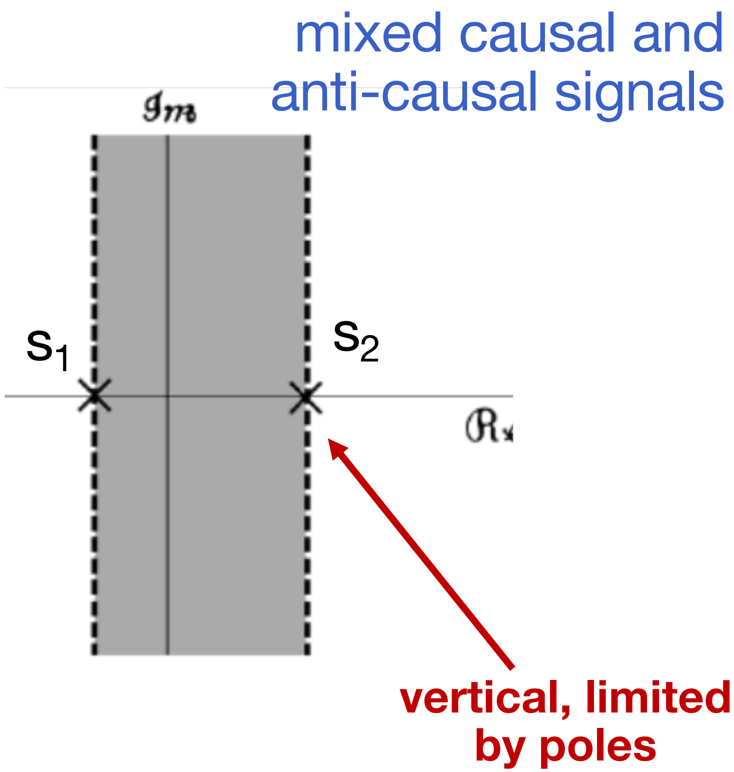
$$x(t) = -(e^{s_1 t} + e^{s_2 t})\,1(-t)$$



$$X(s) = \frac{1}{s - s_1} + \frac{1}{s - s_2}$$

poles (where the transform expression diverges)

General ROC shape



Useful pairs

	time domain	Laplace domain	ROC
constant	1	**does not exist!**	
unit step	$1(t)$	$\frac{1}{s}$	$\Re[s] > 0$
ramp	$\frac{t^k}{k!}1(t)$	$\frac{1}{s^{k+1}}$	
delta	$\delta(t)$	1	any s
delta derivatives	$\delta^{(k)}(t)$	s^k	
exponential	$e^{j\omega_1 t}$	**does not exist!**	
one-sided exp	$e^{p_1 t}1(t)$	$\frac{1}{s-p_1}$	$\Re[s] > \Re[p_1]$
exponential ramp	$\frac{t^k}{k!}e^{p_1 t}1(t)$	$\frac{1}{(s-p_1)^{k+1}}$	
sinusoids	$\cos(\omega_0 1)\,1(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re[s] > 0$
	$\sin(\omega_0 1)\,1(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	

Properties

	time domain	Laplace domain
linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(s) + \beta Y(s)$
time-reversal conjugation	$x(-t)$ $x^*(t)$	$X(-s)$ $X^*(s^*)$
scale	$x(t/a)$	$aX(as)$
time-shift modulation **dual pair**	$x(t-t_0)$ $x(t)\, e^{s_0 t}$	$X(s)e^{-st_0}$ $X(s-s_0)$
convolution	$x * y(t)$	$X(s)Y(s)$
derivation product by t **dual pair**	$x'(t)$ $tx(t)$	$sX(s)$ $-X'(s)$
integration	$x * 1(t)$	$\frac{X(s)}{s}$

