FOUNDATIONS OF SIGNALS AND SYSTEMS
1.2 Homework assignment
Prof. T. Erseghe

Exercises 1.2

Please review your knowledge on complex numbers, and in particular on their representation in either Cartesian form (real and imaginary part) or polar form (absolute value and phase), namely

$$x = a_x + jb_x = |x|\, e^{j\varphi_x} = |x| \cos(\varphi_x) + j|x| \sin(\varphi_x) \ ,$$

where j is the imaginary unit, and where we denoted $\Re[x] = a_x$ and $\Im[x] = b_x$. Keep in mind the relations

$$|x| = \sqrt{a_x^2 + b_x^2} \ , \qquad \varphi_x = \arctan(\tfrac{b_x}{a_x}) + \begin{cases} 0 & a_x \geq 0 \\ \pi & a_x < 0 \end{cases}$$

as well as the concept of complex-conjugate

$$x^* = a_x - jb_x = |x|\, e^{-j\varphi_x} \ .$$

Recall the product rules

$$x \cdot y = |x|\,|y|\, e^{\varphi_x + \varphi_x} \ , \qquad |x \cdot y| = |x| \cdot |y| \ ,$$

as well as Euler's indentities

$$\cos(\alpha) = \tfrac{1}{2} e^{j\alpha} + \tfrac{1}{2} e^{-j\alpha} \ , \quad \sin(\alpha) = \tfrac{1}{2j} e^{j\alpha} - \tfrac{1}{2j} e^{-j\alpha} \ ,$$

that will be extensively used during the course. Remember that, a complex exponential might also have a real-valued part at the exponent, that is

$$x = e^{\alpha + j\beta} = e^{\alpha} \cdot e^{j\beta} \ , \qquad |x| = e^{a} \ , \ \varphi_x = \beta \ .$$

Then solve the following:

1. write $e^{-j\frac{\pi}{2}}$, $je^{j3\pi}$, and $\sqrt{2}\, e^{j\frac{\pi}{4}}$ in Cartesian form;
2. write $1 + j$, $-3j$, and -2 in polar form;
3. write $1/(3 - 2j)$ in Cartesian form;
4. given $x = [e^{j\frac{\pi}{3}} - \cos(\frac{\pi}{3})] \cdot e^{-(2+j\frac{\pi}{3})}$ evaluate its complex-conjugate x^* in Cartesian and polar form;

Solutions.

1. For the first value we have

$$x_1 = e^{j\frac{\pi}{2}} = \cos(-\tfrac{\pi}{2}) + j\sin(-\tfrac{\pi}{2}) = 0 + j \cdot -1 = -j \ .$$

For the second, it is

$$x_2 = je^{j3\pi} = j \cdot [\cos(3\pi) + j\sin(3\pi)] = j \cdot [-1 + j0] = -j \ .$$

since, by the periodicity of the phase, it is $e^{j3\pi} = e^{j(3\pi-2\pi)} = e^{j\pi} = -1$. For the third value we finally have

$$x_3 = \sqrt{2}\, e^{j\frac{\pi}{4}} = \sqrt{2} \cdot [\cos(\tfrac{\pi}{4}) + j\sin(\tfrac{\pi}{4})] = \sqrt{2} \cdot [\tfrac{1}{\sqrt{2}} + j\tfrac{1}{\sqrt{2}}] = 1 + j \ .$$

2. For the first value $x_1 = 1 + j$ we have

$$|x_1| = \sqrt{1+1} = \sqrt{2}\ , \quad \varphi_1 = \arctan(\tfrac{1}{1}) = \tfrac{\pi}{4} \ .$$

For the second value $x_2 = 0 - 3j$ we have

$$|x_2| = \sqrt{0+9} = 3\ , \quad \varphi_2 = \arctan(\tfrac{-3}{0}) = \arctan(-\infty) = -\tfrac{\pi}{2} \ .$$

Finally, for the third value $x_3 = -2 + j0$ we have

$$|x_3| = \sqrt{0+4} = 2\ , \quad \varphi_3 = \arctan(\tfrac{0}{-1}) + \pi = \arctan(0) + \pi = \pi \ .$$

3. In this case it is $x = 1/y$ for $y = 3-2j$, so that we can multiply numerator and denominator by y^* to have

$$x = \frac{1}{y} \cdot \frac{y^*}{y^*} = \frac{y^*}{|y|^2} = \frac{3+2j}{3^2 + (-2)^2} = \tfrac{3}{13} + j\tfrac{2}{13} \ .$$

4. We can write the complex conjugate in the form (where j maps into $-j$)

$$\begin{aligned} x^* &= [e^{-j\frac{\pi}{3}} - \cos(\tfrac{\pi}{3})] \cdot e^{-(2-j\frac{\pi}{3})} \\ &= [\cos(-\tfrac{\pi}{3}) + j\sin(-\tfrac{\pi}{3}) - \cos(\tfrac{\pi}{3})] \cdot e^{-(2-j\frac{\pi}{3})} \\ &= -j\sin(\tfrac{\pi}{3}) \cdot e^{-(2-j\frac{\pi}{3})} \\ &= -j\sin(\tfrac{\pi}{3}) \cdot [e^{-2} \cdot e^{j\frac{\pi}{3}}] \\ &= -je^{-2}\sin(\tfrac{\pi}{3})\, e^{j\frac{\pi}{3}} \end{aligned}$$

In case we are interested to the Cartesian form, we have

$$\begin{aligned} x^* &= -je^{-2}\sin(\tfrac{\pi}{3})\,[\cos(\tfrac{\pi}{3}) + j\sin(\tfrac{\pi}{3})] \\ &= \sin^2(\tfrac{\pi}{3})\, e^{-2} - \cos(\tfrac{\pi}{3})\sin(\tfrac{\pi}{3})\, e^{-2} \end{aligned}$$

In case we are, instead, interested in the polar form we simply need to recall that $-j = e^{-j\frac{\pi}{2}}$, to have

$$x^* = e^{-2}\sin(\tfrac{\pi}{3})\, e^{j\frac{\pi}{3}}\, e^{-j\frac{\pi}{2}} = e^{-2}\sin(\tfrac{\pi}{3})\, e^{j(\frac{\pi}{3}-\frac{\pi}{2})} = e^{-2}\sin(\tfrac{\pi}{3})\, e^{-j\frac{\pi}{6}} \ ,$$

so that $|x^*| = |x| = e^{-2}\sin(\tfrac{\pi}{3})$ and $\varphi(x^*) = -\varphi(x) = -\tfrac{\pi}{6}$.

FOUNDATIONS OF SIGNALS AND SYSTEMS
1.4 Homework assignment
Prof. T. Erseghe

Exercises 1.4

Please review your knowledge on integrals and series with complex numbers, and in particular the geometric series equivalences

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha} \,, \qquad \sum_{n=0}^{\infty} \alpha^n = \begin{cases} \frac{1}{1-\alpha} & , |\alpha| < 1 \\ \text{indeterminate} & , \text{otherwise} \end{cases}$$

and the complex exponential integral

$$\int_{t_0}^{t_1} e^{\alpha t}\, dt = \frac{e^{\alpha t}}{\alpha}\Big|_{t_0}^{t_1} = \frac{e^{\alpha t_1} - e^{\alpha t_0}}{\alpha} \,,$$

which are valid for complex α. Then solve the following:

1. Evaluate the complex integral

$$x(\omega) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j\omega t}\, dt$$

as a function of the real value ω, and write the result in the Cartesian form.

2. Evaluate the complex series

$$x(\beta, \theta) = \sum_{n=0}^{\infty} e^{(\beta - j\theta)n}$$

as a function of the real values β and θ, and write the result in the Cartesian form.

3. Evaluate the two complex series

$$x_1(\theta) = \sum_{n=0}^{9} e^{-j\theta n} \,, \qquad x_2(\theta) = \sum_{n=-9}^{0} e^{-j\theta n}$$

as a function of the real value θ.

4. Evaluate the complex series

$$x(\theta) = \sum_{n=-9}^{9} e^{-j\theta n}$$

as a function of the real value θ. Although this might be challenging, try to write the result in the Cartesian form.

Solutions.

1. We have

$$x(\omega) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j\omega t}\, dt = \frac{e^{j\omega t}}{j\omega}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{j\omega}$$
$$= \frac{2j\sin(\frac{\omega}{2})}{j\omega} = \frac{2\sin(\frac{\omega}{2})}{\omega} \ ,$$

where we used the primitive of a complex exponential with $\alpha = j\omega$, as well as Euler's identity on the sinus. The result is real-valued.

2. This is a in the form of a geometric series with $\alpha = e^{\beta - j\theta}$, that is

$$|\alpha| = |e^{\beta}\, e^{-j\theta}| = |e^{\beta}| \cdot |e^{-j\theta}| = e^{\beta} \cdot 1 = e^{\beta} \ ,$$

so that we have $|\alpha| < 1$ only for $\beta < 0$. Therefore, if $\beta \geq 0$ then the series does not converge, if instead $\beta < 0$ we have

$$x(\beta, \theta) = \frac{1}{1-\alpha} = \frac{1}{1 - e^{\beta - j\theta}} = \frac{1}{1 - \cos(\theta)\, e^{\beta} + j\sin(\theta)\, e^{\beta}} \ .$$

In order to identify the real and imaginary parts, we need to multiply numerator and denominator by the complex conjugate version of the denominator, that is

$$x(\beta, \theta) = \frac{1}{y} = \frac{1}{y} \cdot \frac{y^*}{y^*} = \frac{y^*}{|y|^2}$$
$$= \frac{1 - \cos(\theta)\, e^{\beta} - j\sin(\theta)\, e^{\beta}}{(1 - \cos(\theta)\, e^{\beta})^2 + (-\sin(\theta)\, e^{\beta})^2}$$
$$= \frac{1 - \cos(\theta)\, e^{\beta} - j\sin(\theta)\, e^{\beta}}{1 - 2\cos(\theta)\, e^{\beta} + e^{2\beta}}$$
$$= \frac{1 - \cos(\theta)\, e^{\beta}}{1 - 2\cos(\theta)\, e^{\beta} + e^{2\beta}} - j\frac{\sin(\theta)\, e^{\beta}}{1 - 2\cos(\theta)\, e^{\beta} + e^{2\beta}} \ .$$

3. The first sum is a finite geometric sum with $\alpha = e^{-j\theta}$ and therefore we have

$$x_1(\theta) = \sum_{n=0}^{0} e^{-j\theta n} = \frac{1 - e^{-j10\theta}}{1 - e^{-j\theta}} \ .$$

The second sum is almost written in the form of a finite geometric sum, but the range is negative. We can, however make it positive by replacing variable n with $m = -n$, that is

$$x_2(\theta) = \sum_{n=-9}^{0} e^{-j\theta n} = \sum_{m=0}^{9} e^{j\theta m} = \frac{1 - e^{j10\theta}}{1 - e^{j\theta}} \ .$$

4. We can exploit the result of the previous exercise, to write

$$x(\theta) = x_1(\theta) + x_2(\theta) - 1 = \frac{1 - e^{-j10\theta}}{1 - e^{-j\theta}} + \frac{1 - e^{j10\theta}}{1 - e^{j\theta}} - 1$$

where the contribution -1 takes into account that we are counting twice the contribution for $n = 0$. By taking the common denominator, and by taking some care in rearranging the result, we have

$$\begin{aligned} x(\theta) &= \frac{(1 - e^{-j10\theta})(1 - e^{j\theta}) + (1 - e^{j10\theta})(1 - e^{-j\theta}) - (1 - e^{j\theta})(1 - e^{-j\theta})}{(1 - e^{j\theta})(1 - e^{-j\theta})} \\ &= \frac{-e^{j10\theta} - e^{-j10\theta} + e^{j9\theta} + e^{-j9\theta}}{2 - e^{j\theta} - e^{-j\theta}} \\ &= \frac{\cos(9\theta) - \cos(10\theta)}{1 - \cos(\theta)} \\ &= \frac{-2\sin(\frac{19}{2}\theta)\sin(-\frac{1}{2}\theta)}{-2\sin(\frac{1}{2}\theta)\sin(-\frac{1}{2}\theta)} = \frac{\sin(\frac{19}{2}\theta)}{\sin(\frac{\theta}{2})}\ , \end{aligned}$$

where we used the formula $\cos a - \cos b = -2\sin(\frac{a+b}{2})\sin(\frac{a-b}{2})$ to compact the result. Alternatively, we can work on the series and introduce a new variable m with $n = m - 9$, so that the range of m is $[0, 18]$, to have

$$\begin{aligned} x(\theta) &= \sum_{m=0}^{18} e^{-j\theta(m-9)} = e^{j9\theta} \sum_{m=0}^{18} e^{-j\theta m} \\ &= e^{j9\theta}\, \frac{1 - e^{-j19\theta}}{1 - e^{-j\theta}} = \frac{e^{j9\theta} - e^{-j10\theta}}{1 - e^{-j\theta}} \end{aligned}$$

With some effort we can show that the two results correspond. As a matter of fact

$$\begin{aligned} x(\theta) &= \frac{e^{j9\theta} - e^{-j10\theta}}{1 - e^{-j\theta}} \cdot \frac{e^{j\frac{\theta}{2}}}{e^{j\frac{\theta}{2}}} = \frac{e^{j\frac{19}{2}\theta} - e^{-j\frac{19}{2}\theta}}{e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}}} \\ &= \frac{2j\sin(\frac{19}{2}\theta)}{2j\sin(\frac{\theta}{2})} = \frac{\sin(\frac{19}{2}\theta)}{\sin(\frac{\theta}{2})} \end{aligned}$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
2.2 Solved exercises
Prof. T. Erseghe

Exercises 2.2

Calculate area, mean value, energy, and power for the following signals:

1. unit step $s(t) = 1(t)$,
2. bilateral exponential $s(t) = e^{-a|t|}$ for $a > 0$.

Solutions.

1. For the unit step, the area is

$$\begin{aligned} A_s &= \int_{-\infty}^{\infty} 1(t)\,dt \\ &= \int_{0}^{\infty} 1(t)\,dt \\ &= \int_{0}^{\infty} 1\,dt = t\Big|_0^\infty = \infty\ , \end{aligned}$$

where in the second row we exploited the fact that $1(t)$ has extension $[0, \infty)$ hence the integral can be limited to this interval (it provides zero value outside), and in the third row we exploited the fact that the unit step has values 1 in the interval $[0, \infty)$ (note that the value $\frac{1}{2}$ at $t = 0$ is dropped since we are integrating, i.e., since a single point has zero Lebesgue measure). For the mean value we instead have

$$\begin{aligned} m_s &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 1(t)\,dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} 1(t)\,dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} 1\,dt = \lim_{T\to\infty} \frac{1}{2T} \cdot T = \tfrac{1}{2}\ , \end{aligned}$$

where we exploited the same tricks as for the area. Observe that, a finite and non-zero average value corresponds to an infinite area.

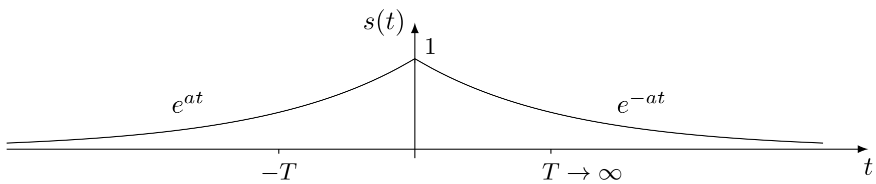
For energy and power we first need to evaluate $|s(t)|^2$, for which we have

$$|s(t)|^2 = \begin{cases} 1 & , t > 0 \\ \frac{1}{4} & , t = 0 \\ 0 & , t < 0 \end{cases} \simeq 1(t)$$

the equivalence being valid everywhere but in $t = 0$. Since energy and power are integral measures, and since a single point has zero Lebesgue measure, we can replace $|s(t)|^2$ with $1(t)$ so that energy and power correspond to the area and the mean value of $1(t)$, to have

$$E_s = \infty\ , \qquad P_s = \tfrac{1}{2}\ .$$

2. For the bilateral exponential, illustrated in the figure below



we have

$$\begin{aligned} A_s &= \lim_{T\to\infty} \int_{-T}^{T} e^{-a|t|}\, dt \\ &= \lim_{T\to\infty} 2\int_{0}^{T} e^{-at}\, dt \\ &= \lim_{T\to\infty} \frac{2e^{-at}}{-a}\Big|_0^T \\ &= \lim_{T\to\infty} 2\frac{e^{-aT}-1}{-a} = \frac{2}{a}\ , \end{aligned}$$

where in the second line we exploited both the symmetry of the bilateral exponential (i.e., the fact that the integral over $[-T,T]$ is twice that over $[0,T]$) and the fact that in the interval $[0,T]$ the bilateral exponential has value e^{-at}, as illustrated in the figure. Note also in the fourth line how the value e^{-aT} tends to 0 as T approaches infinity, as can be appreciated from the figure. Being the area finite, we readily have

$$m_s = 0\ .$$

For energy and power we first need to evaluate $|s(t)|^2$, for which we have

$$|s(t)|^2 = \left(e^{-a|t|}\right)^2 = e^{-2a|t|}\ ,$$

since $e^{-a|t|}$ is real valued. The squared modulo, therefore, has the same structure of the original signal (bilateral exponential) where we should take care of replacing $a \to b = 2a$. Energy and power therefore follow from the results on area and mean value by exploiting this simple replacement, to have

$$E_s = \frac{2}{b} = \frac{1}{a}\ , \qquad P_s = 0\ .$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
2.3 Homework assignment
Prof. T. Erseghe

Exercises 2.3

Calculate area, mean value, energy, and power for the following signals:

1. rectangle $s(t) = \mathrm{rect}(t)$,
2. ramp $s(t) = t \cdot 1(t)$,
3. triangle $s(t) = \mathrm{triang}(t)$,
4. unilateral exponential $s(t) = 1(t) \cdot e^{-at}$ for $a > 0$,
5. complex signal $s(t) = (1 + j) \cdot \mathrm{rect}(t)$,
6. dumped complex exponential $s(t) = e^{p_0 t} 1(t)$ with $p_o = \sigma_0 + j\omega_0$ and $\sigma_0 < 0$.

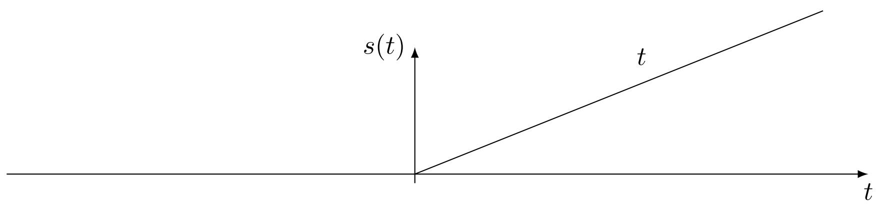
Solutions.
We exploit the same tricks as in Exercises 2.2, and provide short answers.

1. For the rectangular pulse we have

$$A_s = \int_{-\infty}^{\infty} \mathrm{rect}(t)\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1\, dt = 1 \;, \qquad m_s = 0 \;.$$

   Since it is $|s(t)|^2 \simeq \mathrm{rect}(t)$, with inequality only in $t = \pm\frac{1}{2}$, we can replace $|s(t)|^2$ with $\mathrm{rect}(t)$ to have $E_s = 1$ and $P_s = 0$.

2. For the ramp

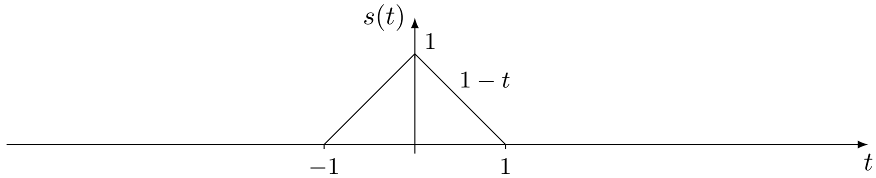


   we have

$$A_s = \int_{-\infty}^{\infty} t \cdot 1(t)\, dt = \int_{0}^{\infty} t\, dt = \infty$$
$$m_s = \lim_{T\to\infty} \frac{1}{2T} \int_0^T t\, dt = \lim_{T\to\infty} \frac{1}{2T} \tfrac{1}{2} t^2 \Big|_0^T = \lim_{T\to\infty} \frac{T^2 - 0}{4T} = \infty$$

   that is for some signals both area and mean value can have no meaning. By observing that $|s(t)|^2 = t^2 \cdot 1(t)$ we also have $E_s = P_s = \infty$.