Time-shift + modulation

$$\int_{-\infty}^{\infty} x(t-t_0)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(u)e^{-s(u+t_0)}du$$

$$= \boxed{X(s)e^{-st_0}}$$

u=t-t_0

$$\int_{-\infty}^{\infty} x(t)e^{s_0t}e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-(s-s_0)t}dt$$

$$= \boxed{X(s-s_0)}$$

Convolution + product-by-t

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(u)y(t-u)du\right)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(u)\left(\int_{-\infty}^{\infty} y(t-u)e^{-st}dt\right)du$$

swap

$$= \int_{-\infty}^{\infty} x(u)Y(s)e^{-su}du$$

time-shift

$$X'(s) = \frac{d}{ds}\left(\int_{-\infty}^{\infty} x(t)e^{-st}dt\right)$$

$$= \int_{-\infty}^{\infty} x(t)\frac{de^{-st}}{ds}dt$$

swap

$$= \int_{-\infty}^{\infty} \boxed{-tx(t)}e^{-st}dt$$

Derivation + integration

$$x'(t) = \frac{d}{dt}\left(\frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds\right)$$

$$= \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} \boxed{sX(s)}e^{st}ds$$

swap

---

by convolution

$$x * 1(t) \Rightarrow \frac{X(s)}{s}$$

the exact inverse!

Exercises

On the Laplace transform

Deriving the **Laplace transform** through its forward rule is a fundamental requirement of this course.

You need to fully memorize the forward rule and the **Laplace properties**, and take particular care in evaluating integrals correctly.

You also need to memorize the **fundamental Laplace pairs**, as these are relevant in what follows.

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Lecture 19

Laplace transform: definition and properties

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19.4 The unilateral Laplace transform

For causal signals

- Definition
- Properties
- Inverting fractional Laplace expressions

The unilateral Laplace transform

(causal part)
$t > 0_-$ only

$$x_+(t) = \frac{1}{j2\pi} \int_{s-j\infty}^{s+j\infty} X_+(s)\, e^{st}\, ds$$

projection onto
an orthogonal
basis

complex valued
$s = \sigma + j\omega$
analytic in the ROC

$$X_+(s) = \int_{0-}^{\infty} x_+(t)\, e^{-st}\, dt$$

signal
reconstruction

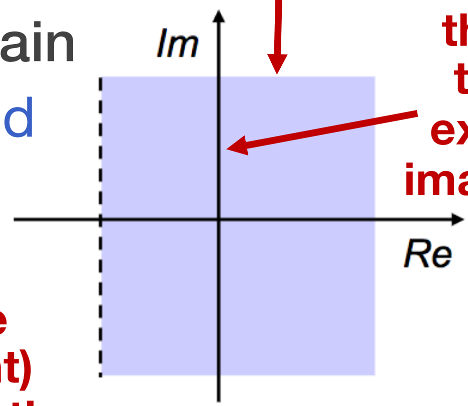
rightmost ROC

time domain

continuous-time
and aperiodic

Laplace domain

continuous and
complex s



the Fourier transform exists in the imaginary axis

$X(j\omega)$

an alternative (and equivalent) signal representation

Properties

	time domain	Laplace domain
linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(s) + \beta Y(s)$
conjugation	$x^*(t)$	$X^*(s^*)$
scale	$x(t/a)$	$aX(as)$
modulation	$x(t)\, e^{s_0 t}$	$X(s - s_0)$
convolution	$x * y(t)$	$X(s)Y(s)$
derivation	$x'(t)$	$s\, X(s) - x(0_-)$
product by t	$tx(t)$	$-X'(s)$
integration	$x * 1(t)$	$\dfrac{X(s)}{s}$

all time-domain signals are causal!!!

Derivation

$$\int_{0^-}^{\infty} x'(t)e^{-st}dt$$

by parts

$$= x(t)e^{-st}\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} x(t)(-se^{-st})dt$$

$$= \left(\lim_{t\to\infty} x(t)e^{-st}\right) - \underline{x(0^-)} + sX(s)$$

$$= \boxed{sX(s) - x(0^-)}$$

the info on the starting level is lost

What if we integrate?

$$x_+(t) - \underline{x(0^-)\,1(t)}$$

$$\frac{1}{s}\left(s\,X(s) - x(0_-)\right) = X(s) - \frac{x(0^-)}{s}$$

Derivation of order k

First derivative $sX(s) - x(0^-)$

Second derivative

$$s\Big(sX(s) - x(0^-)\Big) - x'(0^-)$$

Third derivative

$$s^3X(s) - s^2x(0^-) - sx'(0^-) - x''(0^-)$$

***k*th derivative**

$$\boxed{s^k\,X(s) - \sum_{\ell=0}^{k-1} x^{(\ell)}(0^-)\,s^{k-1-\ell}}$$

Rational functions

Inverting a rational Laplace transform

Improper version (m≥n)

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \ldots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n}$$

Proper counterpart (m≥n)

$$H(s) = q(s) + \frac{r(s)}{a(s)}$$

reminder

quotient

$$= q_0 + q_1 s + \ldots + q_{m-n} s^{m-n}$$

delta
impulses in the
time-domain

$$+ \frac{r_0 + r_1 s + r_2 s^2 + \ldots + r_{n-1} s^{n-1}}{a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n}$$

?

Proper rational functions

The case of distinct poles

m<n

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \ldots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n}$$

zeros

$$= K \frac{(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$

distinct poles

partial fraction

$$= \frac{R_1}{(s - p_1)} + \frac{R_2}{(s - p_2)} + \ldots + \frac{R_n}{(s - p_n)}$$

residue

$$\boxed{R_i = \lim_{s \to p_i} H(s)(s - p_i)}$$

Proper rational functions

The general case

m<n

distinct poles

$$H(s) = K \frac{\prod_i (s - z_i)}{\prod_j (s - p_j)^{\mu_j}}$$

pole multiplicity

$$= \sum_j \sum_{i=1}^{\mu_j} \frac{R_{i,j}}{(s - p_j)^i}$$

partial fractions

pole index

multiplicity index

residues

$$R_{\mu_j - k, j} = \frac{1}{k!} \lim_{s \to p_j} \frac{d^k [H(s)(s - p_j)^{\mu_j}]}{ds^k}$$

$$R_{\mu_j, j} = \lim_{s \to p_j} H(s)(s - p_j)^{\mu_j}$$

highest degree case

An example

$$H(s) = 2s + 1 + \frac{4s-1}{s^2(s-1)}$$

multiplicity 2

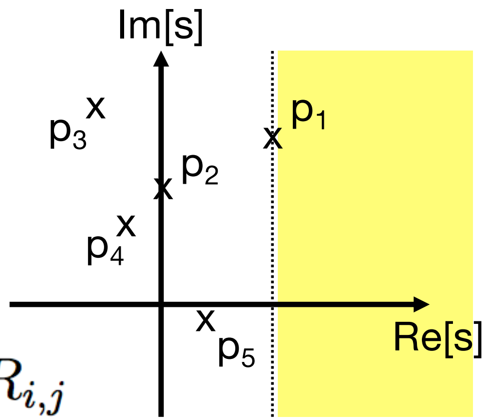
$$= 2s + 1 + \frac{1}{s^2} - \frac{3}{s} + \frac{3}{s-1}$$

Inversion by known transforms

$$h(t) = 2\delta'(t) + \delta(t) + t\,1(t) - 1(t) + 3\,e^{t}\,1(t)$$

causal signal

General inverse transform



$$H(s) = \sum_{j} \sum_{i=1}^{\mu_j} \frac{R_{i,j}}{(s-p_j)^i}$$

Inversion by known transforms

causal signal

$$h(t) = \sum_{j} \sum_{i=1}^{\mu_j} R_{i,j} \frac{t^{i-1} e^{p_j t} 1(t)}{(i-1)!}$$

Exercises

On inverting rational Laplace transforms

Through the **properties of the Laplace transform** we can easily calculate the causal counterpart to any rational expression of the unilateral Laplace transform.

You must get acquainted with this inversion as it will be the basis for solving systems of linear differential equations in the next lecture.

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Lecture 20

Laplace transform: application to differential equations

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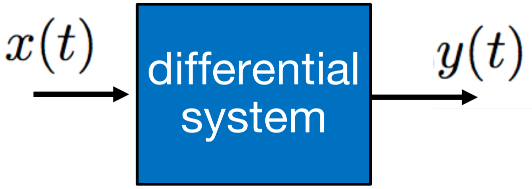
20.1 Differential equations

Solved through the Laplace transform

- Examples of differential equations
- Solution via the transfer function and initial conditions