3. For the triangular pulse



   it is, by symmetry,

$$A_s = 2 \int_0^1 (1-t)\, dt = 2t - t^2 \Big|_0^1 = 1$$

and therefore $m_s = 0$. From the equivalence

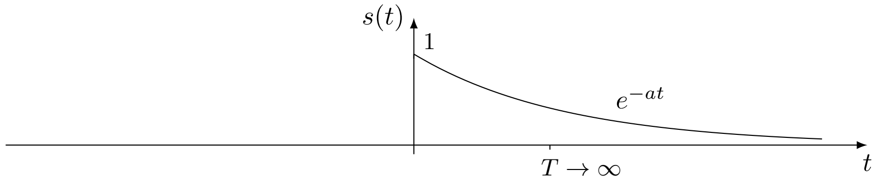
$$|s(t)|^2 = \begin{cases} (1-t)^2 & , 0 < t < 1 \\ (1+t)^2 & , -1 < t < 0 \\ 0 & , \text{otherwise} \end{cases}$$

we readily have, again by symmetry,

$$E_s = 2 \int_0^1 (1-t)^2 \, dt = 2t - 2t^2 + \tfrac{2}{3} t^3 \Big|_0^1 = \tfrac{2}{3}$$

and as a consequence $P_s = 0$.

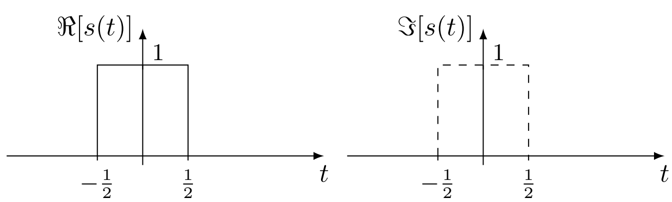
4. For the unilateral exponential



we have a similar result to the bilateral one, but for a factor $\frac{1}{2}$ that takes into accoput for the asymmetry, to have

$$A_s = \frac{1}{a} \;, \quad m_s = 0 \;, \quad E_s = \frac{1}{2a} \;, \quad P_s = 0 \;.$$

5. For the complex signal, which we illustrate separately in its real and imaginary values,



it is $s(t) = B \, \mathrm{rect}(t)$ with $B = 1 + j$, so that by linearity we simply have

$$A_s = B \, A_{\mathrm{rect}} = B = 1 + j \;, \qquad m_s = 0 \;,$$

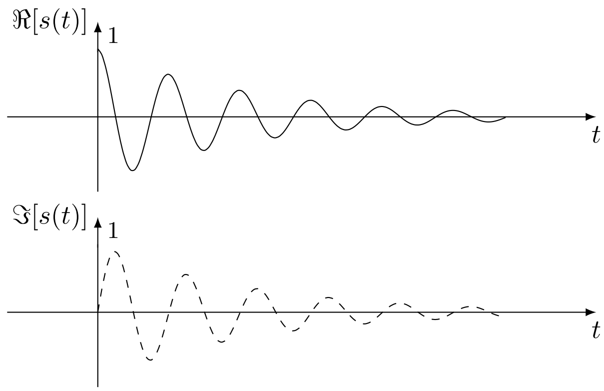
that is the area is in this case complex valued. By instead observing that $|B|^2 = 1^2 + 1^2 = 2$, it then is $|s(t)|^2 = |B|^2 \cdot \mathrm{rect}^2(t) = 2 \, \mathrm{rect}^2(t) \simeq 2 \, \mathrm{rect}(t)$, and therefore, by linearity,

$$E_s = 2 \, A_{\mathrm{rect}} = 2 \;, \qquad P_s = 0 \;.$$

6. For the damped sinusoid we have

$$
\begin{aligned}
s(t) &= e^{\sigma_0 t} e^{j\omega_0 t} 1(t) \\
&= \underbrace{e^{\sigma_0 t} \cos(2\pi f_0 t) 1(t)}_{\Re[s(t)]} + j \underbrace{e^{\sigma_0 t} \sin(2\pi f_0 t) 1(t)}_{\Im[s(t)]}
\end{aligned}
$$

so that its real and imaginary values are of the form



The area is, however, easily calculated directly from the complex expression, to have

$$
A_s = \int_{-\infty}^{\infty} e^{p_0 t} 1(t)\, dt = \int_{0}^{\infty} e^{p_0 t}\, dt = \left. \frac{e^{p_0 t}}{p_0} \right|_0^{\infty} = \frac{0-1}{p_0} = -\frac{1}{p_0}
$$

the value of $e^{p_0 t}$ at $t \to \infty$ being zero as can be observed by the plot (both real and imaginary parts tend to zero at $t \to \infty$). Being the area finite, it is $m_s = 0$. For the power we first need to identify

$$
|s(t)|^2 = |e^{(\sigma_0 + j\omega_0)t} 1(t)|^2 = |e^{\sigma_0 t}|^2 \cdot |e^{j\omega_0 t}|^2 \cdot |1(t)| \simeq e^{2\sigma_0 t} \cdot 1 \cdot 1(t) = e^{2\sigma_0 t} 1(t)
$$

the equivalence being valid everywhere but in $t = 0$, which is fine since a single point has zero Lebesgue measure. This is a unilateral exponential from which we obtain (see previous Exercise 2.3.4)

$$
E_s = \frac{1}{-2\sigma_0} = \frac{1}{2|\sigma_0|} > 0\,,
$$

and accordingly it is $P_s = 0$.

FOUNDATIONS OF SIGNALS AND SYSTEMS
2.5 Solved exercises
Prof. T. Erseghe

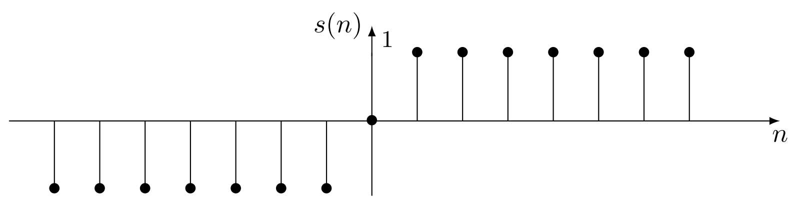
Exercises 2.5

Calculate area, mean value, energy, and power for the following signals:

1. signum $s(n) = \text{sgn}(n)$,
2. exponential $s(n) = a^n\, 1_0(n)$ for $|a| < 1$, and real valued a.

Solutions.

1. For the signum



we have

$$A_s = \lim_{N\to\infty} \sum_{n=-N}^{N} \text{sgn}(n) = \lim_{N\to\infty} N - N = 0$$

and hence $m_s = 0$. For energy and power, instead, we must acknowledge that

$$|s(n)|^2 = \begin{cases} 1 & , n \neq 0 \\ 0 & , n = 0 \end{cases}$$

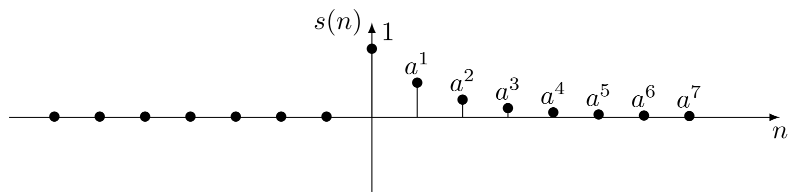
and therefore

$$E_s = \lim_{N\to\infty} \sum_{n=-N}^{N} |s(n)|^2 = \lim_{N\to\infty} 2N = \infty$$

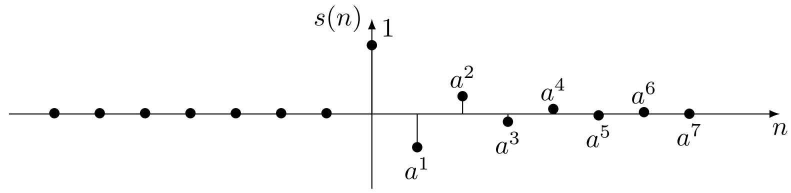
while

$$P_s = \lim_{N\to\infty} \frac{1}{1+2N} \sum_{n=-N}^{N} |s(n)|^2 = \lim_{N\to\infty} \frac{2N}{1+2N} = 1$$

2. For the unilateral exponential, in case $a > 0$ we have



and when $a < 0$, instead,



The area follows from

$$A_s = \sum_{n=-\infty}^{\infty} a^n 1_0(n) = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

which is a result of the geometric series. Therefore we also have $m_s = 0$. For energy and power, instead, we must identify

$$|s(n)|^2 = |a|^{2n} 1_0(n) = b^n 1_0(n) \ , \qquad b = |a|^2$$

revealing that

$$E_s = \sum_{n=-\infty}^{\infty} b^n 1_0(n) = \frac{1}{1-b} = \frac{1}{1-|a|^2}$$

and $P_s = 0$.

FOUNDATIONS OF SIGNALS AND SYSTEMS
2.6 Homework assignment
Prof. T. Erseghe

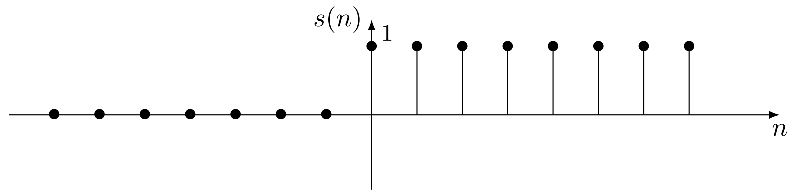
Exercises 2.6

Calculate area, mean value, energy, and power for the following signals:

1. unit step $s(n) = 1_0(n)$,
2. sampled complex exponential $s(n) = e^{j2\pi f_0 nT}\, 1_0(n)$.

Solutions.

1. For the unit step



we have

$$A_s = \lim_{N\to\infty} \sum_{n=-N}^{N} 1_0(n) = 1 + N = \infty$$

and

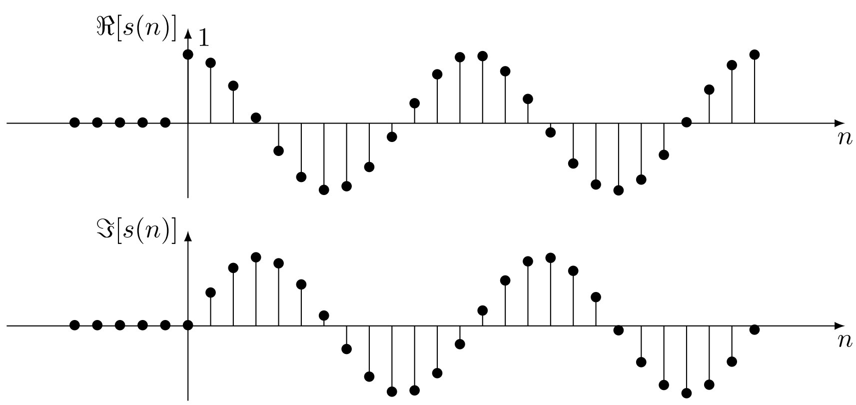
$$m_s = \lim_{N\to\infty} \frac{1}{1+2N} \sum_{n=-N}^{N} 1_0(n) = \frac{1+N}{1+2N} = \tfrac{1}{2} \; .$$

For energy and power, instead, we must acknowledge that

$$|s(n)|^2 = \begin{cases} 1 & , n \geq 0 \\ 0 & , n < 0 \end{cases} = 1_0(n)$$

and therefore $E_s = \infty$ and $P_s = \frac{1}{2}$.

2. For the complex exponential we have a signal of the form



which we can write as

$$s(n) = b^n 1_0(n) \; , \qquad b = e^{j2\pi f_0 T}$$

The area follows from

$$A_s = \sum_{n=-\infty}^{\infty} b^n 1_0(n) = \lim_{N\to\infty} \sum_{n=0}^{N-1} b^n = \lim_{N\to\infty} \frac{1-b^N}{1-a} =?$$

which is a result of the geometric series, but the limit is not defined, hence the area cannot be stated (i.e., it is not a meaningful parameter). For the mean value we instead have

$$m_s = \frac{1}{1+2N} \lim_{N\to\infty} \sum_{n=0}^{N-1} b^n = \lim_{N\to\infty} \frac{1-b^N}{(1-a)(1+2N)} = 0$$

the limit being well defined since the numerator is a number satisfying the triangular inequality $|1-b^N| < 1+|b^N| = 1+1 = 2$ (as a consequence of the fact that $|b| = 1$)), i.e., it is limited, while the denominator tends to infinity. For energy and power, instead, we must identify

$$|s(n)|^2 = |b|^{2n} 1_0(n) = 1_0(n) \ ,$$

which, again, follows from $|b| = 1$. Therefore, from the results obtained with the unit step we have $E_s = \infty$ and $P_s = \frac{1}{2}$.

FOUNDATIONS OF SIGNALS AND SYSTEMS
3.2 Solved exercises
Prof. T. Erseghe

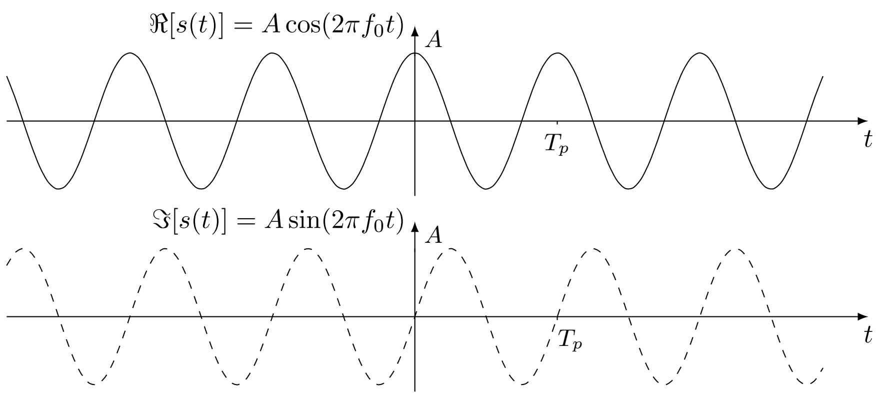
Exercises 3.2

Calculate energy and power for the following signals:

1. complex exponential $s(t) = Ae^{j2\pi f_0 t}$, $f_0 \neq 0$, and $A > 0$ real valued,
2. composition of complex exponentials $s(t) = A_1 e^{j(2\pi f_1 t+\varphi_1)} + A_2 e^{j(2\pi f_2 t+\varphi_2)}$, with $f_1 \neq f_2$, $f_1, f_2 \neq 0$, and $A_1, A_2 > 0$,
3. composition of complex exponentials
$$s(t) = A_0 + \sum_{k=1}^{K} A_k e^{j2\pi f_k t}$$
for $f_k \neq 0$, $f_k \neq f_j$ for $k \neq j$, and A_k complex valued.

Solutions.

1. For the complex exponential, which is a periodic signal, we separately draw its real and imaginary values



where the period is $T_p = 1/|f_0|$ (common to both real and imaginary parts). Therefore, we have

$$\begin{aligned} A_s(T_p) &= \int_0^{T_p} Ae^{j2\pi f_0 t}\, dt \\ &= \frac{A}{j2\pi f_0} e^{j2\pi f_0 t}\Big|_0^{T_p} \\ &= A\frac{e^{j2\pi f_0 T_p} - e^{j2\pi f_0 0}}{j2\pi f_0} = A\frac{1-1}{j2\pi f_0} = 0 \end{aligned}$$

since it is $e^{j2\pi f_0 0} = e^{j0} = 1$ (zero phase) and also $e^{j2\pi f_0 T_p} = e^{\pm j2\pi} = 1$ (phase of either 2π or -2π). Please check your knowledge on complex numbers for this result. As a consequence it also is

$$\boxed{m_s = \frac{A_s(T_p)}{T_p} = 0\ .}$$

For the value of power we need to first identify the value of $|s(t)|^2$, which in the present context is simply

$$|s(t)|^2 = |A|^2 \cdot |e^{j2\pi f_0 t}|^2 = A^2 \cdot 1^2 = A^2$$

and therefore we have

$$E_s(T_p) = \int_0^{T_p} A^2\, dT = A^2 T_p\ , \qquad \boxed{P_s = \frac{E_s(T_p)}{T_p} = A^2\ .}$$

2. For the composition of complex exponentials, which we write in the form

$$s(t) = B_1 e^{j2\pi f_1 t} + B_2 e^{j2\pi f_2 t}\ , \qquad B_i = A_i e^{j\varphi_i}$$

we are not confident on wether $s(t)$ is periodic or not (it is only if f_1 and f_2 are in a rational relation), and therefore we do not exploit this result. However, from the linearity of the mean value it is

$$m_s = B_1 m_1 + B_2 m_2$$

where m_1 and m_2 are the average values of, respectively, $s_1(t) = e^{j2\pi f_1 t}$ and $s_2(t) = e^{j2\pi f_2 t}$. Since form the previous exercise we have that for $f_1, f_2 \neq 0$ it is $m_1 = m_2 = 0$, then it is

$$\boxed{m_s = 0\ .}$$

For the power we need to investigate

$$\begin{aligned}
|s(t)|^2 &= (B_1 s_1(t) + B_2 s_2(t)) \cdot (B_1 s_1(t) + B_2 s_2(t))^* \\
&= |B_1|^2 |s_1(t)|^2 + |B_2|^2 |s_2(t)|^2 + B_1 B_2^* s_1(t) s_2^*(t) + B_1^* B_2 s_1^*(t) s_2(t) \\
&= |B_1|^2 + |B_2|^2 + B_1 B_2^* e^{j2\pi(f_1 - f_2)t} + B_1^* B_2 e^{-j2\pi(f_1 - f_2)t}
\end{aligned}$$

whose last two contributions are complex exponentials with nonzero frequencies $\pm(f_1 - f_2) \neq 0$, hence we know that their average value is zero. Therefore, given that the average value of a constant is the constant itself, we obtain

$$\boxed{P_s = |B_1|^2 + |B_2|^2 = A_1^2 + A_2^2\ .}$$

3. For the second composition of complex integrals, we proceed by linearity and state that

$$m_s = A_0 + \sum_{k=1}^{K} A_k \text{mean}\left(e^{j2\pi f_k t}\right) = A_0$$

since we already learned that the complex exponential $e^{j2\pi f_k t}$ has zero mean value when $f_k \neq 0$. For the power, we need to identify

$$\begin{aligned}
|s(t)|^2 &= \left(\sum_{k=0}^{K} A_k e^{j2\pi f_k}\right)\left(\sum_{\ell=0}^{K} A_k e^{j2\pi f_k}\right)^* \\
&= \sum_{k,\ell=0}^{K} A_k A_\ell^* e^{j2\pi(f_k - f_\ell)t} \\
&= \sum_{k=0}^{K} |A_k|^2 + \sum_{\substack{k,\ell=0 \\ ,k\neq\ell}}^{K} A_k A_\ell^* e^{j2\pi(f_k - f_\ell)t}
\end{aligned}$$

where we assumed $f_0 = 0$. Now, frequencies $f_k - f_\ell \neq 0$, by assumption, hence the corresponding average value is zero. This implies that

$$\boxed{P_s = \sum_{k=0}^{K} |A_k|^2}$$

that is that, in the composition of complex sinusoids, the power is the sum of the individual powers of each component, which is a notable result that must be kept in mind.

FOUNDATIONS OF SIGNALS AND SYSTEMS
3.3 Homework assignment
Prof. T. Erseghe

Exercises 3.3

Identify the periodicity, mean value, and power for the following signals:

1. sinusoid $s(t) = A\cos(2\pi f_0 t + \vartheta_0)$, $f_0 \neq 0$, and $A > 0$ real valued;
2. composition of sinusoids

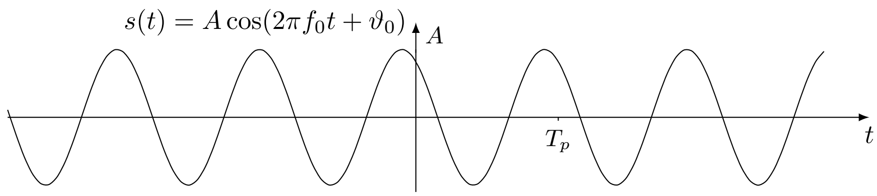
$$s(t) = A_0 + \sum_{k=1}^{K} A_k \cos(2\pi f_k t + \vartheta_k)$$

   for $f_k > 0$, $f_k \neq f_j$ for $k \neq j$, and A_k real valued;
3. $s(t) = A\ \cos(2\pi t + \frac{\pi}{2}) + B\ \sin(8\pi t + \pi)$;
4. $s(t) = e^{j20\pi t}$;
5. $s(t) = e^{j20t}$;
6. $s(t) = e^{-j2\pi t}\ 1(t)$;
7. $s(t) = \cos(10\pi t) + \sin(\frac{8}{3}\pi t)$;
8. $s(t) = A\ \sin^2(10\pi t + \frac{\pi}{4})$;
9. $s(t) = \frac{\sin(5t)}{\sin(t)}$.

Suggestion: here you might want to exploit Euler's identity $\cos(a) = \frac{1}{2}e^{ja} + \frac{1}{2}e^{-ja}$, as well as the identity $e^{j(a+b)} = e^{ja} \cdot e^{jb}$.

Solutions.

1. For the real-valued sinusoid



the period is $T_p = 1/|f_0|$. Therefore, we have

$$\begin{aligned} A_s(T_p) &= \int_0^{T_p} A\cos(2\pi f_0 t + \vartheta_0)\, dt \\ &= \frac{-A}{2\pi f_0} \sin(2\pi f_0 t + \vartheta_0)\Big|_0^{T_p} \\ &= -A\frac{\sin(2\pi f_0 T_p + \vartheta_0) - \sin(\vartheta_0)}{2\pi f_0} = A\frac{\sin(\vartheta_0) - \sin(\vartheta_0 \pm 2\pi)}{2\pi f_0} = 0 \end{aligned}$$

since $\sin(\vartheta_0 \pm 2\pi) = \sin(\vartheta_0)$. As a consequence it also is

$$\boxed{m_s = \frac{A_s(T_p)}{T_p} = 0\ .}$$

For the value of power we need to first identify the value of $|s(t)|^2$, which in the present context requires use of

$$\cos^2(\alpha) = \tfrac{1}{2} + \tfrac{1}{2}\cos(2\alpha)$$

to have

$$|s(t)|^2 = s^2(t) = \tfrac{1}{2}A^2 + \tfrac{1}{2}A^2\cos(2\pi 2 f_0 t + 2\vartheta_0)$$

By exploiting this result we can then write

$$E_s(T_p) = \int_0^{T_p} \tfrac{1}{2}A^2\, dt + \int_0^{T_p} \tfrac{1}{2}A^2 \cos(2\pi 2 f_0 t + 2\vartheta_0)\, dt = \tfrac{1}{2}A^2 T_p + 0$$

where the zero value is a consequence of the fact that the second contribution is a sinusoid with frequency $2f_0$, hence its period is $1/2|f_0| = \frac{1}{2}T_p$, and its integral over $[0, T_p]$, i.e., over two periods, is 0 because of the result stated for the area. We finally obtain

$$\boxed{P_s = \frac{E_s(T_p)}{T_p} = \tfrac{1}{2}A^2\ .}$$

Incidentally note that, this result can also be derived by exploiting the outcome of Exercise 3.2.2 since by Euler's formula we have

$$s(t) = A\cos(2\pi f_0 t + \varphi_0) = \tfrac{1}{2}Ae^{j(2\pi f_0 t+\varphi_0)} + \tfrac{1}{2}Ae^{-j(2\pi f_0 t+\varphi_0)}$$

which corresponds to the present context by assuming, in Exercise 3.2.2, that $f_1 = -f_2 = f_0$, that $A_1 = \frac{1}{2}Ae^{j\varphi_0}$, and that $A_2 = \frac{1}{2}Ae^{-j\varphi_0}$. Therefore, it readily follows that $m_s = 0$ and $P_s = |A_1|^2 + |A_2|^2 = (\frac{1}{2}A)^2 + (\frac{1}{2}A)^2 = \frac{1}{2}A^2$.

2. For the composition of sinusoids, by resorting to Euler's formula we can write the signal in the form

$$s(t) = A_0 + \sum_{k=1}^{K} \tfrac{1}{2}A_k e^{j\vartheta_k} e^{j2\pi f_k t} + \sum_{k=1}^{K} \tfrac{1}{2}A_k e^{-j\vartheta_k} e^{-j2\pi f_k t}$$

where frequencies f_k and $-f_k$ are naturally different. As a consequence we can exploit the results of Exercise 3.2.3 to have $m_s = A_0$ and

$$P_s = A_0^2 + \sum_{k=1}^{K} |\tfrac{1}{2}A_k e^{j\vartheta_k}|^2 + \sum_{k=1}^{K} |\tfrac{1}{2}A_k e^{j\vartheta_k}|^2 = A_0^2 + \tfrac{1}{2}\sum_{k=1}^{K} A_k^2$$

3. In this case it is $\omega_1 = 2\pi = 2\pi f_1$ with $f_1 = 1$ and $T_1 = 1/|f_1| = 1$, and $\omega_2 = \pi = 2\pi f_2$ with $f_2 = 4$ and $T_2 = 1/|f_2| = \frac{1}{4}$. Hence we must solve

$$T_p = kT_1 = mT_2 \quad \Longrightarrow \quad \frac{k}{m} = \frac{T_2}{T_1} = \tfrac{1}{4}$$

which reveals $k = 1$ and therefore $T_p = 1$. From the results of Exercise 3.3.2 we readily have $m_s = 0$ and $P_s = \frac{1}{2}A^2 + \frac{1}{2}B^2$.

4. In this case it is $\omega_0 = 20\pi = 2\pi f_0$ with $f_0 = 10$ and $T_p = 1/|f_0| = \frac{1}{10}$. From the results of Exercise 3.2.3 we readily have $m_s = 0$ and $P_s = 1$.

5. In this case it is $\omega_0 = 20 = 2\pi f_0$ with $f_0 = 10/\pi$ and $T_p = 1/|f_0| = \frac{\pi}{10}$. From the results of Exercise 3.2.3 we readily have $m_s = 0$ and $P_s = 1$.

6. This signal is aperiodic because of the presence of the unit step $1(t)$. For the mean value in this case we need to calculate it directly, to have

$$m_s = \lim_{T\to\infty} \frac{1}{2T}\int_0^T e^{-j2\pi t}\,dt = \lim_{T\to\infty} \frac{1}{2T}\,\frac{e^{-j2\pi T}-1}{-j2\pi} = 0$$

since $e^{-j2\pi T} - 1$ is limited, and in fact by triangular inequality we have $|e^{-j2\pi T} - 1| < |e^{-j2\pi T}| + 1 = 1 + 1 = 2$, while the denominator $2T$ diverges. For the power, instead, we must identify

$$|s(t)|^2 = |e^{-j2\pi t}|^2 \cdot |1(t)|^2 = 1^2(t) \simeq 1(t)$$

hence form the results of the unit step we readily have $P_s = \frac{1}{2}$.

7. In this case it is $\omega_1 = 10\pi = 2\pi f_1$ with $f_1 = 5$ and $T_1 = 1/|f_1| = \frac{1}{5}$, and $\omega_2 = \frac{8}{3}\pi = 2\pi f_2$ with $f_2 = \frac{4}{3}$ and $T_2 = 1/|f_2| = \frac{3}{4}$. Hence we must solve

$$T_p = kT_1 = mT_2 \quad \Longrightarrow \quad \frac{k}{m} = \frac{T_2}{T_1} = \tfrac{3}{4} \cdot 5 = \frac{15}{4}$$

which reveals $k = 15$ and therefore $T_p = 3$. From the results of Exercise 3.3.2 we readily have $m_s = 0$ and $P_s = \frac{1}{2} + \frac{1}{2} = 1$.

8. In this case it is

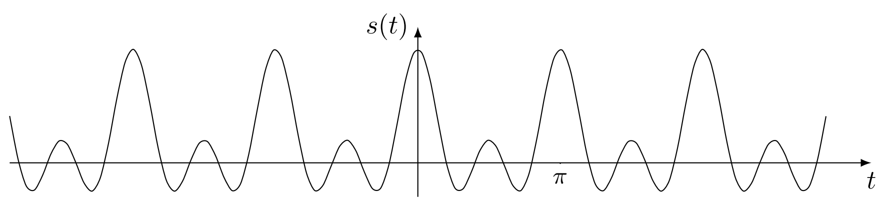
$$s(t) = A \sin^2(10\pi t + \tfrac{\pi}{4}) = \tfrac{1}{2}A - \tfrac{1}{2}A\cos(20\pi t + \tfrac{\pi}{2})$$

so that its periodicity corresponds to the one of the cosine, where $\omega_0 = 20\pi = 2\pi f_0$ with $f_0 = 10$ and $T_p = 1/|f_0| = \frac{1}{10}$. This is consistent with the fact that $\sin^2$ is periodic of period π. From the results of Exercise 3.3.2 we readily have $m_s = \frac{1}{2}A$ and $P_s = (\frac{1}{2}A)^2 + \frac{1}{2}(\frac{1}{2}A)^2 = \frac{3}{8}A^2$.

9. In this case it is $\omega_1 = 5 = 2\pi f_1$ with $f_1 = 5/2\pi$ and $T_1 = 1/|f_1| = \frac{2}{5}\pi$, and $\omega_2 = 1 = 2\pi f_2$ with $f_2 = 1/2\pi$ and $T_2 = 1/|f_2| = 2\pi$. Hence we must solve

$$T_p = kT_1 = mT_2 \quad \Longrightarrow \quad \frac{k}{m} = \frac{T_2}{T_1} = \tfrac{5}{2}$$

which reveals $k = 5$ and therefore $T_p = 2\pi$. However, as can be seen from the graph



2π is certainly a periodicity, but it is not the minimum one, which instead is π. This is due to the fact that, in the division, two negative signs (at the numerator and at the denominator) appear at distance π, and they naturally simplify. Therefore, remember that the mcm approach reveals one periodicity, but not necessarily the minimum one. Mean value and power are in this case hard to evaluate... it is in fact impossible to solve the integral since the primitive is not known in this case. We will see later on, by use of Fourier transforms, how this exercise can be efficiently completed.

FOUNDATIONS OF SIGNALS AND SYSTEMS
3.5 Solved exercises
Prof. T. Erseghe

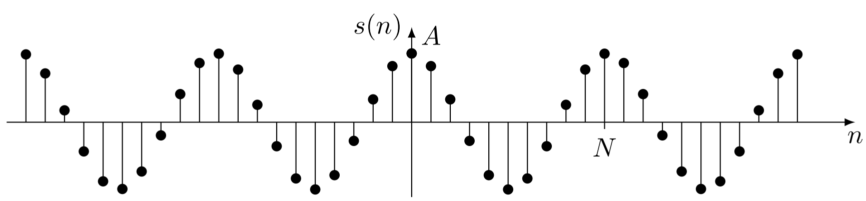
Exercises 3.5

Calculate energy and power for the following signals:

1. the sampled sinusoid $s(n) = A\cos(2\pi f_0 nT)$ periodic of period N, with f_0NT an integer value,
2. the complex sampled exponential $s(n) = e^{j2\pi f_0 nT}$ with generic $f_0 \neq 0$,
3. the composition of complex exponentials $s(n) = ae^{j2\pi f_1 nT} + be^{j2\pi f_2 nT}$, with $f_1 \neq f_2 + k/T$ and $f_1, f_2 \neq k/T$ (non-zero frequencies).

Solutions.

1. In this case we have a signal of the form



For the mean value we identify the area in a period, that is, by Euler's formula,

$$\begin{aligned} A_s(N) &= \sum_{n=0}^{N-1} A\cos(2\pi f_0 nT) \\ &= \tfrac{1}{2}A\sum_{n=0}^{N-1}(e^{j2\pi f_0 T})^n + \tfrac{1}{2}A\sum_{n=0}^{N-1}(e^{-j2\pi f_0 T})^n \\ &= \tfrac{1}{2}A\frac{1-e^{j2\pi f_0 NT}}{1-e^{j2\pi f_0 T}} + \tfrac{1}{2}A\frac{1-e^{-j2\pi f_0 NT}}{1-e^{-j2\pi f_0 T}} \\ &= \tfrac{1}{2}A\frac{1-1}{1-e^{j2\pi f_0 T}} + \tfrac{1}{2}A\frac{1-1}{1-e^{-j2\pi f_0 T}} \\ &= 0 \end{aligned}$$

since $e^{j2\pi f_0 NT} = e^{-j2\pi f_0 NT} = 1$ because of the assumption that f_0NT is an integer. Hence, $m_s = 0$. For the power we first need to investigate

$$|s(t)|^2 = \tfrac{1}{2}A^2 + \tfrac{1}{2}A^2\cos(2\pi 2 f_0 nT)$$

where the latter contribution has $2f_0NT$ integer valued, hence its average value is zero. Therefore, we readily have

$$P_s = \tfrac{1}{2}A^2$$

that is the result perfectly corresponds to the continuous case.

2. For the complex exponential we do not know wether it is periodic or not, hence we need to resort to the general definition of mean value. We have

$$\begin{aligned} m_s &= \lim_{N\to\infty} \frac{1}{1+2N} \sum_{n=-N}^{N} a^n , \qquad a = e^{j2\pi f_0 T} \\ &= \lim_{N\to\infty} \frac{1}{1+2N} a^{-N} \sum_{\ell=0}^{2N} a^\ell \\ &= \lim_{N\to\infty} \frac{1}{1+2N} a^{-N} \frac{1-a^{2N+1}}{1-a} \\ &= 0 \end{aligned}$$

where the fact that the limit is 0 can be explained by the fact that $1-a \neq 0$ since $f_0 \neq 0$ is a given value (finite), and also the numerator is finite since $|a^{-N}| = 1$ because $|a| = 1$, and by triangular inequality it also is $|1-a^{1+2N}| < 1 + |a^{1+2N}| = 1 + 1 = 2$, hence the numerator is limited in absolute value by 2. The fact that the denominator $1 + 2N$ diverges to infinity, ensures the final result. For the power, we need to observe that $|s(n)|^2 = 1$, hence it also is $P_s = 1$.

3. For the composition of complex exponentials the mean value follows from the previous exercise by linearity, and we have $m_s = a \cdot 0 + b \cdot 0 = 0$. For the power we need to evaluate

$$|s(n)|^2 = s(n)s^*(n) = |a|^2 + |b|^2 + ab^* e^{j2\pi(f_1-f_2)nT} + a^* b e^{-j2\pi(f_1-f_2)nT}$$

where $\pm(f_1 - f_2) \neq k/T$ (i.e., it is. a non-zero frequency) so that the two latter contributions have zero mean value. As a consequence it is

$$P_s = |a|^2 + |b|^2 ,$$

which is equivalent to the continuous case. Incidentally note that this result can be used to solve Exercise 3.5.1 for any value of f_0, since by Euler's identity we have

$$y(n) = A\cos(2\pi f_0 nT) = \tfrac{1}{2}Ae^{j2\pi f_0 nT} + \tfrac{1}{2}Ae^{-j2\pi f_0 nT}$$

that is we can set $f_1 = -f_2 = f_0$ and $A_1 = A_2 = \frac{1}{2}A$ to obtain $m_y = 0$ and $P_y = (\frac{1}{2}A)^2 + (\frac{1}{2}A)^2 = \frac{1}{2}A^2$.

FOUNDATIONS OF SIGNALS AND SYSTEMS
3.6 Homework assignment
Prof. T. Erseghe

Exercises 3.6

Identify the periodicity, mean value, and power for the following signals:

1. composition of sampled complex exponentials

$$s(n) = A_0 + \sum_{k=1}^{K} A_k e^{j2\pi f_k n T}$$

   for $f_k \neq \ell/T$ (non-zero frequencies), $f_k \neq f_j + \ell/T$ for $k \neq j$ (different frequencies), and A_k complex valued;

2. composition of sinusoids

$$s(n) = A_0 + \sum_{k=1}^{K} A_k \cos(2\pi f_k n T + \vartheta_k)$$

   for $f_k > 0$, $f_k \neq \ell/T$ (non-zero frequencies), $f_k \neq f_j + \ell/T$ for $k \neq j$ (different frequencies), and A_k real valued;

3. $s(n) = \cos(\frac{4}{3}\pi n)$;
4. $s(n) = \cos(2\pi n/\sqrt{3})$;
5. $s(n) = \cos(2n) - e^{j\frac{\pi}{4}n}$;
6. $s(n) = e^{j\frac{3}{2}\pi n}\cos(\frac{5}{2}\pi n) + j\sin(\pi n)$;
7. $s(n) = e^{jn}\sin(n)$;
8. $s(n) = e^{j\pi n}\sin(\pi n)$.

Suggestion: also here you might want to exploit Euler's identity $\cos(a) = \frac{1}{2}e^{ja} + \frac{1}{2}e^{-ja}$, as well as the identity $e^{j(a+b)} = e^{ja} \cdot e^{jb}$.

Solutions.

1. We proceed as in the continuous case. For the mean value we exploit the result of Exercise 3.5.3 and linearity to acknowledge that all active sampled exponentials have zero average value, hence $m_s = A_0$. For the power, instead, we write

$$
\begin{aligned}
|s(n)|^2 &= \left(\sum_{k=0}^{K} A_k e^{j2\pi f_k nT}\right)\left(\sum_{\ell=0}^{K} A_\ell e^{j2\pi f_\ell nT}\right)^* \\
&= \sum_{k,\ell=0}^{K} A_k A_\ell^* e^{j2\pi(f_k - f_\ell)nT} \\
&= \sum_{k=0}^{K} |A_k|^2 + \sum_{\substack{k,\ell=0 \\ k\neq\ell}}^{K} A_k A_\ell^* e^{j2\pi(f_k - f_\ell)nT}
\end{aligned}
$$

where the second summation contains exponentials of non-null frequencies, hence their average value is zero. As a consequence we have

$$
P_s = \sum_{k=0}^{K} |A_k|^2
$$

2. We first exploit Euler's identity

$$
s(n) = A_0 + \sum_{k=1}^{K} \tfrac{1}{2} A_k e^{j\vartheta_k} e^{j2\pi f_k nT} + \sum_{k=1}^{K} \tfrac{1}{2} A_k e^{-j\vartheta_k} e^{-j2\pi f_k nT}
$$

which identifies a sum of sampled complex exponentials, all with different frequency values. Hence, from the results from Exercise 3.6.1 we immediately have $m_s = A_0$ and

$$
P_s = A_0^2 + 2\sum_{k=1}^{K} (\tfrac{1}{2} A_k)^2 = A_0^2 + \tfrac{1}{2} \sum_{k=1}^{K} A_k^2 \ .
$$

This also corresponds to the continuous case.

3. We have $\omega_0 T = 2\pi f_0 T = \frac{4}{3}\pi$, so that $f_0 T = \frac{2}{3}$ and $N = 3$. By the results of Exercise 3.6.2 the mean value is $m_s = 0$ and the power $P_s = \frac{1}{2}$.

4. We have $\omega_0 T = 2\pi f_0 T = 2\pi/\sqrt{3}$, so that $f_0 T = 1/\sqrt{3}$ which is not rational, hence the sampled sinusoid is non-periodic. By the results of Exercise 3.6.2 the mean value is $m_s = 0$ and the power $P_s = \frac{1}{2}$.

5. For the first contribution we have $\omega_1 T = 2\pi f_1 T = 2$, so that $f_1 T = \frac{1}{\pi}$ which is not rational, hence the sampled sinusoid is non-periodic. By combining the results of Exercise 3.6.1 and Exercise 3.6.2 the mean value is $m_s = 0$ and the power $P_s = \frac{1}{2} + 1 = \frac{3}{2}$.

6. We need to reinterpret the signal first. By Euler's identity we have

$$\begin{aligned} s(n) &= e^{j\frac{3}{2}\pi n} \cdot \tfrac{1}{2}(e^{j\frac{5}{2}\pi n} + e^{j\frac{5}{2}\pi n}) + j\sin(\pi n) \\ &= \tfrac{1}{2}e^{j4\pi n} + \tfrac{1}{2}e^{-j\pi n} + j\sin(\pi n) \\ &= \tfrac{1}{2} + \tfrac{1}{2}(-1)^n + j0 \\ &= \tfrac{1}{2} + \tfrac{1}{2}e^{-j\pi n} \end{aligned}$$

   The signal is evidently periodic of period $N = 2$, as in a period carries the values $s(0) = 1$ and $s(1) = 0$. Moreover, from the results of Exercise 3.6.1 we have $m_s = \frac{1}{2}$ and $P_s = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$.

7. We need to reinterpret the signal first. By Euler's identity we have

$$s(n) = e^{jn} \cdot \tfrac{1}{2j}(e^{jn} - e^{-jn}) = \tfrac{1}{2j}e^{j2n} - \tfrac{1}{2j}\ .$$

   The second contribution is constant, hence periodic of any period. For first contribution we have $\omega_1 T = 2\pi f_1 T = 2$, so that $f_1 T = \frac{1}{\pi}$ which is not rational, hence the sampled signal is non-periodic. By the results of Exercise 3.6.1 the mean value is $m_s = -\frac{1}{2j} = \frac{1}{2}j$ and the power $P_s = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

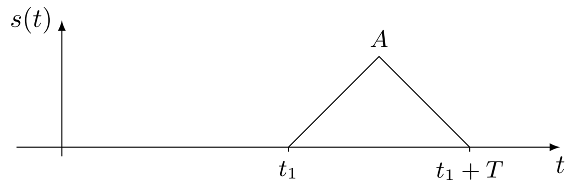
8. Since $\sin(\pi n) = 0$ it is $s(n) = 0$, hence it is periodic of any period with $m_s = P_s = 0$.

FOUNDATIONS OF SIGNALS AND SYSTEMS
4.2 Solved exercises
Prof. T. Erseghe

Exercises 4.2

Solve the following:

1. express the signal in figure as a function of triang(t) by using basic transformations,



2. express rect(t) as a function of the unit step $1(t)$ by exploiting linear combinations and basic transformations,

3. draw the discrete time signal
$$s(n) = \text{rect}\left(\frac{n}{1+2N}\right) .$$

Solutions.