Differential equations

A linear (constant) model



differential equation

$$a_0 y(t) + a_1 y'(t) + a_2 y''(t) + \ldots + a^n y^{(n)}(t)$$

$$= b_0 x(t) + b_1 x'(t) + b_2 x''(t) + \ldots + b^m x^{(m)}(t)$$

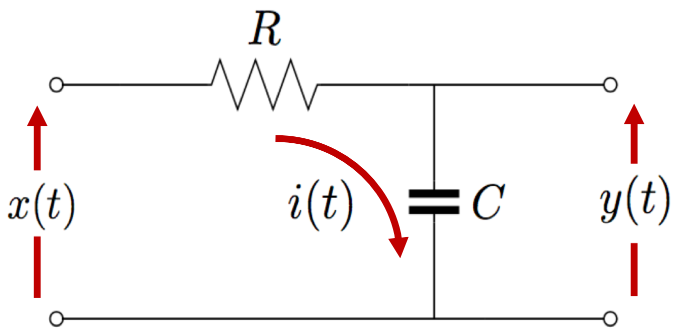
input signal (known) $x(t), t > 0^-$

initial conditions (known)

$$y(0^-), y'(0^-), \ldots, y^{(n-1)}(0^-)$$

RC filter

An example from electric circuits



differential equation

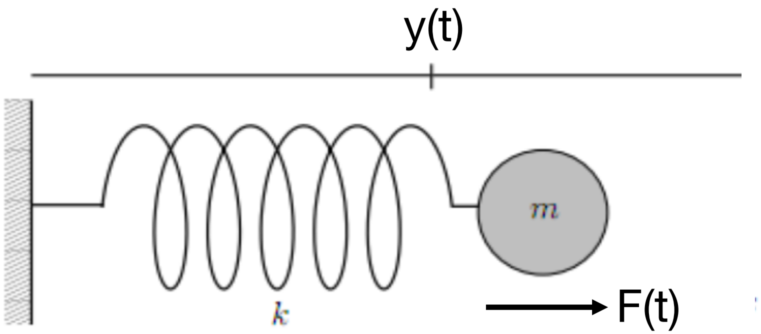
$$x(t) = Ri(t) + y(t)$$

$$y'(t) = \frac{i(t)}{C}$$

$$\boxed{y(t) + RC\,y'(t) = x(t)}$$

Spring-mass system

An example from physics



differential equation

$$F(t) = x(t) - ky(t)$$

$$F(t) = m\,y''(t)$$

$$\boxed{m\,y''(t) + ky(t) = x(t)}$$

Laplace counterpart

Of a differential equation

$$\sum_{k=0}^{m} b_k x^{(k)}(t) = \sum_{i=0}^{n} a_i y^{(i)}(t)$$

$$\Big\Downarrow \text{ unilateral Laplace transform}$$

$$\overset{b_m(s)}{\boxed{\sum_{k=0}^{m} b_k s^k}} X(s) - \overset{x_{m-1}(s)}{\boxed{\sum_{k=1}^{m} b_k \sum_{\ell=0}^{k-1} x^{(\ell)}(0^-) s^{k-1-\ell}}}$$

$$= \overset{a_n(s)}{\boxed{\sum_{i=0}^{n} a_i s^i}} Y(s) - \overset{y_{n-1}(s)}{\boxed{\sum_{i=1}^{n} \sum_{j=0}^{i-1} a_i y^{(j)}(0^-) s^{i-1-j}}}$$

Reinterpreting the result

Of a differential equation

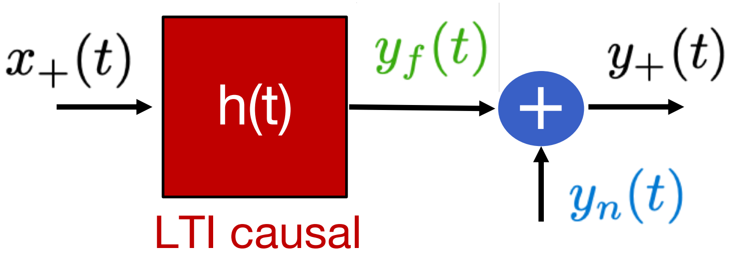
$$Y(s) = \boxed{\boxed{\frac{b_m(s)}{a_n(s)}} X(s)} + \boxed{\frac{y_{n-1}(s) - x_{m-1}(s)}{a_n(s)}}$$