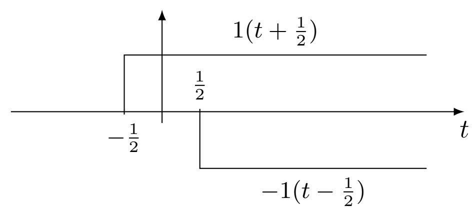
1. We observe that $s(t)$ is a triangle of height A, centred at $t_0 = \frac{1}{2}t_1 + \frac{1}{2}(t_1 + T) = t_1 + \frac{1}{2}T$, with basis of length T. Since triang(t) has a basis of length 2 (its extension is $[-1, 1]$), in order to scale it to length T we need a scaling factor a that maps the extension $[-1, 1]$ into $[-a, a]$ of length $2a = T$, that is we need $a = \frac{1}{2}T$. Therefore, by exploiting the notation $x((t - t_0)/a)$ whose meaning we have learned during the lectures, we have
$$\begin{aligned} s(t) &= A\,\text{triang}\left(\frac{t-t_0}{a}\right) \\ &= A\,\text{triang}\left(\frac{t-t_1-\frac{1}{2}T}{\frac{1}{2}T}\right) \\ &= A\,\text{triang}\left(\frac{2(t-t_1)}{T}-1\right) \end{aligned}$$
As a check it is
$$\begin{aligned} s(t_1) &= A\text{triang}\left(\frac{2(t_1-t_1)}{T}-1\right) = A\text{triang}(-1) \\ s(t_1+T) &= A\text{triang}\left(\frac{2(t_1+T-t_1)}{T}-1\right) = A\text{triang}(1) \end{aligned}$$

which evidences the (linear) map $t_1 \to -1$ and $t_1 + T \to 1$, which is correct.

2. The rectangular signal can be expressed through the expression

$$\text{rect}(t) = 1(t + \tfrac{1}{2}) - 1(t - \tfrac{1}{2}) = \begin{cases} 0 - 0 = 0 & , t < -\frac{1}{2} \\ 1 - 0 = 1 & , -\frac{1}{2} < t < \frac{1}{2} \\ 1 - 1 = 0 & , t > \frac{1}{2} \end{cases}$$

as can be appreciated from the following figure

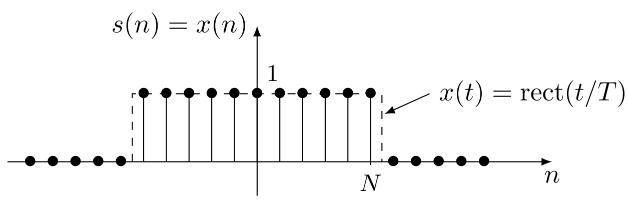


where $1(t + \frac{1}{2}) = 1(t - (-\frac{1}{2}))$ is a unit step shifted to the left by $\frac{1}{2}$, and $1(t - \frac{1}{2})$ is a unit step shifted to the right by $\frac{1}{2}$.

3. We can interpret this signal as the sampled version $s(n) = x(n)$ of

$$x(t) = \text{rect}(t/T)\ , \quad T = 1 + 2N$$

with extension $[-\frac{1}{2}T, \frac{1}{2}T]$ and where $\frac{1}{2}T = N + \frac{1}{2}$. The result can be better understood graphically, providing



that is we have

$$s(n) = \begin{cases} 1 & , |n| \leq N \\ 0 & , \text{otherwise} \end{cases}$$

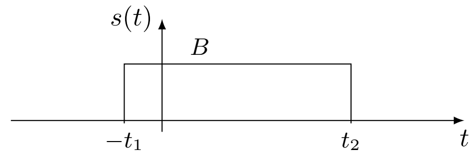
and the discrete-time signal is a rectangle with $1 + 2N$ samples centred at the origin.

FOUNDATIONS OF SIGNALS AND SYSTEMS
4.3 Homework assignment
Prof. T. Erseghe

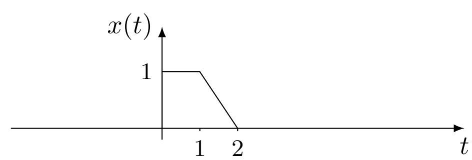
Exercises 4.3

Solve the following:

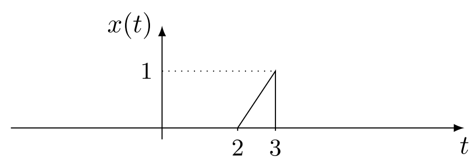
1. express the signal in figure as a function of $\mathrm{rect}(t)$ by using basic transformations,



2. draw $s(t) = x(\frac{3}{2}t + 1)$ for $x(t)$ as given in figure,



3. draw $s(t) = x(-t + 2)$ for $x(t)$ as given in figure,



4. express $\mathrm{sgn}(t)$ as a function of the unit step $1(t)$ by exploiting linear combinations and basic transformations,

5. draw the signal
$$s(t) = \begin{cases} t-1 & 1 < t < 3 \\ 0 & \text{otherwise} \end{cases}$$
then express it as a linear combination of the unit step $1(t)$, the ramp $\mathrm{ramp}(t) = t \cdot 1(t)$, and their time shifts,

6. consider the signal $s(t) = x(-2t+1)$ where $x(t)$ has period $T_x = 2$. Is $s(t)$ a periodic signal? If so, what is its period T_s?

7. consider $s(t) = x(t/a)$ a time scaled version of $x(t)$. What is the connection between the area, mean, energy and power of $x(t)$ and those of $s(t)$?

8. consider $s(n) = x(n - n_0)$ a time shifted version of $x(n)$. What is the connection between the area, mean, energy and power of $x(n)$ and those of $s(n)$?

9. draw the discrete-time signal

$$s(n) = -\operatorname{rect}\left(\frac{n - \frac{1}{2}}{2N}\right)$$

then evaluate its area, mean value, energy, and power.

Solutions.
We exploit the same tricks as in Exercises 4.2, and provide short answers.

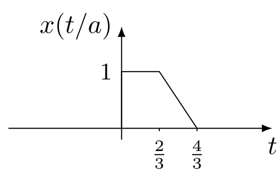
1. We observe that $s(t)$ is a rectangle of height B, centred at $t_0 = \frac{1}{2}(t_2 - t_1)$, with basis of length $T = t_2 + t_1$. Since $\text{rect}(t)$ has a basis of length 1 (its extension is $[-\frac{1}{2}, \frac{1}{2}]$), in order to scale it to length T we need a scaling factor a that maps the extension $[-\frac{1}{2}, \frac{1}{2}]$ into $[-\frac{1}{2}a, \frac{1}{2}a]$ of length $a = T$, that is we need $a = t_1 + t_2$. Therefore, by exploiting the notation $x((t - t_0)/a)$ whose meaning we have learned during the lectures, we have

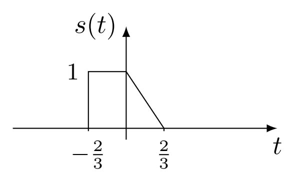
$$\begin{aligned} s(t) &= B \ \text{rect}\left(\frac{t - t_0}{a}\right) \\ &= A \ \text{rect}\left(\frac{t + \frac{1}{2}t_1 - \frac{1}{2}t_2}{t_1 + t_2}\right) \ . \end{aligned}$$

2. We observe that $s(t)$ can be reinterpreted in the form

$$s(t) = x\left(\frac{t - t_0}{a}\right) = x\left(\frac{t - (-\frac{2}{3})}{\frac{2}{3}}\right)$$

hence it is the result of first scaling $x(t)$ by $a = \frac{2}{3}$, then shifting it by $t_0 = -\frac{2}{3}$ (i.e., a shift on the left by $\frac{2}{3}$). We therefore have the following

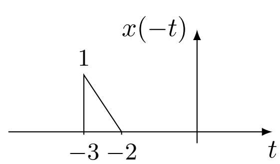


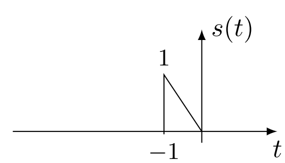


3. We observe that $s(t)$ can be reinterpreted in the form

$$s(t) = x(-(t - 2)) = x_-(t - 2)$$

hence it is the result of first reversing it in time to obtain $x_-(t) = x(-t)$, then shifting the result (to the right) by $t_0 = 2$. We therefore have the following

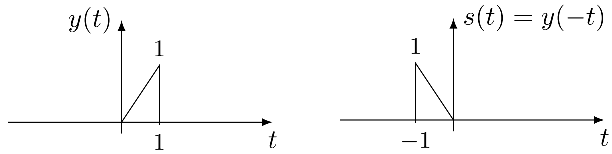




The time reversal and shift operations can be also swapped, by interpreting the signal in the form

$$s(t) = y(-t) \ , \qquad y(t) = x(t + 2) = x(t - (-2))$$

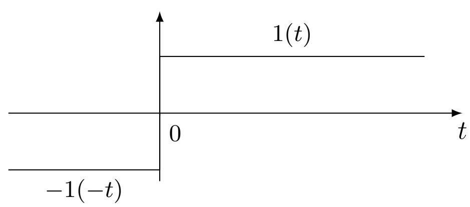
that is as the result of first shifting the signal by $t_1 = -2$ (a shift on the left by 2), and by then applying a time reversal, as illustrated below



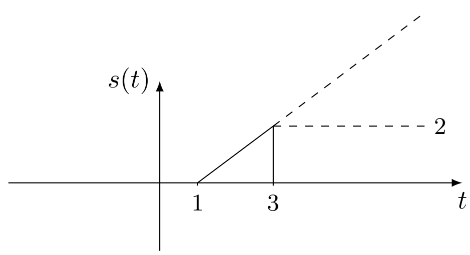
4. The signum signal can be expressed through the expression

$$\mathrm{sgn}(t) = 1(t) - 1(-t) = \begin{cases} 0 - 1 = -1 & , t < 0 \\ 1 - 0 = 1 & , t > 0 \end{cases}$$

as can be appreciated from the following figure



5. The signal is illustrated in figure



and can be expressed in the form

$$s(t) = \mathrm{ramp}(t-1) - 2 \cdot 1(t-3) - \mathrm{ramp}(t-3) \ .$$

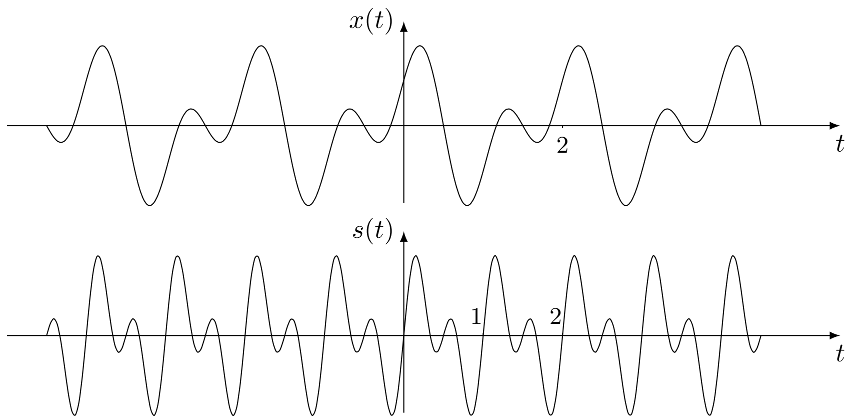
6. Observe that

$$s(t) = x(-(2t-1)) = x_-(2t-1) = x_-\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)$$

that is it is a scaled (by $a = \frac{1}{2}$) and shifted (by $t_0 = \frac{1}{2}$) version of $x(-t)$. Now, if a signal is periodic of period T_p so is its time reversed counterpart (by symmetry), as well as any time-shift counterpart. In fact we have

$$\begin{aligned} x_-(t + T_p) &= x(-t - T_p) = x(-t) = x_-(t) \\ x_{t_0}(t + T_p) &= x(t - t_0 + T_p) = x(t - t_0) = x_{t_0}(t) \ , \end{aligned}$$

where we used $x_{t_0}(t) = x(t - t_0)$. The only basic transformation that changes periodicity is the scaling factor a, which naturally multiplies the time axis, and therefore it simply is $T_s = aT_x = 1$. We show this with an example of a signal periodic of period 2, that is $x(t) = \sin(2\pi t) + \cos(\pi t)$, for which $s(t) = x(-2t + 1)$ is illustrated in figure



7. For the area, by a change of variable $u = t/a$, we have

$$A_s = \int_{-\infty}^{\infty} x(t/a)\, dt = a \int_{-\infty}^{\infty} x(u)\, du = aA_x \ ,$$

so the area is scaled by a. So is the energy, by the same rationale, and in fact

$$E_s = \int_{-\infty}^{\infty} |x(t/a)|^2\, dt = a \int_{-\infty}^{\infty} |x(u)|^2\, du = aE_x \ .$$

Mean value and power, instead, follow a different rule, as they are robust to scaling. We have

$$\begin{aligned} m_s &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t/a)\, dt \\ &= \lim_{T\to\infty} \frac{a}{2T} \int_{-T/a}^{T/a} x(u)\, du \\ &= \lim_{T'\to\infty} \frac{1}{2T'} \int_{-T'}^{T'} x(u)\, du = m_x \ , \qquad T' = T/a \end{aligned}$$

and by an identical argument it also is $P_s = P_x$.

8. For the area, by a change of variable $m = n - n_0$, we have

$$A_s = \sum_{n=-\infty}^{\infty} x(n - n_0) = \sum_{m=-\infty}^{\infty} x(m) = A_x \ ,$$

and so is for the energy, as it is simply the area of $|s(n)|^2$, that is

$$E_s = \sum_{n=-\infty}^{\infty} |x(n-n_0)|^2 = \sum_{m=-\infty}^{\infty} |x(m)|^2 = E_x \ ,$$

Mean value and power are instead more tricky, but they lead to the same result. For the mean, in case $n_0 > 0$, we have

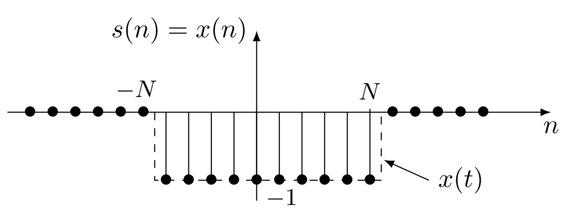
$$\begin{aligned} m_s &= \lim_{N\to\infty} \frac{1}{1+2N} \sum_{n=-N}^{N} x(n-n_0) \\ &= \lim_{N\to\infty} \frac{1}{1+2N} \sum_{m=-N-n_0}^{N-n_0} x(m) \\ &= \lim_{N\to\infty} \frac{1}{1+2N} \left(\sum_{m=-N}^{N} x(m) + \sum_{m=-N-n_0}^{-N-1} x(m) - \sum_{m=N-n_0+1}^{N} x(m) \right) \\ &= m_x \end{aligned}$$

the final equivalence being valid since the second and third series are sums of n_0 values, and are divided by value $1+2N$ that grows to infinity, hence their contribution goes to zero if the signal values are limited as n grows. A similar argument can be used when $n_0 < 0$. Equivalently we can show that $P_s = P_x$.

9. We can interpret this signal as the sampled version $s(n) = x(n)$ of

$$x(t) = -\operatorname{rect}((t-\tfrac{1}{2})/T) \ , \quad T = 2N$$

which is a rectangle with extension $[-N, N]$ shifted by $\frac{1}{2}$, hence its extension is $[-N+\frac{1}{2}, N+\frac{1}{2}]$. The result can be better understood graphically, providing



that is we have

$$s(n) = \begin{cases} -1 & , -N < n \le N \\ 0 & , \text{otherwise} \end{cases}$$

or, in other words, the signal has $2N$ active values each one associated with value -1. Therefore, its area is $A_s = -2N$, its energy $E_s = 2N$, while mean value and power are $m_s = P_s = 0$.

FOUNDATIONS OF SIGNALS AND SYSTEMS

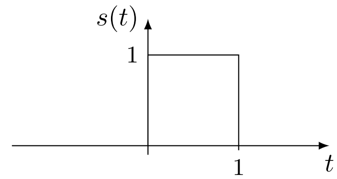
4.5 Solved exercises

Prof. T. Erseghe

Exercises 4.5

Solve the following:

1. identify, and draw, the even and odd parts of signal,



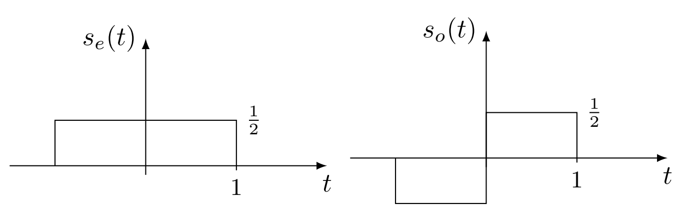
2. identify the even and odd parts of $s(t) = \cos(2\pi f_0 t + \varphi_0)$,
3. identify the even and odd parts of $s(t) = e^{j2\pi f_0 t}$,
4. prove that every even and Hermitian signal $s(t)$ is necessarily real-valued.

Solutions.

1. Since we have $s(t) = \mathrm{rect}(t - \frac{1}{2})$, by definition it is

$$\begin{aligned}
s_e(t) &= \tfrac{1}{2}s(t) + \tfrac{1}{2}s(-t) \\
&= \tfrac{1}{2}\,\mathrm{rect}(t - \tfrac{1}{2}) + \tfrac{1}{2}\,\mathrm{rect}(-t - \tfrac{1}{2}) \\
&= \tfrac{1}{2}\,\mathrm{rect}(t - \tfrac{1}{2}) + \tfrac{1}{2}\,\mathrm{rect}(t + \tfrac{1}{2}) \qquad \text{(since rect is even)} \\
&= \tfrac{1}{2}\;\mathrm{rect}(\tfrac{1}{2}t) \\
s_o(t) &= \tfrac{1}{2}s(t) - \tfrac{1}{2}s(-t) \\
&= \tfrac{1}{2}\,\mathrm{rect}(t - \tfrac{1}{2}) - \tfrac{1}{2}\,\mathrm{rect}(t + \tfrac{1}{2}) \\
&= \tfrac{1}{2}\;\mathrm{rect}(\tfrac{1}{2}t)\,\mathrm{sgn}(t)
\end{aligned}$$

as illustrated in figure



2. In this case it is easier to proceed by exploiting standard rules on sinusoids, to have
$$\begin{aligned} s(t) &= \cos(2\pi f_0 t + \varphi_0) \\ &= \underbrace{\cos(\varphi_0)\cos(2\pi f_0 t)}_{s_e(t)} \underbrace{- \sin(\varphi_0)\sin(2\pi f_0 t)}_{s_o(t)} \end{aligned}$$
to recall that there might exist simpler ways than to apply the rule as-it-is.

3. In this case it is easier to proceed by exploiting Euler's identity, to have
$$\begin{aligned} s(t) &= e^{j2\pi f_0 t + \varphi_0} \\ &= \underbrace{\cos(2\pi f_0 t)}_{s_e(t)=s_{re}(t)} + \underbrace{j\sin(2\pi f_0 t)}_{s_o(t)=s_{im}(t)} \end{aligned}$$

4. An even and Hermitian signal $s(t)$ satisfies
$$\begin{aligned} s(t) &= s(-t) \\ s(t) &= s^*(-t) \end{aligned}$$
so that by exploiting the first equality in the second we obtain
$$s(t) = [s(-t)]^* = [s(t)]^* = s^*(t) \ ,$$
which proves the assertion. Clearly, since an Hermitian signal has an even real part and an odd imaginary part, being the signal even its odd imaginary part must be equal to zero. This property is valid in both continuous and discrete-time.

FOUNDATIONS OF SIGNALS AND SYSTEMS
4.6 Homework assignment
Prof. T. Erseghe

Exercises 4.6

Solve the following:

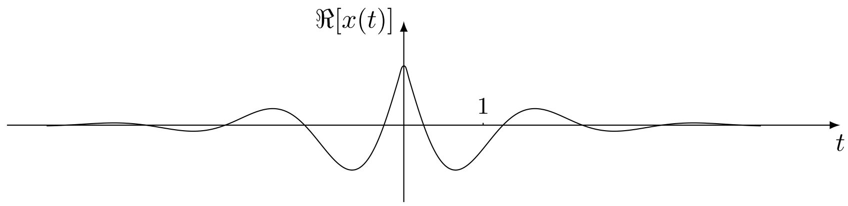
1. identify and sketch the real and imaginary parts of $s(t) = (1+j)\, e^{-|t|+j\pi|t|}$; is the signal real and/or Hermitian?
2. identify and sketch the real and imaginary parts of $s(t) = (1-j)\, e^{(\sigma+j\omega)t}$,
3. identify the even and odd parts of $s(n) = 1_0(n)$,
4. what is the symmetry (even, odd, or none) of the product among:
   a) two even signals,
   b) two odd signals,
   c) an even and an odd signal?
5. prove that every odd and Hermitian signal $s(t)$ is necessarily imaginary-valued.
6. what are the symmetries of $s(n) = n^2 + jn$? (even/odd, real/imaginary, Hermitian/anti-Hermitian)
7. what are the symmetries of $s(t) = je^{jt}$?
8. what are the symmetries of $s(n) = e^{jn}\cos(n)$?
9. what are the symmetries of $s(t) = e^{jt}\sin(t)$?
10. identify a signal that is real, odd and Hermitian,
11. prove that the only signal that is both even and odd is the all-zero signal,
12. prove that the only signal that is both Hermitian and anti-Hermitian is the all-zero signal.

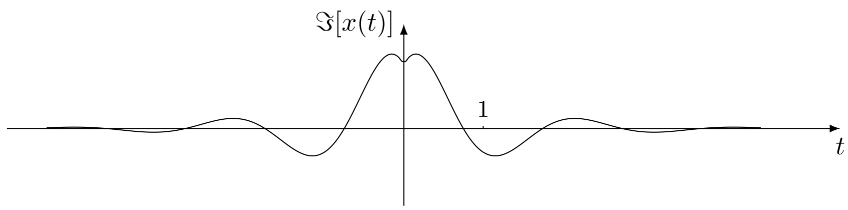
Solutions.

1. We first write $s(t)$ in the more readable form

$$\begin{aligned} s(t) &= \sqrt{2}\, e^{j\frac{\pi}{4}}\, e^{j\pi|t|}\, e^{-|t|} \\ &= \sqrt{2}\, e^{-|t|}\, \cos(\pi|t| + \tfrac{\pi}{4}) + j\sqrt{2}\, e^{-|t|}\, \sin(\pi|t| + \tfrac{\pi}{4}) \end{aligned}$$

which evidences the presence of both a real and an imaginary part, sketched in the figure below.



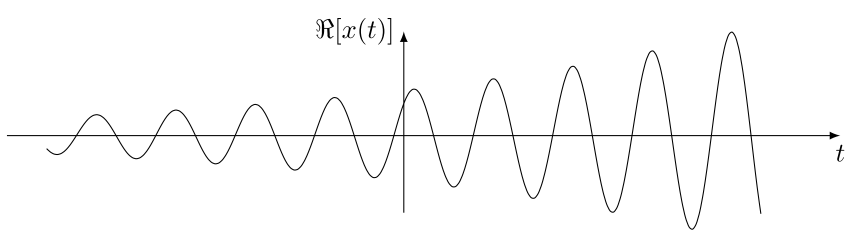


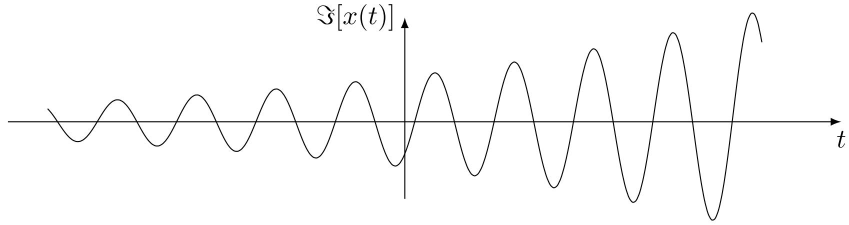
As can be appreciated from the figure, because of the presence of the map $|t|$ the signal is even, so that both its real and imaginary parts (both active) are even, hence it is not an Hermitian signal.

2. We write $s(t)$ in the more readable form

$$\begin{aligned} s(t) &= \sqrt{2}\, e^{-j\frac{\pi}{4}}\, e^{j\omega t}\, e^{\sigma t} \\ &= \sqrt{2}\, e^{\sigma t}\, \cos(\omega t - \tfrac{\pi}{4}) + j\sqrt{2}\, e^{\sigma t}\, \sin(\omega t - \tfrac{\pi}{4}) \end{aligned}$$

which evidences the presence of both a real and an imaginary part, sketched in the figure below for $\sigma > 0$.





3. For the discrete-time unit step we have

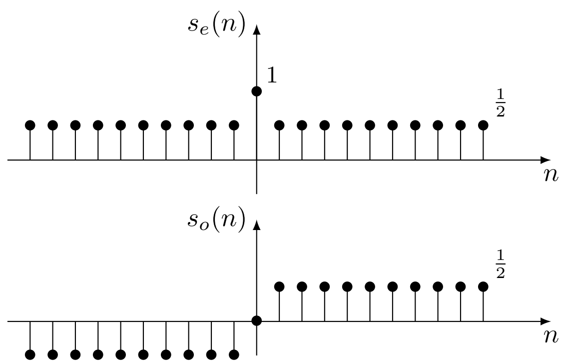
$$1_0(-n) = \begin{cases} 1 & , n \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

hence it is

$$s_e(n) = \frac{1}{2}1_0(n) + 1\frac{1}{2}1_0(-n) = \begin{cases} \frac{1}{2} & , n < 0 \\ 1 & , n = 0 \\ \frac{1}{2} & , n > 0 \end{cases}$$

$$s_o(n) = \frac{1}{2}1_0(n) - 1\frac{1}{2}1_0(-n) = \begin{cases} \frac{1}{2} & , n < 0 \\ 0 & , n = 0 \\ -\frac{1}{2} & , n > 0 \end{cases} = \frac{1}{2}\text{sgn}(n)$$

which is slightly different from the continuous case since, as illustrated below, the even part is not constant.



4. When a) $a(t) = a(-t)$ and $b(t) = b(-t)$ then the product $s(t) = a(t)b(t)$ satisfies

$$s(-t) = a(-t)b(-t) = a(t)b(t) = s(t) \ ,$$

hence the product is even; when, instead, b) it is $a(t) = -a(-t)$ and $b(t) = -b(-t)$ then we have

$$s(-t) = a(-t)b(-t) = [-a(-t)] \cdot [-b(-t)] = a(t)b(t) = s(t) \ ,$$

and the product is still even; when, finally, c) t is $a(t) = a(-t)$ and $b(t) = -b(-t)$ then we have

$$s(-t) = a(-t)b(-t) = -a(-t) \cdot [-b(-t)] = -a(t)b(t) = -s(t) \ ,$$

so that the product is in this case odd. The above properties are valid in both continuous and discrete-time.

5. An odd and Hermitian signal $s(t)$ satisfies

$$s(t) = -s(-t)$$
$$s(t) = s^*(-t)$$

so that by exploiting the first equality in the second we obtain

$$s(t) = [s(-t)]^* = [-s(t)]^* = -s^*(t) ,$$

which proves the assertion. Clearly, since an Hermitian signal has an even real part and an odd imaginary part, being the signal odd its even real part must be equal to zero. This property is valid in both continuous and discrete-time.

6. Signal $s(n) = n^2 + jn$ has an even real part (n^2) and an odd imaginary part (n), hence it is Hermitian.

7. Signal $s(t) = je^{jt} = j\cos(t) - \sin(t)$ has an odd real part ($-\sin(t)$) and an even imaginary part ($\cos(t)$), hence it is anti-Hermitian.

8. Signal $s(n) = e^{jn}\cos(n) = \cos^2(n) + j\sin(n)\cos(n)$ has an even real part ($\cos^2(n)$) and an odd imaginary part ($\cos(n)\sin(n)$), hence it is Hermitian.

9. Signal $s(t) = e^{jt}\sin(t) = \cos(t)\sin(t) + j\sin^2(t)$ has an odd real part ($\cos(t)\sin(t)$) and an even imaginary part ($\sin^2(t)$), hence it is anti-Hermitian.

10. An Hermitian signal has an even real part and an imaginary off part, therefore no signal can be real, odd, and Hermitian, except for $s(t) = 0$, which satisfies any symmetry.

11. The odd symmetry implies $s(t) = -s(-t)$ or, equivalently, $s(-t) = -s(t)$. By using this result in the odd symmetry statement we obtain $s(t) = s(-t) = -s(t)$ which identifies the signal $s(t) = 0$. This property is valid in both continuous and discrete-time.

12. The anti-Hermitian symmetry implies $s(t) = -s^*(-t)$ or, equivalently, $s^*(-t) = -s(t)$. By using this result in the Hermitian symmetry statement we obtain $s(t) = s^*(-t) = -s(t)$ which identifies the signal $s(t) = 0$. This property is valid in both continuous and discrete-time.

FOUNDATIONS OF SIGNALS AND SYSTEMS
5.2 Solved exercises
Prof. T. Erseghe

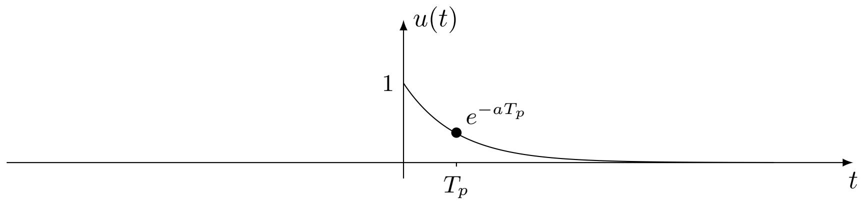
Exercises 5.2

Solve the following:

1. calculate the periodic repetition of period T_p for $u(t) = e^{-at}1(t)$ with $a > 0$,
2. prove that the period repetition of $u(-t)$ is equivalent to the time-reversed counterpart of the periodic repetition $s(t) = \text{rep}_{T_p} u(t)$ of $u(t)$.

Solutions.

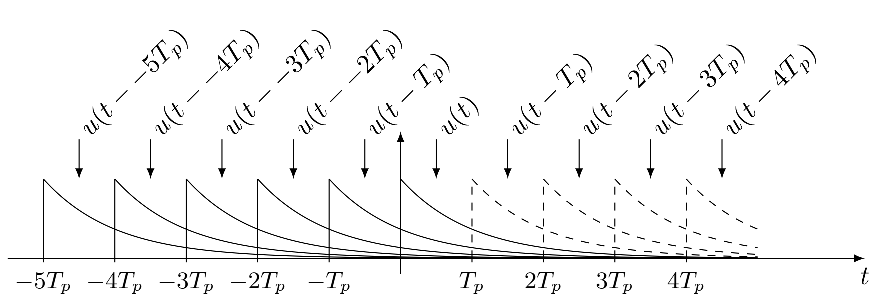
1. We first plot the signal $u(t) = e^{-at}1(t)$ and observe that it has infinite extension, which will naturally introduce aliasing.



If we then investigate the periodic repetition

$$s(t) = \sum_{n=-\infty}^{\infty} u(t - nT_p)$$

we see that the contributions $u(t - nT_p)$ superpose (aliasing effect)



Now, the correct approach to periodic repetition is to identify *one* specific period, which we choose to be the period $(0, T_p)$. In this reference period, as can be observed from the figure, because of the presence of the unit step $1(t)$, only the shifted contributions $u(t - nT_p)$ with $n \leq$ contribute to $s(t)$. These are highlighted in solid lines in the figure, while the contributions for $n > 0$ are highlighted in dashed lines. Specifically, we have

$$s(t) = \sum_{n=-\infty}^{0} u(t - nT_p) = \sum_{n=-\infty}^{0} e^{-a(t-nT_p)} \; , \qquad t \in (0, T_p)$$

which identifies a (truncated) geometric series that can be easily solved as

$$\begin{aligned} s(t) &= e^{-at} \sum_{n=-\infty}^{0} e^{naT_p} \\ &= e^{-at} \sum_{m=0}^{\infty} e^{-maT_p} \\ &= e^{-at} \sum_{m=0}^{\infty} (e^{-aT_p})^m = \frac{e^{-at}}{1 - e^{-aT_p}} \; , \qquad t \in (0, T_p) \end{aligned}$$

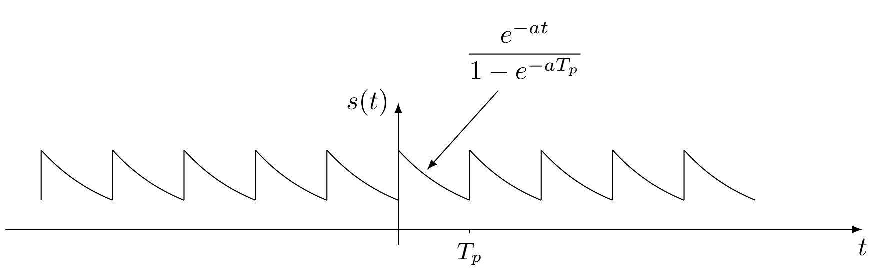
where we also note that $e^{-aT_p} < 1$. Knowing the signal expression in one period naturally allows for extending it to any period by means of a simple time-shift, that is

$$s(t) = \begin{cases} v(t) = Be^{-at} & , \; t \in (0, T_p) \\ v(t - T_p) = Be^{-a(t-T_p)} & , \; t \in (T_p, 2T_p) \\ v(t + T_p) = Be^{-a(t+T_p)} & , \; t \in (-T_p, 0) \\ \vdots & \end{cases} \; , \qquad B = \frac{1}{1 - e^{-aT_p}}$$

and in general we have

$$s(t) = v(t - nT_p) = Be^{-a(t-nT_p)} \; , \quad t \in (nT_p, (n+1)T_p) \; ,$$

as illustrated in the figure below



2. We want to prove that

$$z(t) = \operatorname*{rep}_{T_p} u(-t) = s(-t)\ , \qquad s(t) = \operatorname*{rep}_{T_p} u(t)$$

To do so we expand $z(t)$, to have

$$z(t) = \sum_{n=-\infty}^{\infty} u(-(t-nT_p)) = \sum_{n=-\infty}^{\infty} u(-t+nT_p) = \sum_{m=-\infty}^{\infty} u(-t-mT_p) = s(-t)$$

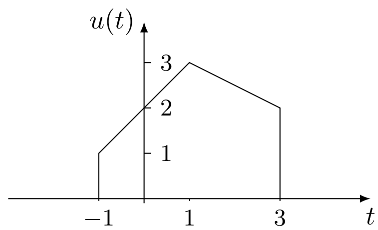
where we replaced $n = -m$.

FOUNDATIONS OF SIGNALS AND SYSTEMS
5.3 Homework assignment
Prof. T. Erseghe

Exercises 5.3

Solve the following:

1. evaluate the periodic repetition of period T_p for $u(t) = e^{-a|t|}$ with $a > 0$,
2. evaluate the periodic repetition of period $T_p = 2$ for the signal



3. evaluate the periodic repetition of period T_p for the signal

$$u(t) = \frac{|t|}{T} \operatorname{rect}\left(\frac{t}{2T}\right)$$

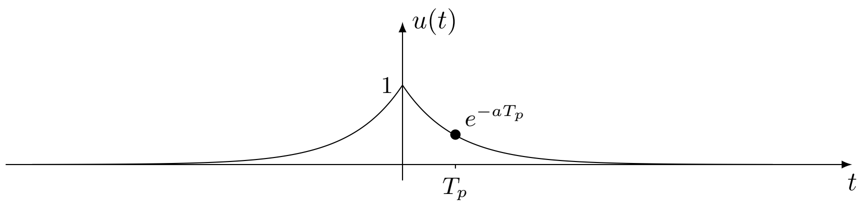
   by considering $T_p \in (T, 2T)$,

4. evaluate the periodic repetition of period N for the discrete-time signal $u(n) = a^{-n} 1_0(n)$ for $|a| > 1$.

Solutions.

1. We can solve the exercise by exploiting the result of Exercise 5.2.1 and the properties of the periodic repetition. We first observe that

$$u(t) = e^{-a|t|} = \begin{cases} e^{at} & , \ t < 0 \\ e^{-at} & , \ t > 0 \end{cases}$$



can be written in the form

$$u(t) = u_1(t) + u_1(-t) \ , \qquad u_1(t) = e^{-at} 1(t)$$

so that by the properties (linearity and time reversal) of the periodic repetition we have

$$s(t) = \operatorname*{rep}_{T_p} u(t) = s_1(t) + s_1(-t) \ , \qquad s_1(t) = \operatorname*{rep}_{T_p} u_1(t)$$

where from Exercise 5.1.1 we know that

$$s_1(t) = B\, e^{-at} \ , \qquad B = \frac{1}{1 - e^{-aT_p}} \ , \quad t \in (0, T_p) \ .$$

Now, for the time-reversed version of $s_1(t)$ we have (by symmetry)

$$s_1(-t) = B\, e^{at} \ , \qquad t \in (-T_p, 0) \ ,$$

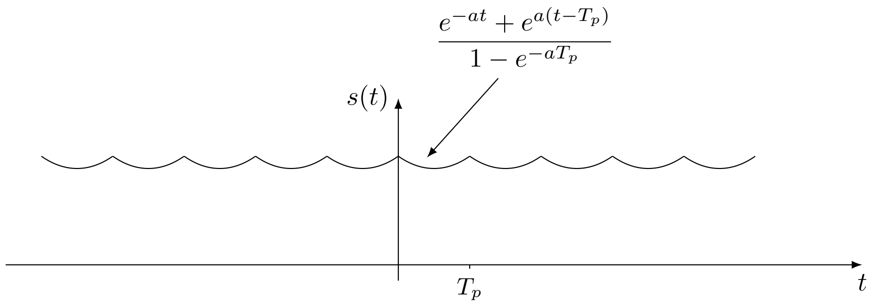
from which we obtain, by time-shift,

$$s_1(-t) = B\, e^{a(t-T_p)} \ , \qquad t \in (0, T_p) \ ,$$

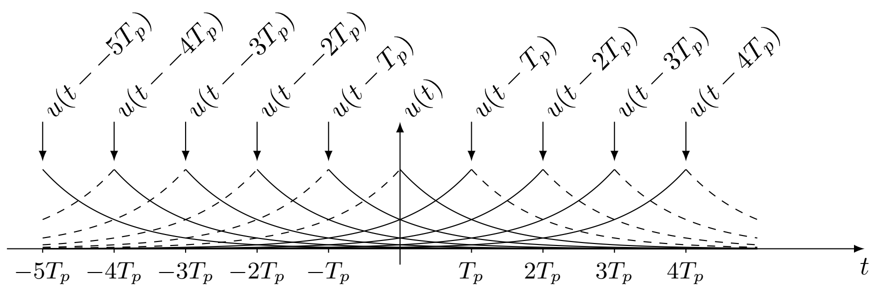
and, therefore,

$$s(t) = B\left(e^{-at} + e^{a(t-T_p)}\right) \ , \qquad t \in (0, T_p) \ ,$$

as illustrated in the figure below



Alternatively, we can proceed without exploiting any property, by observing the behaviour of $u(t - nT_p)$

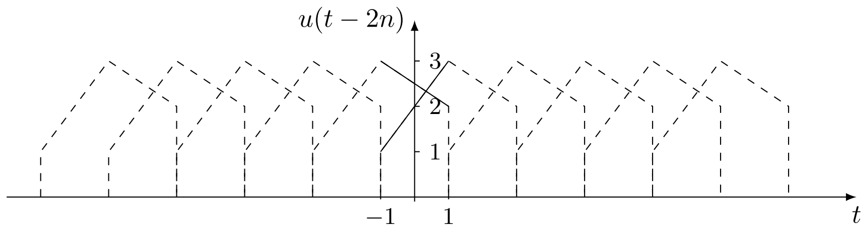


where the solid lines indicate the contributions that are active in the interval $(0, T_p)$, that is, the contributions $e^{-a(t-nT_p)}$ for $n \leq 0$, and the contributions $e^{a(t-nT_p)}$ for $n > 0$. Hence, we have

$$\begin{aligned}
s(t) &= \sum_{n=-\infty}^{0} u(t - nT_p) \\
&= \sum_{n=-\infty}^{0} e^{-a(t-nT_p)} + \sum_{n=1}^{\infty} e^{a(t-nT_p)} \\
&= e^{-at} \sum_{n=-\infty}^{0} (e^{-aT_p})^{-n} + e^{at} \sum_{n=1}^{\infty} (e^{-aT_p})^{n} \\
&= Be^{-at} + (B-1)\, e^{at} \\
&= B\Big(e^{-at} + e^{a(t-T_p)}\Big) \, , \qquad t \in (0, T_p) \, ,
\end{aligned}$$

which correctly leads to the same result.