transfer function $H(s)$

forced response $Y_f(s)$

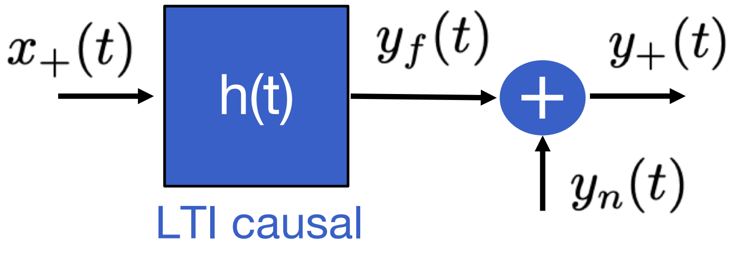
natural response $Y_n(s)$ (initial conditions only)

inverse unilateral Laplace



BIBO stability conditions

Of a differential system



1. **Filter h(t) BIBO stable**
   - **m≤n**, otherwise delta derivatives
   - **Re[pi]<0**, otherwise h(t) not absolutely integrable
2. **Natural response $y_n(t)$ limited**
   - m≤n, otherwise deltas appear
   - Re[pi]<0, guarantees a limit

More on BIBO stability

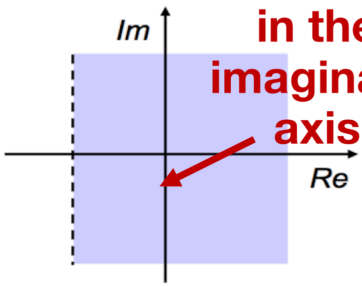
Of a proper rational transfer function

Filter h(t) BIBO stable → Re[p_i]<0

$$|H(j\omega)| = \left|\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt\right|$$

converges

$$\leq \int_{-\infty}^{\infty} |h(t)| \underbrace{|e^{-j\omega t}|}_{1} d$$



it exists in the imaginary axis

Re[p_i]<0 → Filter h(t) BIBO stable

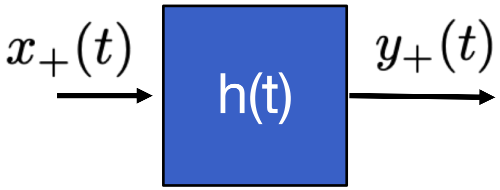
$$\int_{-\infty}^{\infty} |h(t)|dt \leq \sum_j \sum_{i=1}^{\mu_j} |R_{i,j}| \int_{-\infty}^{\infty} \frac{t^{i-1}|e^{p_j t}|1(t)}{(i-1)!} dt$$

$$= \sum_j \sum_{i=1}^{\mu_j} \frac{|R_{i,j}|}{(0-\Re[p_j])^i} < \infty$$

BIBO stability properties

At steady state t≫0

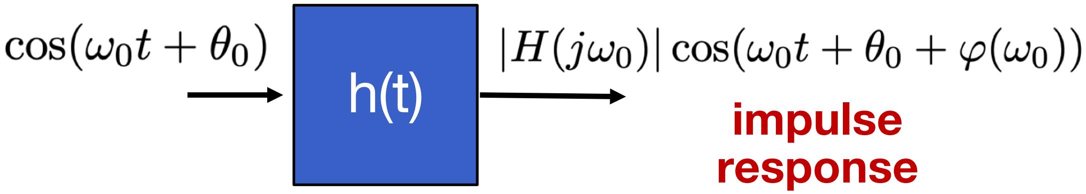
at t≫0



BIBO stable

$\cancel{y_n(t)}$ **natural response vanishes for large t**

at t≫0



BIBO stable, and real

impulse response vanishes for large t

Exercises

On differential equations

Solving **differential equations** by use of the **Laplace transform** is a fundamental requirement of this course.

You need to fully memorize the method, and take particular care in applying it correctly.

You also need to memorize the **fundamental Laplace pairs**, as these are relevant in what follows.