2. For the signal at hand, the time-shifted counterparts $u(t - 2n)$ take the form



where we highlighted in solid lines the only two contributions that are active in the period $(-1,1)$, namely those for $u(t)$ and for $u(t+2)$. Since it is

$$u(t) = \begin{cases} 2+t & , t \in (-1,1) \\ \frac{7}{2} - \frac{1}{2}t & , t \in (1,3) \\ 0 & \text{otherwise} \end{cases}$$

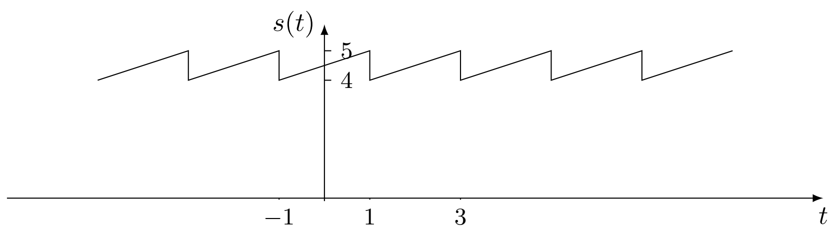
we have, by time-shift,

$$u(t+2) = \frac{7}{2} - \frac{1}{2}(t+2) = \frac{5}{2} - \frac{1}{2}t \, , \qquad t \in (-1,1)$$

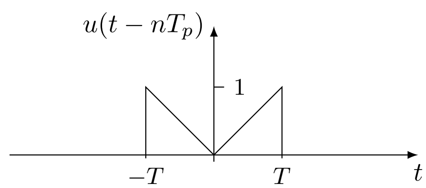
and therefore

$$s(t) = u(t) + u(t+2) = 2 + t + \frac{5}{2} - \frac{1}{2}t = 4 + \frac{1}{2}(t+1) \, , \quad t \in (-1,1)$$

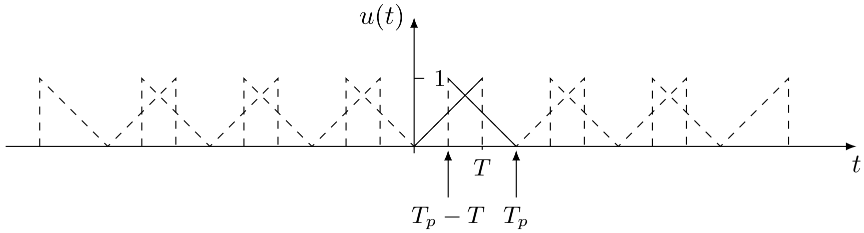
as illustrated in figure



3. We first plot the signal



then illustrate its time-shifts $u(t-nT_p)$ for $T_p \in (T, 2T)$



where we denoted in solid lines the only two contributions that are active in the period $(0, T_p)$, namely those for $u(t)$ and for $u(t - T_p)$. Since its is

$$u(t) = \begin{cases} t/T & , t \in (0, T) \\ -t/T & , t \in (-T, 0) \\ 0 & , \text{otherwise} \end{cases}$$

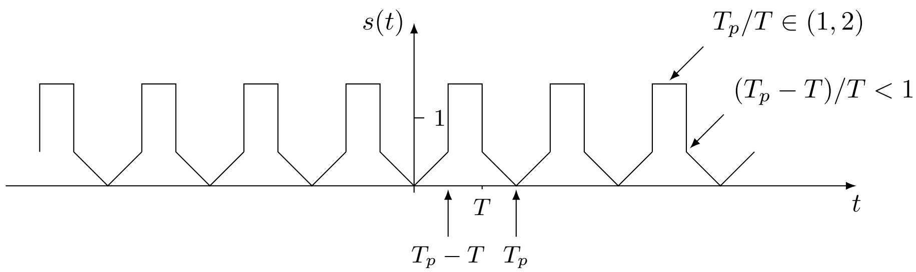
and

$$u(t - T_p) = \begin{cases} (t - T_p)/T & , t \in (T_p, T_p + T) \\ -(t - T_p)/T & , t \in (T_p - T, T_p) \\ 0 & , \text{otherwise} \end{cases}$$

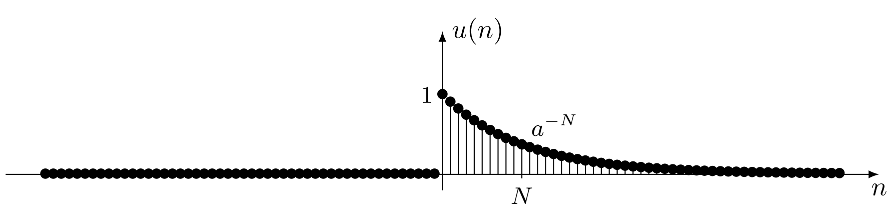
and since $T_p - T < T < T_p$, then in the period $(0, T_p)$ we have

$$s(t) = u(t) + u(t - T_p) = \begin{cases} t/T + 0 & , t \in (0, T_p - T) \\ t/T - (t - T_p)/T = T_p/T & , t \in (T_p - T, T) \\ -(t - T_p)/T & , t \in (T, T_p) \end{cases}$$

as illustrated in the figure below



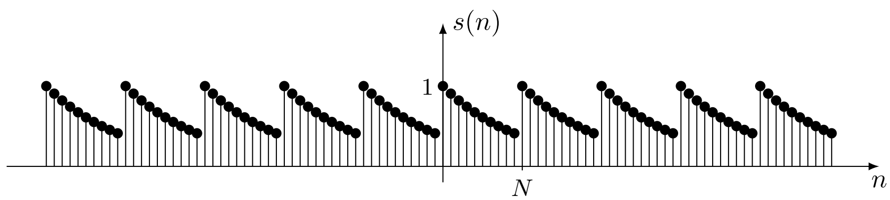
4. This is the discrete counterpart to Exercise 5.2.1. The signal $u(n) = a^{-n} 1_0(n)$ corresponds to a unilateral exponential signal in discrete time



whose periodic repetition, in the period $[0, N)$, only includes the time-shify counterparts $u(n - kN)$ for $k \leq 0$, that is it assumes the form

$$
\begin{aligned}
s(n) &= \sum_{k=-\infty}^{\infty} u(n - kN) \\
&= \sum_{k=-\infty}^{0} u(n - kN) \\
&= \sum_{k=-\infty}^{0} a^{-(n-kN)} \\
&= a^{-n} \sum_{k=-\infty}^{0} (a^{-N})^{-}k \\
&= B\, a^{-n}\,, \qquad B = \frac{1}{1 - a^{-N}}\,, \qquad n \in [0, N)
\end{aligned}
$$

the result being displayed in the figure below for $a > 1$.



FOUNDATIONS OF SIGNALS AND SYSTEMS
5.5 Solved exercises
Prof. T. Erseghe

Exercises 5.5

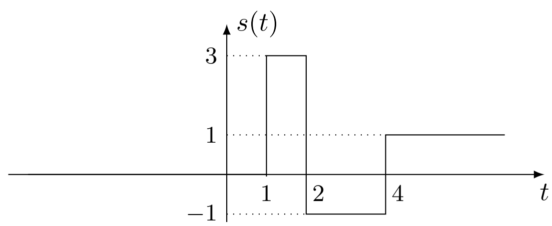
Solve the following:

1. apply the sifting property to the following expressions
$$s_1(t) = \cos(t)\delta(t) + \sin(t)\delta(t-\pi)$$
$$a_1 = \int_0^{40} \delta(t+3) - \delta(t-3) + 2\delta(t-10)\, dt$$
$$s_2(n) = \sin(n+1)\delta(n+1) + e^{-n}\delta(n-3)$$
$$a_2 = \sum_{n=-5}^{5} \delta(n+3) - \delta(n-3) + 2\delta(n-10)\ ,$$

2. prove that the first derivative $\delta'(t)$ is a generalised function that satisfies the sifting property
$$\int_{-\infty}^{\infty} s(t)\delta'(t-t_0)\, dt = -s'(t_0)\ ,$$

3. evaluate the generalised derivative for $\mathrm{rect}(t)$ and $\mathrm{sgn}(t)$,

4. evaluate the generalised derivative for the signal



5. evaluate the generalised derivative for $s(t) = \cos(t)1(t)$.

Solutions.

1. For the first signal we simply have
$$s_1(t) = \cos(0)\delta(t) + \sin(\pi)\delta(t-\pi) = 1\delta(t) + 0\delta(t-\pi) = \delta(t)$$
The second is an integral over the range $[0, 40]$ hence the sifting property must be applied only to those deltas belonging to the interval, that is we have
$$a_1 = \int_0^{40} -\delta(t-3) + 2\delta(t-10)\, dt = -1 + 2 = 1$$

where $\delta(t+3)$ was dropped since it is centred in -3, i.e., outside the interval. For the third signal we have

$$s_2(n) = \sin(0)\delta(n+1) + e^{-3}\delta(n-3) = e^{-3}\delta(n-3)$$

Finally, the last sum has range $[-5, 5]$, hence any delta outside this interval must be discarded, and we have

$$a_2 = \sum_{n=-5}^{5} \delta(n+3) - \delta(n-3) = 1 - 1 = 0$$

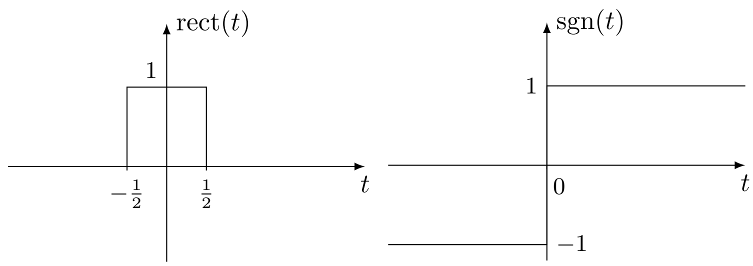
where we discarded $\delta(n-10)$ since it is centred in 10.

2. In this case we solve the integral by parts considering the couples $\delta \to \delta'$ and $s \to s'$, to have

$$\int_{-\infty}^{\infty} s(t)\delta'(t-t_0)\,dt = s(t)\delta'(t-t_0)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} s'(t)\delta(t-t_0)\,dt$$
$$= (0-0) - s'(t_0)$$

where in the last equality we exploited the sifting property.

3. The generalised derivatives can be easily identified from the plots



from which we observe that the rectangle has a discontinuity of $D = 1$ in $t = -\frac{1}{2}$ (the value increases), and one of $D = -1$ in $t = \frac{1}{2}$ (the value decreases), while for the signum there is only one discontinuity $d = 2$ at $t = 0$. Elsewhere, both signals are constant (they are piecewise constant signals), hence their derivative in the constant regions is simply zero. Hence, it is

$$\text{rect}'(t) = \delta(t + \tfrac{1}{2}) - \delta(t - \tfrac{1}{2})\,, \qquad \text{sgn}'(t) = 2\delta(t)\,.$$

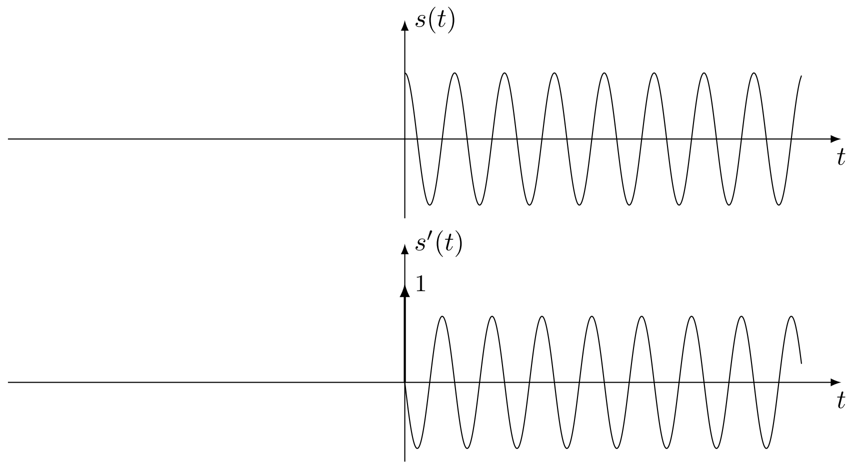
One could also exploit the link with the unit step, to write

$$\text{rect}(t) = 1(t + \tfrac{1}{2}) - 1(t - \tfrac{1}{2})\,, \qquad \text{sgn}(t) = 1(t) - 1(-t)\,,$$

and then obtain the same result by derivation and by exploiting $1' = \delta$.

4. The generalized derivative here is easily obtained by simply reading the plot. The signal is a piecewise constant signal, hence its derivative is zero, except at discontinuities. The discontinuities are: one with $D = 3$ at $t = 1$ (the signal value increases by 3), a second with $d = -4$ at $t = 2$ (the signal value decreases by 4), and, finally, a third one with $D = 2$ at $t = 4$. Hence, we have
$$s'(t) = 3\delta(t-1) - 4\delta(t-2) + 2\delta(t-4) \; .$$

5. Here we simply need to exploit the rule of the derivative of a product, to have
$$\begin{aligned} s'(t) &= \cos'(t)1(t) + \cos(t)1'(t) \\ &= -\sin(t)1(t) + \cos(t)\delta(t) \\ &= -\sin(t)1(t) + \cos(0)\delta(t) = -\sin(t)1(t) + \delta(t) \end{aligned}$$
as illustrated graphically in the figure below.



FOUNDATIONS OF SIGNALS AND SYSTEMS
5.6 Homework assignment
Prof. T. Erseghe

Exercises 5.6

Solve the following:

1. prove that the second derivative $\delta''(t)$ is a generalised function that satisfies the sifting property
$$\int_{-\infty}^{\infty} s(t)\delta''(t-t_0)\,dt = s'(t_0)\ ,$$
2. evaluate the generalised derivative for $s(t) = \mathrm{sgn}(t)\, e^{(2+j)t}$,
3. evaluate the generalised derivative for $s(t) = \frac{1}{2}\mathrm{sgn}(t) - 1(t) + t^2 \cdot 1(t)$,
4. evaluate the generalised derivative for a signal $s(t)$ periodic of period $T_p = 2$ and such that, in a period,
$$s(t) = \begin{cases} 3t & ,\ t \in (0,1) \\ 0 & ,\ t \in (1,2) \end{cases}$$

Solutions.

1. In this case we solve the integral by parts considering the couples $\delta' \to \delta''$ and $s \to s'$, to have

$$\begin{aligned}\int_{-\infty}^{\infty} s(t)\delta''(t-t_0)\,dt &= s(t)\delta''(t-t_0)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} s'(t)\delta'(t-t_0)\,dt \\ &= (0-0) - [-s''(t_0)]\end{aligned}$$

where in the last equality we exploited the result of Exercise 5.5.2. Using the same rationale, one can also easily prove, by induction, that

$$\int_{-\infty}^{\infty} s(t)\delta^{(k)}(t-t_0)\,dt = (-1)^k s^{(k)}(t_0)$$

where $^{(k)}$ denotes the derivative of order k.

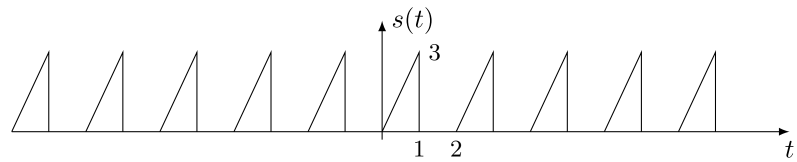
2. Here we simply need to exploit the rule of the derivative of a product, to have

$$\begin{aligned}s'(t) &= \mathrm{sgn}'(t)\,e^{(2+j)t} + \mathrm{sgn}(t)\,(2+j)\,e^{(2+j)t} \\ &= 2\,e^{(2+j)t}\delta(t) + (2+j)\,\mathrm{sgn}(t)\,e^{(2+j)t} \\ &= 2\,\delta(t) + (2+j)\,\mathrm{sgn}(t)\,e^{(2+j)t}\end{aligned}$$

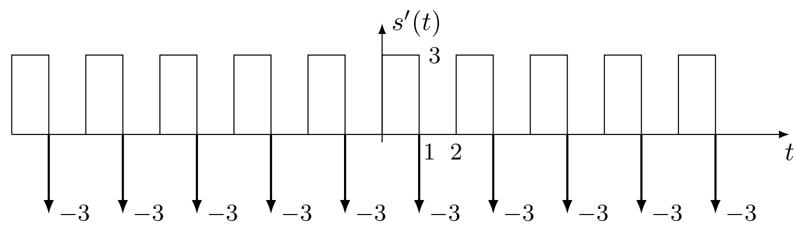
3. Here, again, we simply need to exploit the rule of the derivative of a product, to have

$$\begin{aligned}s'(t) &= \tfrac{1}{2}\mathrm{sgn}'(t) - 1'(t) + 2t\cdot 1(t) + t^2\cdot 1'(t) \\ &= \tfrac{1}{2}\,2\delta(t) - \delta(t) + 2t\cdot 1(t) + t^2\delta(t) \\ &= \delta(t) - \delta(t) + 2t\cdot 1(t) + 0\delta(t) \\ &= 2t\cdot 1(t)\end{aligned}$$

4. In this case we can either draw the signal



from which its derivative, graphically, is



or we can exploit formulas, and write the signal in the form

$$s(t) = \text{rep}_2 u(t) , \qquad u(t) = 3t \ \text{rect}(t - \tfrac{1}{2})$$

so that

$$\begin{aligned} s'(t) = \text{rep}_2 u'(t) , \qquad u'(t) &= 3 \ \text{rect}(t - \tfrac{1}{2}) + 3t \, \text{rect}'(t - \tfrac{1}{2}) \\ &= 3 \ \text{rect}(t - \tfrac{1}{2}) + 3t \, \delta(t - \tfrac{1}{2} + \tfrac{1}{2}) - 3t \, \delta(t - \tfrac{1}{2} - \tfrac{1}{2}) \\ &= 3 \ \text{rect}(t - \tfrac{1}{2}) + 3t \, \delta(t) - 3t \, \delta(t - 1) \\ &= 3 \ \text{rect}(t - \tfrac{1}{2}) + 0 \, \delta(t) - 3 \, \delta(t - 1) \\ &= 3 \ \text{rect}(t - \tfrac{1}{2}) - 3 \, \delta(t - 1) \end{aligned}$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
6.2 Solved exercises
Prof. T. Erseghe

Exercises 6.2

Discuss reality, memory (static, causal, anti-causal, finite-memory), and BIBO stability properties for the following systems:

1.
$$y(n) = \sum_{k=-5}^{5} e^{|k|} |x(n-k)|^2 \ ,$$

2.
$$y(t) = \begin{cases} 0 & , \ t \leq 2 \\ \cos(t+2) \int_{-1}^{t-2} x(u)\, du & , \ t > 2, \end{cases}$$

3.
$$y(n) = \begin{cases} \operatorname{sgn}(1/x(n)) & , \ x(n) \neq 0 \\ 0 & , \ x(n) = 0, \end{cases}$$

4.
$$y(n) = \min(|x(n)|, |n|) \ .$$

Solutions.

1. The system is real, as, independently of the values of $x(n)$, the output is a linear combination of real-valued positive contributions thanks to the presence of the absolute value. With respect to memory, the output at time n gathers together input values in the range $[n-5, n+5]$, hence the system has finite-memory. With respect to BIBO stability, for $|x(n)| < L_x$ we have
$$\begin{aligned} |y(n)| &= \left| \sum_{k=-5}^{5} e^{|k|} |x(n-k)|^2 \right| \\ &= \sum_{k=-5}^{5} e^{|k|} |x(n-k)|^2 \\ &\leq \sum_{k=-5}^{5} e^{|k|} L_x \\ &= L_y < \infty \end{aligned}$$
where it is evident that L_y is limited (it is a limited sum) even without the need to calculate its exact value. Hence, the system is BIBO stable.

2. The system is real, as it only involves multiplications by real-valued signals ($\cos(t+2)$). With respect to memory, the output at time t gathers together input values in the range $[-1, t-2]$ when $t > 2$, hence it is causal. With respect to BIBO stability, we show with a counterexample that it is not BIBO stable. In fact, by choosing $x(t) = 1(t)$ we obtain

$$\begin{aligned} y(t) &= \begin{cases} 0 & , t \leq 2 \\ \cos(t+2) \int_{-1}^{t-2} 1(u)\, du = (t-2)\, \cos(t+2) & , t > 2, \end{cases} \\ &= (t-2)\, \cos(t+2)\, 1(t-2) \end{aligned}$$

which is an oscillating signal that gets larger and larger as t increases.

3. Note that this is a static system of the form $y(n) = f(x(n))$ with

$$f(x) = \begin{cases} \text{sgn}(1/x) & , x \neq 0 \\ 0 & , x = 0 \end{cases} = \text{sgn}(x)$$

It is real by construction, since the definition does not make sense for complex $x(n)$. It is also BIBO stable since $|y(n)| = |\text{sgn}(x(n))| \leq 1$ from the properties of the signum.

4. Also this is a static system, whose function, however, is updated at each time-step n, that is $y(n) = f_n(x(n))$. It is a real system, thanks to the presence of the absolute value. It is also a BIBO stable system since, for $|x(n)| < L_x$ we have

$$|y(n)| = y(n) = \min(|x(n)|, |n|) < \min(L_x, |n|) \leq L_x \,.$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
6.3 Homework assignment
Prof. T. Erseghe

Exercises 6.3

Discuss reality, memory (static, causal, anti-causal, finite-memory), and BIBO stability properties for the following systems:

1. $$y(t) = \int_{t-1}^{t+1} |t-u| x(u)\, du \;,$$

2. $$y(n) = \sum_{k=1}^{\infty} 2^k x(n-k) \;,$$

3. $$y(t) = \int_{t-1}^{t} e^{t+u} x(u)\, du \;,$$

4. $$y(n) = \sum_{k=n-10}^{n+10} x(k) \;,$$

5. $$y(t) = x(t-2) \;,$$

6. $$y(n) = \sum_{k=-\infty}^{n-1} 3^k x(k) \;,$$

7. $$y(t) = \cos(t-2)\, x(t) \;,$$

8. $$y(t) = x(t+5)\, x(t-1) \;,$$

9. $$y(t) = \int_{-\infty}^{2t} |t-u|^2 x(u)\, du \;,$$

Solutions.