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Lecture 20

Laplace transform: application to differential equations

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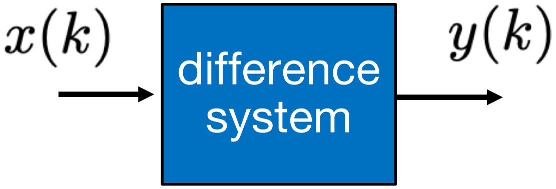
20.4 The unilateral Z transform

For causal discrete-time signals

- Definition and properties
- Solution to difference equations

Difference equations

Or discrete differential equations



$k \geq 0$

differential equation

$$a_0 y(k) + a_1 y(k-1) + \ldots + a_n y(k-n)$$
$$= b_0 x(k) + b_1 x(k-1) + \ldots + b_m x(k-m)$$

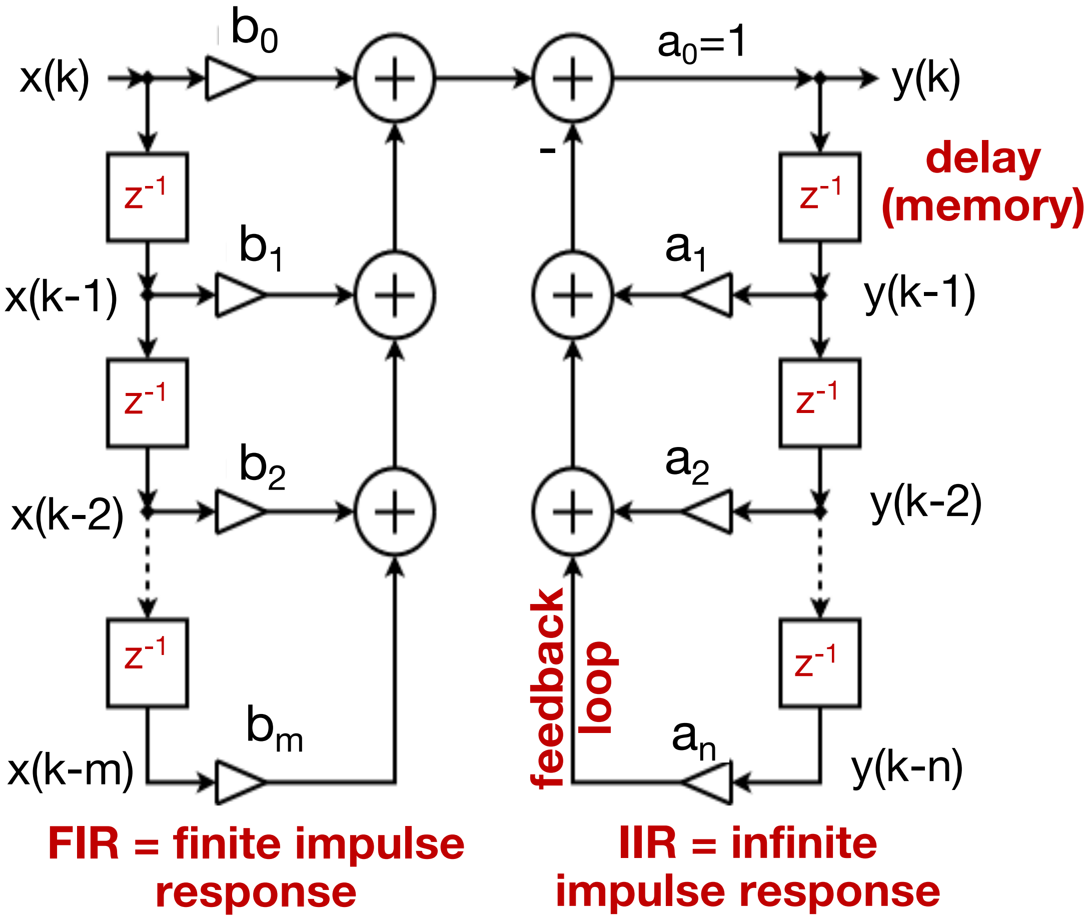
input signal (known) $x(k)\ ,\ k \geq -m$

initial conditions (known)

$$y(-n), \ldots, y(-2), y(-1)$$

The ARMA model

Auto-regressive moving-average



The unilateral Z transform

(causal part)
n ≥ 0 only

$$x(n) = \frac{1}{j2\pi}\oint_r X(z) z^{n-1} dz$$

signal reconstruction

projection onto an orthogonal basis

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

complex valued
$z = re^{j\theta}$
analytic in the ROC

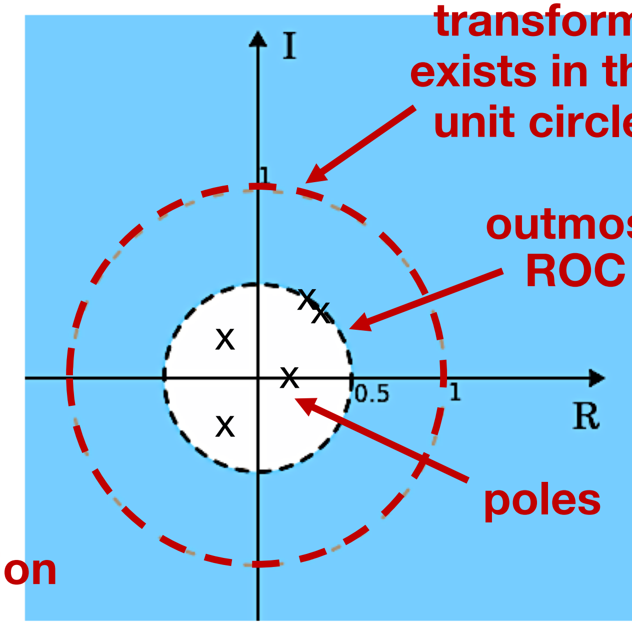
time domain

discrete-time and aperiodic

Z domain

continuous and complex z

an alternative (and equivalent) signal representation



the Fourier transform exists in the unit circle

outmost ROC

poles

Properties

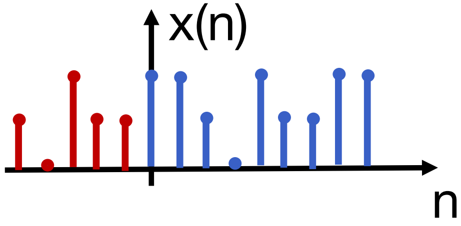
	time domain	Z domain
linearity	$\alpha x(n) + \beta y(n)$	$\alpha X(z) + \beta Y(z)$
modulation	$p_0^n x(n)$	$X(z/p_0)$
convolution	$x * y(n)$	$X(z)Y(z)$
product by n	$nx(n)$	$-zX'(z)$
time-shift	$x(n-n_0)$	$X(z)z^{-n_0} + \sum_{m=-n_0}^{-1} x(m)z^{-(m+n_0)}$

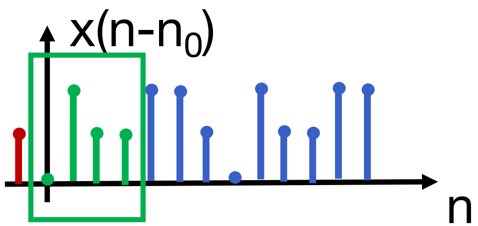
all time-domain signals are causal!!!

Modulation + time-shift

$$\sum_{n=0}^{\infty} x(n)\, p_0^n\, z^{-n} = \sum_{n=0}^{\infty} x(n)\, (z/p_0)^{-n}$$
$$= X(z/p_0)$$

$$\sum_{n=0}^{\infty} x(n-n_0) z^{-n} = \sum_{m=-n_0}^{\infty} x(m) z^{-(m+n_0)}$$
$$= X(z) z^{-n_0} + \boxed{\sum_{m=-n_0}^{-1} x(m) z^{-(m+n_0)}}$$





Convolution + product-by-n

$$\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x_+(k)y_+(n-k)\right) z^{-n}$$
$$= \sum_{k=-\infty}^{\infty} x_+(k)\left(\sum_{n=-\infty}^{\infty} y_+(n-k)z^{-n}\right)$$
$$= \sum_{k=-\infty}^{\infty} x_+(k)Y(z)z^{-k}$$