1. The system is real, as, for real $x(t)$ the output is a real-valued linear combination of real-valued signal samples. With respect to memory, the output at time t gathers together input values in the range $[t-1, t+1]$, hence the system has finite-memory. With respect to BIBO stability, for $|x(n)| < L_x$ we have

$$\begin{aligned} |y(t)| &= \left| \int_{t-1}^{t+1} |t-u| x(u) \, du \right| \\ &\leq \int_{t-1}^{t+1} |t-u||x(u)| \, du \\ &< \int_{t-1}^{t+1} |t-u| L_x \, du \\ &= L_x \int_{-1}^{1} |v| \, dv = L_x < \infty \end{aligned}$$

   where $v = t - u$. Hence, the system is BIBO stable.

2. The system is real, as, for real $x(n)$ the output is a real-valued linear combination of real-valued signal samples. With respect to memory, the output at time n gathers together input values in the range $(-\infty, n-1]$, hence the system is causal. The system is non BIBO stable, as we can verify using $x(n) = 1$, for which we have

$$y(n) = \sum_{k=1}^{\infty} 2^k = \infty \; .$$

3. The system is real, as, for real $x(t)$ the output is a real-valued linear combination of real-valued signal samples. With respect to memory, the output at time t gathers together input values in the range $[t-1, t]$, hence the system has finite-memory and it is also causal. The system is non BIBO stable, as we can verify using $x(t) = 1$, for which we have

$$\begin{aligned} y(t) &= \int_{t-1}^{t} e^{t+u} \, du \\ &= e^t \int_{t-1}^{t} e^u \, du \\ &= e^t \left(e^t - e^{t-1} \right) = e^{2t} \left(1 - e^{-1}\right) \end{aligned}$$

   which gets to infinite value as t approaches infinity.

4. The system is real, as, for real $x(n)$ the output is a real-valued linear combination of real-valued signal samples. With respect to memory, the

output at time n gathers together input values in the range $[n-10, n+10]$, hence the system has finite memory. The system is BIBO stable, as we can verify assuming $|x(n)| < L_x$, for which we have

$$\begin{aligned} |y(n)| &= \left| \sum_{k=n-10}^{n+10} x(k) \right| \\ &\leq \sum_{k=n-10}^{n+10} |x(k)| \\ &< \sum_{k=n-10}^{n+10} L_x = 21 L_x < \infty \end{aligned}$$

5. The system is real, as, for real $x(t)$ the output is evidently real-valued. With respect to memory, the output at time t gathers together input values in the range $[t-2]$, hence the system has finite memory, and it is also causal. The system is BIBO stable, as we can verify assuming $|x(n)| < L_x$, for which we have $|y(t)| = |x(t-2)| < L_x$.

6. The system is real, as, for real $x(n)$ the output is a real-valued linear combination of real-valued signal samples. With respect to memory, the output at time n gathers together input values in the range $(-\infty, n-1]$, hence the system is causal. The system is non BIBO stable, as we can verify assuming $x(n) = 1$, for which we have

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{n-1} 3^k \\ &= \sum_{m=0}^{\infty} 3^{n-1-m} \\ &= 3^{n-1} \frac{1}{1-\frac{1}{3}} = \tfrac{1}{2} \cdot 3^n \end{aligned}$$

   where we used $m = n - 1 - k$.

7. The system is real, as, for real $x(t)$ the output is evidently real-valued. With respect to memory, the output at time t gathers together input values in the range $[t]$, hence the system is instantaneous (both causal and anti-causal). The system is BIBO stable, as we can verify assuming $|x(n)| < L_x$, for which we have $|y(t)| = |\cos(t-2)| \cdot |x(t)| < 1 \cdot L_x$.

8. The system is real, as, for real $x(t)$ the output is evidently real-valued. With respect to memory, the output at time t gathers together input values in the range $[t-1, t+5]$, hence it has finite memory. The system is BIBO stable, as we can verify assuming $|x(n)| < L_x$, for which we have $|y(t)| = |x(t+5)| \cdot |x(t-1)| < L_x^2 < \infty$.

9. The system is real, as, for real $x(t)$ the output is a real-valued linear combination of real-valued signal samples. With respect to memory, the output at time t gathers together input values in the range $(-\infty, 2t]$, hence the system is simply dynamic. The system is non BIBO stable, as we can verify assuming $x(n) = 1(t)$, for which we have

$$\begin{aligned} y(t) &= \int_{-\infty}^{2t} |t-u|^2 1(u)\,du \\ &= 1(t) \int_{0}^{2t} (t-u)^2\,du \\ &= 1(t) \int_{-t}^{t} v^2\,dv = 1(t)\,\tfrac{2}{3}t^3 \end{aligned}$$

where we used $v = u - t$

FOUNDATIONS OF SIGNALS AND SYSTEMS
6.5 Solved exercises
Prof. T. Erseghe

Exercises 6.5

For each of the following systems state if they are linear and/or time-invariant, and evaluate their impulse response $h(t)$, as well as the response $h_{-1}(t)$ to the unit step:

1.
$$y(n) = \sum_{k=-5}^{5} e^{|k|} |x(n-k)|^2 \ ,$$

2.
$$y(t) = \begin{cases} 0 & , t \leq 2 \\ \cos(t+2) \int_{-1}^{t-2} x(u)\, du & , t > 2, \end{cases}$$

3.
$$y(n) = \begin{cases} \operatorname{sgn}(1/x(n)) & , x(n) \neq 0 \\ 0 & , x(n) = 0, \end{cases}$$

4.
$$y(n) = \min(|x(n)|, |n|) \ .$$

Solutions.

1. The system is not linear, since it involves the absolute value, which is not a linear mapping. With respect to time invariance, we need to compare

$$y(n-n_0) = \sum_{k=-5}^{5} e^{|k|} |x(n-n_0-k)|^2$$
$$\Sigma[x(n-n_0)] = \sum_{k=-5}^{5} e^{|k|} |x(n-k-n_0)|^2$$

which are equal, hence the system is time-invariant. The impulse response is

$$\begin{aligned} h(n) &= \sum_{k=-5}^{5} e^{|k|} |\delta(n-k)|^2 \\ &= \sum_{k=-5}^{5} e^{|k|} \delta(n-k) \\ &= \sum_{k=-5}^{5} e^{|k|} \delta(k-n) \quad \text{since } \delta(n) \text{ is even} \\ &= e^{|n|} \sum_{k=-5}^{5} \delta(k-n) = \begin{cases} e^{|n|} & , n \in [-5,5] \\ 0 & , \text{otherwise} \end{cases} \end{aligned}$$

while the response to the unit step is

$$\begin{aligned}
h_{-1}(n) &= \sum_{k=-5}^{5} e^{|k|} |1_0(n-k)|^2 \\
&= \sum_{k=-5}^{5} e^{|k|} 1_0(n-k)
\end{aligned}$$

where $1_0(n-k)$ is active for $n-k \geq 0$, that is, for $k \leq n$. Hence, we obtain

$$h_{-1}(n) = \begin{cases} 0 & , n < -5 \\ \sum_{k=-5}^{n} e^{|k|} & , n \in [-5, 5] \\ \sum_{k=-5}^{5} e^{|k|} & , n > 5 \end{cases}$$

2. The system is linear, since it involves a product by a known waveform and an integral, both linear. With respect to time invariance, we need to compare

$$\begin{aligned}
y(t-t_0) &= \begin{cases} 0 & , t-t_0 \leq 2 \\ \cos(t-t_0+2) \int_{-1}^{t-t_0-2} x(u)\, du & , t-t_0 > 2, \end{cases} \\
\Sigma[x(t-t_0)] &= \begin{cases} 0 & , t \leq 2 \\ \cos(t+2) \int_{-1}^{t-2} x(u-t_0)\, du & , t > 2, \end{cases} \\
&= \begin{cases} 0 & , t \leq 2 \\ \cos(t+2) \int_{-1-t_0}^{t-2-t_0} x(v)\, dv & , t > 2, \end{cases}
\end{aligned}$$

which are evidently different, hence the system is not time-invariant. For the impulse response we have

$$\begin{aligned}
h(t) &= \begin{cases} 0 & , t \leq 2 \\ \cos(t+2) \int_{-1}^{t-2} \delta(u)\, du & , t > 2, \end{cases} \\
&= \begin{cases} 0 & , t \leq 2 \\ \cos(t+2) & , t > 2, \end{cases} \\
&= \cos(t+2)\, 1(t-2)\ ,
\end{aligned}$$

while the response to the unit step is

$$\begin{aligned}
h(t) &= \begin{cases} 0 & , t \leq 2 \\ \cos(t+2) \int_{-1}^{t-2} 1(u)\, du & , t > 2, \end{cases} \\
&= \begin{cases} 0 & , t \leq 2 \\ \cos(t+2) \int_{0}^{t-2} 1\, du & , t > 2, \end{cases} \\
&= \cos(t+2)\, (t-2)\, 1(t-2)\ .
\end{aligned}$$

3. The system is evidently non-linear since $\text{sgn}(x)$ is not a linear function. Recalling that this is a mapping $y(n) = f(x(n))$, with respect to time

invariance, we need to compare

$$y(n-n_0) = f(x(n-n_0)) \ , \qquad f(x) = \text{sgn}(x) = \begin{cases} \text{sgn}(1/x) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$
$$\Sigma[x(n-n_0)] = f(x(n-n_0))$$

which are equal, hence the system is time-invariant (every mapping of the form $y(n) = f(x(n))$ is time-invariant by construction). The impulse response is

$$h(n) = f(\delta(n)) = \begin{cases} 1 & , n = 0 \\ 0 & , n \neq 0 \end{cases} = \delta(n) \ ,$$

while for the response to the unit step we have

$$h_{-1}(n) = f(1_0(n)) = \begin{cases} 1 & , n \geq 0 \\ 0 & , n < 0 \end{cases} = 1_0(n) \ .$$

4. The system is evidently non-linear since $|x|$ is not a linear function. With respect to time invariance, we need to compare

$$\begin{aligned} y(n-n_0) &= \min(|x(n-n_0)|, |n-n_0|) \\ \Sigma[x(n-n_0)] &= \min(|x(n-n_0)|, |n|) \end{aligned}$$

which are evidently different, hence the system is non time invariant either. The impulse response is

$$\begin{aligned} h(n) = \min(|\delta(n)|, |n|) &= \begin{cases} \min(1,0) = 0 & , n = 0 \\ \min(0,|n|) = 0 & , n \neq 0 \end{cases} \\ &= 0 \ , \end{aligned}$$

while for the response to the unit step we have

$$\begin{aligned} h_{-1}(n) = \min(|1_0(n)|, |n|) &= \begin{cases} \min(1,|n|) = 1 & , n > 0 \\ \min(0,0) = 0 & , n = 0 \\ \min(0,|n|) = 0 & , n < 0 \end{cases} \\ &= 1_0(n) - \delta(n) \ . \end{aligned}$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
6.6 Homework assignment
Prof. T. Erseghe

Exercises 6.6

For each of the following systems state if they are linear and/or time-invariant, and evaluate their impulse response $h(t)$, as well as the response $h_{-1}(t)$ to the unit step:

1. $$y(t) = \int_{t-1}^{t+1} |t-u| x(u)\, du \ ,$$

2. $$y(n) = \sum_{k=1}^{\infty} 2^k x(n-k) \ ,$$

3. $$y(t) = \int_{t-1}^{t} e^{t+u} x(u)\, du \ ,$$

4. $$y(n) = \sum_{k=n-10}^{n+10} x(k) \ ,$$

5. $$y(t) = x(t-2) \ ,$$

6. $$y(n) = \sum_{k=-\infty}^{n-1} 3^k x(k) \ ,$$

7. $$y(t) = \cos(t-2)\, x(t) \ ,$$

8. $$y(t) = x(t+5)\, x(t-1) \ ,$$

9. $$y(t) = \int_{-\infty}^{2t} |t-u|^2 x(u)\, du \ ,$$

Solutions.

1. The system is linear, since it involves a product by a known waveform and an integral, both linear. With respect to time invariance, we need to compare

$$\begin{aligned} y(t-t_0) &= \int_{t-t_0-1}^{t-t_0+1} |t-t_0-u| x(u)\, du \\ \Sigma[x(t-t_0)] &= \int_{t-1}^{t+1} |t-u| x(u-t_0)\, du \\ &= \int_{t-1-t_0}^{t+1-t_0} |t-(v+t_0)| x(v)\, dv \end{aligned}$$

where $v = u - t_0$, which are equal, hence the system is time-invariant. For the impulse response we have

$$\begin{aligned} h(t) &= \int_{t-1}^{t+1} |t-u| \delta(u)\, du \\ &= |t| \int_{t-1}^{t+1} \delta(u) du = \begin{cases} 0 & , t < -1 \\ |t| & , t \in (-1, 1) \\ 0 & , t > 1 \end{cases} \\ &= |t| \operatorname{rect}(\tfrac{1}{2}t) \end{aligned}$$

while the response to the unit step is

$$\begin{aligned} h_{-1}(t) &= \int_{t-1}^{t+1} |t-u| 1(u)\, du \\ &= \begin{cases} 0 & , t < -1 \\ \int_0^{t+1} |t-u| du = \int_{-t}^{1} |v| dv = \frac{1}{2} - \frac{1}{2}t^2 & , t \in (-1, 0) \\ \int_0^{t+1} |t-u| du = \int_{-t}^{1} |v| dv = \frac{1}{2} + \frac{1}{2}t^2 & , t \in (0, 1) \\ \int_{t-1}^{t+1} |t-u| du = \int_{-1}^{1} |v| dv = 1 & , t > 1 \end{cases} \end{aligned}$$

2. The system is linear, since it involves a product by a known waveform and an integral, both linear. With respect to time invariance, we need to compare

$$\begin{aligned} y(n-n_0) &= \sum_{k=1}^{\infty} 2^k x(n-n_0-k) \\ \Sigma[x(n-n_0)] &= \sum_{k=1}^{\infty} 2^k x(n-k-n_0) \end{aligned}$$

which are equal, hence the system is time-invariant. For the impulse response we have

$$\begin{aligned} h(n) &= \sum_{k=1}^{\infty} 2^k \delta(n-k) = \sum_{k=1}^{\infty} 2^k \delta(k-n) = 2^n \sum_{k=1}^{\infty} \delta(k-n) \\ &= 2^n \, 1_0(n-1) \end{aligned}$$

while the response to the unit step is

$$\begin{aligned}
h_{-1}(n) &= \sum_{k=1}^{\infty} 2^k 1_0(n-k) \\
&= \begin{cases} 0 & , n \le 0 \\ \sum_{k=1}^{n} 2^k = \frac{1-2^{n+1}}{1-2} - 1 = 2^{n+1} - 2 & , n > 0 \end{cases} \\
&= 2(2^n - 1)\, 1_0(n-1) \ ,
\end{aligned}$$

since $1_0(n-k)$ is active for $n-k \ge 0$, that is $k \le n$.

3. The system is linear, since it involves a product by a known waveform and an integral, both linear. With respect to time invariance, we need to compare

$$\begin{aligned}
y(t-t_0) &= \int_{t-t_0-1}^{t-t_0} e^{t-t_0+u} x(u)\, du \\
\Sigma[x(t-t_0)] &= \int_{t-1}^{t} e^{t+u} x(u-t_0)\, du \\
&= \int_{t-1-t_0}^{t-t_0} e^{t+v+t_0} x(v)\, dv
\end{aligned}$$

which differ in the exponential. hence the system is non BIBO stable. For the impulse response we have

$$\begin{aligned}
h(t) &= \int_{t-1}^{t} e^{t+u} \delta(u)\, du = e^t \int_{t-1}^{t} \delta(u)\, du \\
&= e^t \ \mathrm{rect}(t - \tfrac{1}{2}) = \begin{cases} e^t & , t \in (0,1) \\ 0 & , \text{otherwise} \end{cases}
\end{aligned}$$

while the response to the unit step is

$$\begin{aligned}
h_{-1}(t) &= \int_{t-1}^{t} e^{t+u} 1(u)\, du \\
&= \begin{cases} 0 & , t < 0 \\ e^t \int_0^t e^u\, du = e^t(e^t - 1) & , t \in (0,1) \\ e^t \int_{t-1}^t e^u\, du = e^{2t}(1 - e^{-1}) & , t > 1 \end{cases}
\end{aligned}$$

4. The system is linear, since it involves a summation, which is a linear mapping. With respect to time invariance, we need to compare

$$\begin{aligned}
y(n-n_0) &= \sum_{k=n-n_0-10}^{n-n_0+10} x(k) \\
\Sigma[x(n-n_0)] &= \sum_{k=n-10}^{n+10} x(k-n_0) \\
&= \sum_{m=n-10-n_0}^{n+10-n_0} x(m)
\end{aligned}$$

which are equivalent, hence the system is time-invariant. The impulse response is

$$h(n) = \sum_{k=n-10}^{n+10} \delta(k) = \begin{cases} 1 & , n \in [-10, 10] \\ 0 & , \text{otherwise} \end{cases} = \text{rect}(n/21) \ ,$$

while the response to the unit step is

$$h_{-1}(n) = \sum_{k=n-10}^{n+10} 1_0(k) = \begin{cases} 0 & , n < -10 \\ \sum_{k=0}^{n+10} 1 = n + 11 & , n \in [-10, 10] \\ \sum_{k=n-10}^{n+10} 1 = 21 & , n > 10 \end{cases}$$

5. The system is evidently linear. With respect to time invariance, we need to compare

$$y(t - t_0) = x(t - t_0 - 2)$$
$$\Sigma[x(t - t_0)] = x(t - 2 - t_0)$$

which are equivalent, hence the system is time-invariant. The impulse response is $y(t) = \delta(t - 2)$, and the response to the unit step is $h_{-1}(t) = 1_0(t - 2)$.