$$X'(z) = \sum_{n=0}^{\infty} x(n)\frac{dz^{-n}}{dz}$$

-n z^{-n-1}

$$= -z^{-1}\sum_{n=0}^{\infty} \boxed{nx(n)}z^{-n}$$

Useful pairs

	time domain	Z domain	ROC
unit step	$1_0(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
ramp	$\frac{1}{k!}\,(n+k)\ldots(n+2)(n+1)\,1_0(n)$	$\frac{1}{(1-z^{-1})^{k+1}}$	
delta	$\delta(n)$	1	
shifted delta	$\delta(n-n_0)$	z^{-n_0}	$\lvert z\rvert > 0$
one-sided exp	$p_0^{n+1}\,1_0(n)$	$\frac{1}{p_0^{-1}-z^{-1}}$	$\lvert z\rvert > \lvert p_0\rvert$
exponential ramp	$\frac{1}{k!}\,(n+k)\ldots(n+2)(n+1)\,p_0^{n+k+1}\,1_0(n)$	$\frac{1}{(p_0^{-1}-z^{-1})^{k+1}}$	

Z counterpart

Of a difference equation

$$\sum_{\ell=0}^{m} b_\ell\, x(k-\ell) = \sum_{i=0}^{n} a_i\, y(k-i)$$

unilateral Z transform

$$\overset{b_m(z)}{\boxed{\sum_{\ell=0}^{m} b_\ell z^{-\ell}}} X(z) + \overset{x_{m-1}(z)}{\boxed{\sum_{\ell=1}^{m} \sum_{j=-\ell}^{-1} b_\ell x(j) z^{-(j+\ell)}}}$$

$$= \overset{a_n(z)}{\boxed{\sum_{i=0}^{n} a_i z^{-i}}} Y(z) + \overset{y_{n-1}(z)}{\boxed{\sum_{i=1}^{n} \sum_{k=-i}^{-1} a_i y(k) z^{-(k+i)}}}$$

Reinterpreting the result

Of a difference equation

fractional functions in z^{-1}

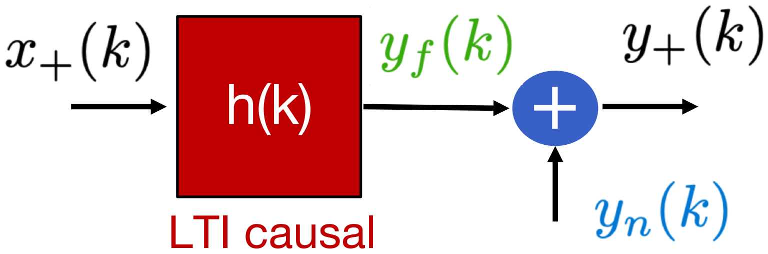
$$Y(z) = \boxed{\boxed{\frac{b_m(z)}{a_n(z)}} X(z)} + \boxed{\frac{y_{n-1}(z) - x_{m-1}(z)}{a_n(z)}}$$

transfer function H(z)

forced response $Y_f(z)$

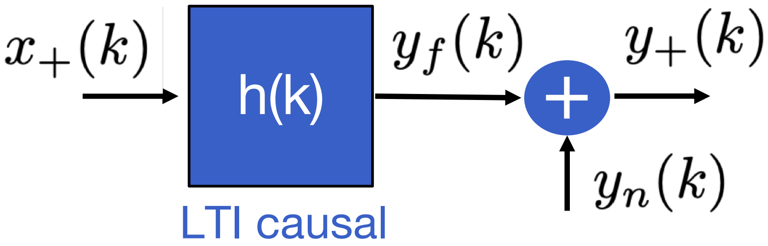
natural response $Y_n(z)$ (initial conditions only)

inverse unilateral Z transform



BIBO stability conditions

Of a difference system



1. **Filter h(k) BIBO stable**
   - $|p_i|<1$, otherwise h(k) not absolutely integrable

2. **Natural response $y_n(k)$ limited**
   - $|p_i|<1$, guarantees a limit

here deltas are not a problem

Exercises

On difference equations

Solving **difference equations** by use of the **Z transform** is a fundamental requirement of this course.

You need to fully memorize the method, and take particular care in applying it correctly.

You also need to memorize the **fundamental Z pairs** , as these are relevant in what follows. Beware of the fact that now **fractional expressions are in z^{-1}**, and that particular care is needed to correctly identify the poles

MIME

Master's degree ICT Internet Multimedia Engineering

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