6. The system is linear, since it involves a product by a known waveform and a summation, both linear. With respect to time invariance, we need to compare

$$y(n - n_0) = \sum_{k=-\infty}^{n-n_0-1} 3^k x(k)$$
$$\Sigma[x(n - n_0)] = \sum_{k=-\infty}^{n-1} 3^k x(k - n_0)$$
$$= \sum_{k=-\infty}^{n-1-n_0} 3^k x(m)$$

which are equivalent, hence the system is time-invariant. The impulse response is

$$h(n) = \sum_{k=-\infty}^{n-1} 3^k \delta(k) = \sum_{k=-\infty}^{n-1} \delta(k) = 1_0(n - 1)$$

while the response to the unit step is

$$h(n) = \sum_{k=-\infty}^{n-1} 3^k 1_0(k) = \begin{cases} 0 & n \leq 0 \\ \sum_{k=0}^{n-1} 3^k = \frac{1-3^n}{1-3} & , n > 0 \end{cases}$$
$$= \tfrac{1}{2}(3^n - 1) \, 1_0(n - 1)$$

7. The system is linear, since it involves a product by a known waveform, and in fact

$$\cos(t-2)\,[ax_1(t)+bx_2(y)] = a\,[\cos(t-2)x_1(t)] + b\,[\cos(t-2)x_2(t)]\ .$$

With respect to time invariance, we need to compare

$$\begin{aligned} y(t-t_0) &= \cos(t-t_0-2)\,x(t-t_0) \\ \Sigma[x(t-t_0)] &= \cos(t-2)\,x(t-t_0) \end{aligned}$$

which are different, hence the system is not time-invariant. Impulse response and the response to the unit step are

$$\begin{aligned} h(t) &= \cos(t-2)\,\delta(t) = \cos(2)\,\delta(t) \\ h_{-1}(t) &= \cos(t-2)\,1(t)\ . \end{aligned}$$

8. The system is not linear since it involves a product of the signal by itself (quadratic function). With respect to time invariance, we need to compare

$$\begin{aligned} y(t-t_0) &= x(t-t_0+5)\,x(t-t_0-1) \\ \Sigma[x(t-t_0)] &= x(t+5-t_0)\,x(t-1-t_0) \end{aligned}$$

which are equivalent, hence the system is time-invariant. Impulse response and the response to the unit step are

$$\begin{aligned} h(t) &= \delta(t+5)\,\delta(t-1) = 0 \\ h_{-1}(t) &= 1(t+5)\,1(t-1) = 1(t-1)\ . \end{aligned}$$

9. The system is linear, since it involves a product by a known waveform and an integral, both linear. With respect to time invariance, we need to compare

$$\begin{aligned} y(t-t_0) &= \int_{-\infty}^{2t-2t_0} |t-t_0-u|^2 x(u)\,du \\ \Sigma[x(t-t_0)] &= \int_{-\infty}^{2t} |t-u|^2 x(u-t_0)\,du \\ &= \int_{-\infty}^{2t-t_0} |t-v-t_0|^2 x(v)\,dv \end{aligned}$$

which differ in the upper extreme of the integral, hence the system is not time-invariant. The impulse response is

$$h(t) = \int_{-\infty}^{2t} |t-u|^2 \delta(u)\,du = |t| \int_{-\infty}^{2t} \delta(u)\,du = |t|\,1(t) = t\,1(t)\ ,$$

and the response to the unit step provides

$$\begin{aligned} h_{-1}(t) &= \int_{-\infty}^{2t} |t-u|^2 1(u)\,du \\ &= \begin{cases} 0 & ,\ t<0 \\ \int_0^{2t} |t-u|^2\,du = \int_{-t}^{t} |v|^2\,dv = \frac{2}{3}t^3 & ,\ t>0 \end{cases} \\ &= \tfrac{2}{3}t^3\,1(t)\ . \end{aligned}$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
7.2 Solved exercises
Prof. T. Erseghe

Exercises 7.2

Solve the following:

1. evaluate the convolution $z(n) = x * y(n)$ for $x(n) = A + \cos(\theta_0 n)$ and $y(n) = 1_0(n)\,\alpha^n$, $-1 < \alpha < 1$,

2. evaluate the convolution $z(t) = x * y(t)$ for $x(t) = A + \cos(\omega_0 t)$ and $y(t) = 1(t)\,e^{-\alpha t}$, $\alpha > 0$,

3. evaluate the convolution $z(t) = x * y(t)$ for $x(t) = \mathrm{rect}(t/4D)$ and $y(t) = \mathrm{rect}(t/2D)$,

4. express the following signals

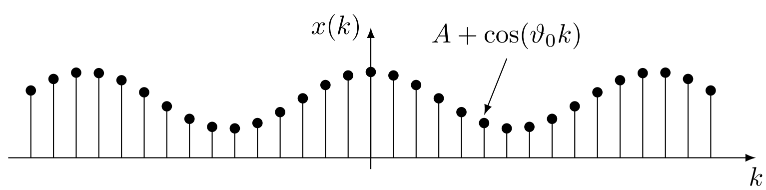
$$z_1(t) = \int_{-\infty}^{\infty} e^{-|u|} \sin(t-u)\,du$$
$$z_2(t) = \int_{0}^{\infty} e^{t-u} \sin(u+2)\,du$$
$$z_3(n) = \sum_{-\infty}^{n} e^{k} \sin(n-k+2)$$
$$z_4(t) = \begin{cases} 0 & ,\ t<0 \\ \int_0^t e^{t-u}\sin(u+2)\,du & ,\ t>0 \end{cases}$$

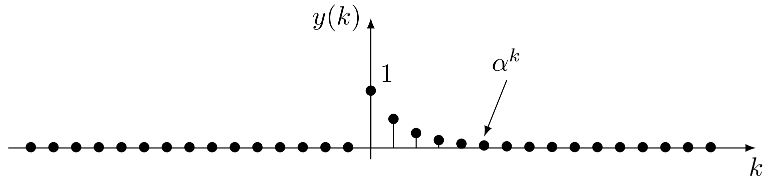
as a convolution $z = x * y$,

5. prove that the convolution $z(n) = x * y(n)$ between an aperiodic signal $x(n)$ and a periodic signal $y(n+N) = y(n)$, is periodic of period N, the same of $y(n)$, that is we have $z(n+N) = z(n)$.

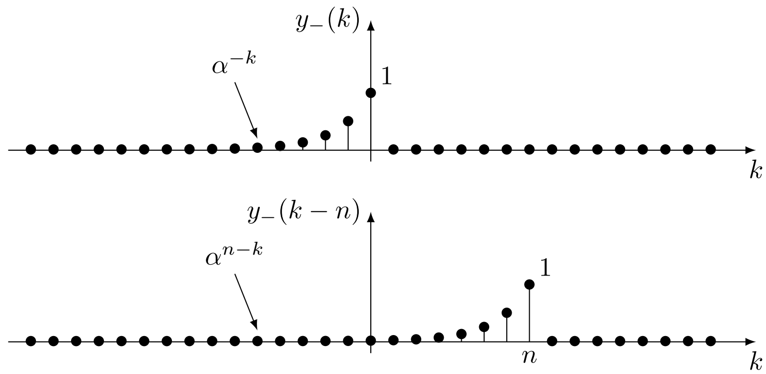
Solution.

1. We illustrate the signals first, which we do with respect to the time-variable k, as illustrated in the following figure





Then, we need to visualise the time-reversed and time-shifted version of y, to have



where it becomes evident that $y_{-}(k-n) = y(n-k)$ has extension $(-\infty, n]$, which equivalently constrains the extension of the product. Hence, we can easily interpret the convolution operation in the form

$$\begin{aligned} z(n) &= \sum_{k=-\infty}^{\infty} x(k) y_{-}(k-n) \\ &= \sum_{k=-\infty}^{n} x(k) y_{-}(k-n) \\ &= \sum_{k=-\infty}^{n} \left[A + \cos(\varphi_0 k)\right] \alpha^{n-k} \end{aligned}$$

which we can solve by expressing the cosine through Euler's identity, and by a further change of variable $m = n - k$ in the sum in order to evidence the presence of a geometric series. We have

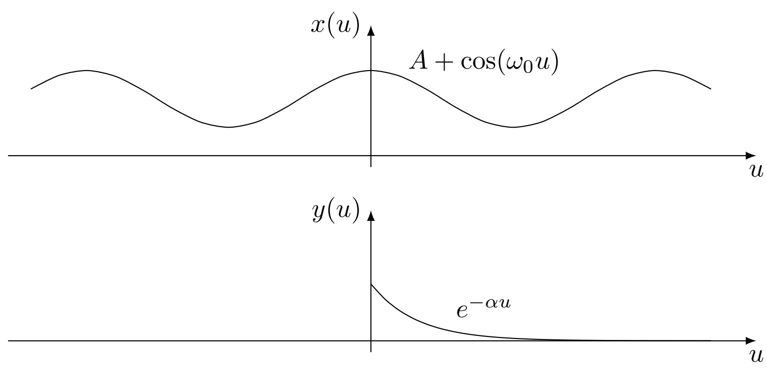
$$\begin{aligned}
z(n) &= \sum_{k=-\infty}^{n} \left[A + \tfrac{1}{2}e^{j\varphi_0 k} + \tfrac{1}{2}e^{-j\varphi_0 k}\right]\alpha^{n-k} \\
&= \sum_{m=0}^{\infty} \left[A + \tfrac{1}{2}e^{j\varphi_0 (n-m)} + \tfrac{1}{2}e^{j\varphi_0 (m-n)}\right]\alpha^{m} \\
&= A\sum_{m=0}^{\infty}\alpha^m + \tfrac{1}{2}e^{j\varphi_0 n}\sum_{m=0}^{\infty}[\alpha\, e^{-j\varphi_0}]^m + \tfrac{1}{2}e^{-j\varphi_0 n}\sum_{m=0}^{\infty}[\alpha\, e^{j\varphi_0}]^m \\
&= \frac{A}{1-\alpha} + \frac{1}{2}\frac{e^{j\varphi_0 n}}{1-\alpha\, e^{-j\varphi_0}} + \frac{1}{2}\frac{e^{-j\varphi_0 n}}{1-\alpha\, e^{j\varphi_0}} \\
&= \frac{A}{1-\alpha} + \Re\left[\frac{e^{j\varphi_0 n}}{1-\alpha\, e^{-j\varphi_0}}\right]
\end{aligned}$$

where in the last equivalence we exploited the equivalence $2\Re[x] = x + x^*$. Note that the geometric series converge since $|\alpha e^{\pm j\varphi_0}| = |\alpha| < 1$. Incidentally observe that, if we define $\beta = 1 - \alpha\, e^{-j\varphi_0} = |\beta| e^{j\varphi_\beta}$, then it also is

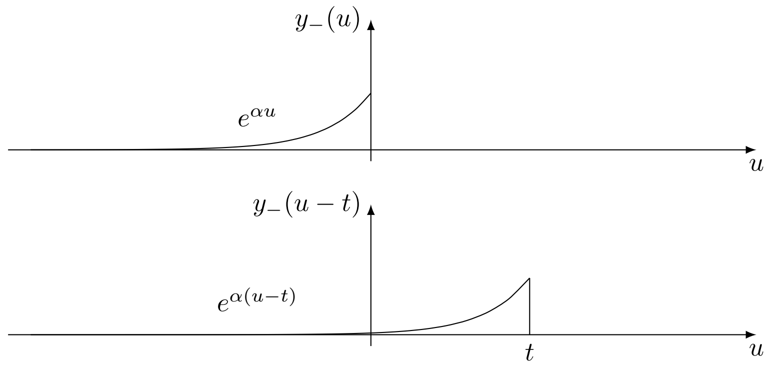
$$z(n) = \frac{A}{1-\alpha} + \frac{\cos(\varphi_0 n - \varphi_\beta)}{|\beta|}$$

where we observe that the original constant contribution A to $x(n)$ has been scaled, and so has the contribution $\cos(\varphi_0 n)$, with an additional change in phase. That is, the original contributions are somehow kept by convolution, which, as we will better learn later on in the course, is a fundamental result.

2. This is the continuous-time counterpart to the previous exercise. We illustrate the signals first, which we do with respect to the time-variable u, as illustrated in the following figure



Then, we need to visualise the time-reversed and time-shifted version of y, to have



where it becomes evident that $y_-(u-t) = y(t-u)$ has extension $(-\infty, t]$, which equivalently constrains the extension of the product. Hence, we can easily interpret the convolution operation in the form

$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} x(u) y_-(u-t)\, du \\ &= \int_{-\infty}^{t} x(u) y_-(u-t)\, du \\ &= \int_{-\infty}^{t} [A + \cos(\omega_0 u)]\, e^{\alpha(u-t)}\, du \end{aligned}$$

which we can solve by expressing the cosine through Euler's identity, and by a further change of variable $v = t - u$ in the integral. We have

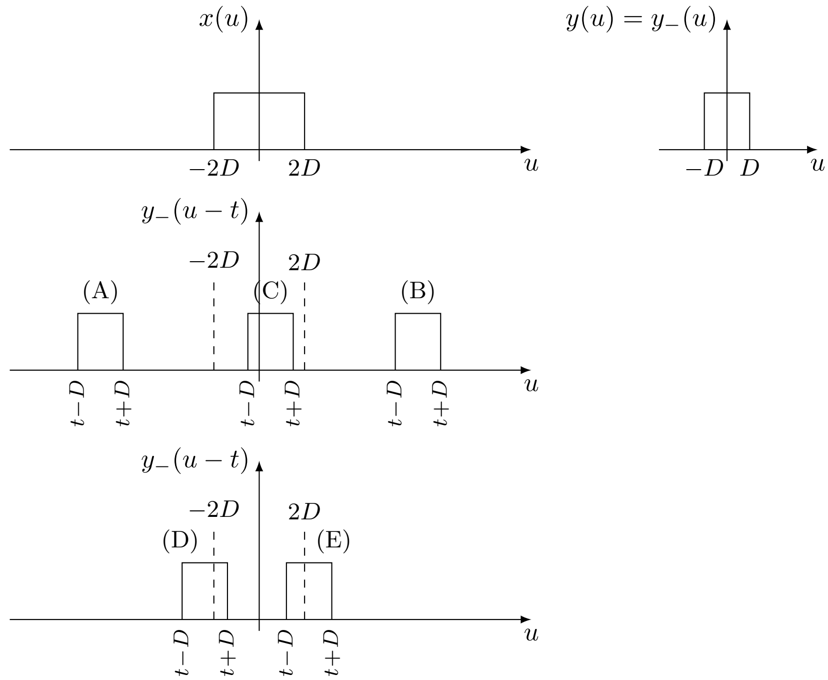
$$\begin{aligned} z(t) &= \int_{-\infty}^{t} [A + \tfrac{1}{2} e^{j\omega_0 u} + \tfrac{1}{2} e^{-j\omega_0 u}]\, e^{\alpha(u-t)}\, du \\ &= \int_{0}^{\infty} [A + \tfrac{1}{2} e^{j\omega_0 (t-v)} + \tfrac{1}{2} e^{j\omega_0 (v-t)}]\, e^{-\alpha v}\, dv \\ &= A \int_0^\infty e^{-\alpha v}\, dv + \tfrac{1}{2} e^{j\omega_0 t} \int_0^\infty e^{-(\alpha + j\omega_0) v}\, dv + \tfrac{1}{2} e^{-j\omega_0 t} \int_0^\infty e^{-(\alpha - j\omega_0) v}\, dv \\ &= \frac{A}{\alpha} + \frac{1}{2} \frac{e^{j\omega_0 t}}{\alpha + j\omega_0} + \frac{1}{2} \frac{e^{-j\omega_0 t}}{\alpha - j\omega_0} \\ &= \frac{A}{\alpha} + \Re\left[\frac{e^{j\omega_0 t}}{\alpha + j\omega_0}\right] \end{aligned}$$

Note that the integrals converge since $\Re[-\alpha \pm j\omega_0] = -\alpha < 0$. Incidentally observe that, if we define $\beta = \alpha + j\omega_0 = |\beta| e^{j\varphi_\beta}$, then it also is

$$z(n) = \frac{A}{\alpha} + \frac{\cos(\omega_0 t - \varphi_\beta)}{|\beta|}$$

where we observe that the original constant contribution A to $x(t)$ has been scaled, and so has the contribution $\cos(\omega_0 t)$, with an additional change in phase. That is, the original contributions are somehow kept by convolution, which, as we will better learn later on in the course, is a fundamental result perfectly equivalent to the discrete-time case.

3. For the convolution of the two rectangles, given that $y_-(t) = y(t)$ since the rectangle is an even signal, we have the following



where we highlighted the five cases (A to E) of interest in the relative position between $x(u)$ and $y(t-u)$. We have

(A) In this case $y(t-u)$ is at the left of $x(u)$, hence their product is zero, and we have $z(t) = 0$. The range of validity is $t + D < -2D$, that is $t < -3D$.

(B) In this case $y(t-u)$ is at the right of $x(u)$, hence their product is zero, and we have $z(t) = 0$. The range of validity is $t - D > 2D$, that is $t > 3D$.

(C) In this case $y(t-u)$ is inside $x(u)$, hence their product is $x(u)y(t-u) = y(t-u)$, and since the area of y is $2D$ we have $z(t) = 2D$. The range of validity is $t + D < 2D$ and $t - D > -2D$, that is $-D < t < D$.

(D) In this case $y(t-u)$ enters $x(u)$ from the left, and we have

$$z(t) = \int_{-\infty}^{\infty} x(u)y(t-u)\,du = \int_{-2D}^{t+D} 1\,du = t + D - (-2D) = t + 3D\ .$$

The range of validity is $t - D < -2D < t + D$, that is $-3D < t < -D$.

(E) In this case $y(t-u)$ exits $x(u)$ from the right, and we have

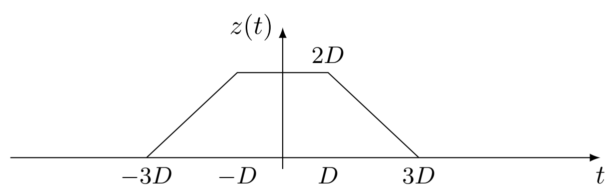
$$z(t) = \int_{-\infty}^{\infty} x(u)y(t-u)\,du = \int_{t-D}^{2D} 1\,du = 2D - (t-D) = 3D - t\ .$$

The range of validity is $t - D < 2D < t + D$, that is $D < t < 3D$.

By putting the results together, we obtain

$$z(t) = \begin{cases} 3D + t & , t \in (-3D, -D) \\ 2D & , t \in (-D, D) \\ 3D - t & , t \in (D, 3D) \\ 0 & , \text{otherwise} \end{cases}$$

which is the trapezoidal shape illustrated in the figure below.



4. In this exercise we wish to write the integrals in the form

$$z(t) = \int_{-\infty}^{\infty} x(u)y(t-u)\ ,du\ , \qquad z(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$$

for some x and y. In the first expression the solution is trivial

$$z_1(t) = \int_{-\infty}^{\infty} \underbrace{e^{-|u|}}_{x(u)} \underbrace{\sin(t-u)}_{y(t-u)}\,du$$

so that $x(t) = e^{-|t|}$ and $y(t) = \sin(t)$. For the second integral, instead we first need to extend the integral to $(-\infty, \infty)$, which is possible by introducing a unit step (appropriately shifted and/or reversed) in the following form

$$\begin{aligned} z_2(t) &= \int_0^{\infty} e^{t-u} \sin(u+2)\,du \\ &= \int_{-\infty}^{\infty} \underbrace{e^{t-u}}_{y(t-u)} \underbrace{\sin(u+2) 1(u)}_{x(u)}\,du \end{aligned}$$

so that $x(t) = \sin(t+2)\,1(t)$ and $y(t) = e^t$. For the third signal, instead, we have

$$\begin{aligned} z_3(n) &= \sum_{-\infty}^{n} e^k \sin(n-k+2) \\ &= \sum_{-\infty}^{\infty} \underbrace{e^k}_{x(k)} \underbrace{\sin(n-k+2)\,1_0(n-k)}_{y(n-k)} \end{aligned}$$

where $1_0(n-k) = 1_{0-}(k-n)$ is active for $n-k \geq 0$, that is for $k \leq n$, as we wish. Therefore, it is $x(n) = e^k$ and $y(n) = \sin(n+2)\, 1_0(n)$. Finally, the integral expression in the last signal suggests writing it, for $t > 0$, in the form

$$\begin{aligned} z_4(t) &= \int_0^t e^{t-u} \sin(u+2)\, du \\ &= \int_{-\infty}^{\infty} \underbrace{e^{t-u} 1(t-u)}_{y(t-u)} \underbrace{\sin(u+2) 1(u)}_{x(u)}\, du \end{aligned}$$

so that $x(t) = \sin(t+2)1(t)$ and $y(t) = e^t 1(t)$, which provides the correct result since for $t < 0$ the product $1(t-u)1(u)$ is zero.

5. Assume that, in the discrete-time convolution, we have $y(n+N) = y(n)$. Hence, we can write

$$x * y(n+N) = \sum_{k=-\infty}^{\infty} x(k)y(n+N-k) = \sum_{k=-\infty}^{\infty} x(k)y(n-k) = x * y(n)$$

since $y(n+N-k) = y(n-k)$ by periodicity.

FOUNDATIONS OF SIGNALS AND SYSTEMS
7.3 Homework assignment
Prof. T. Erseghe

Exercises 7.3

Solve the following:

1. prove that the convolution $z(t) = x * y(t)$ between an aperiodic signal $x(t)$ and a periodic signal $y(t+T_p) = y(t)$, is periodic of period T_p, that is, we have $z(t+T_p) = z(t)$,
2. show that $\mathrm{rect} * \mathrm{rect}(t) = \mathrm{triang}(t)$,
3. show that $1 * 1(t) = t \cdot 1(t)$,
4. evaluate the convolution between $x(t) = 1(t)$ and $y(t) = \mathrm{rect}(t)$,
5. evaluate the convolution between $x(t) = e^{-\alpha t}\, 1(t)$ and $y(t) = \mathrm{rect}(t)$,
6. evaluate the convolution between $x(t) = A\cos(\omega_0 t)$ and $y(t) = \mathrm{rect}(t/2D)$,
7. evaluate the convolution between $x(n) = a^{-|n|}$ and $y(n) = \mathrm{rect}(\frac{1}{1+2N} n)$,
8. evaluate the convolution between $x(t) = \mathrm{rect}(t+\frac{1}{2})$ and $y(t) = \mathrm{sgn}(t)\, e^{-|t|}$,
9. evaluate the convolution between $x(n) = \mathrm{rect}(\frac{1}{1+2N} n)$ and $y(n) = \mathrm{sgn}(n)$,
10. evaluate the convolution between $x(t) = \mathrm{rect}(t)$ and $y(t) = |t|\ \mathrm{rect}(\frac{1}{2} t)$,
11. evaluate the convolution between the two discrete-time rectangles

$$x(n) = \begin{cases} 1 & , n \in [0, N) \\ 0 & , \text{otherwise} \end{cases} \qquad y(n) = \begin{cases} 1 & , n \in [0, M) \\ 0 & , \text{otherwise} \end{cases}$$

    where $N \geq M$,

12. express the following signals as convolutions

$$z_1(n) = \sum_{-\infty}^{n-1} 3^k\ , \quad z_2(t) = \int_{t-4}^{t+4} \sin(u)\, du\ .$$

Solutions.

1. Assume that, in the continuous-time convolution, we have $y(t+T_p) = y(t)$. Hence, we can write

$$x * y(t+T_p) = \int_{-\infty}^{\infty} x(u)y(t+T_p-u)\,du = \int_{-\infty}^{\infty} x(u)y(t-u)\,du = x * y(t)$$

since $y(t+T_p-u) = y(t-u)$ by periodicity.

2. For the case $x(t) = \mathrm{rect}(t)$ and $y(t) = y_-(t) = \mathrm{rect}(t)$ we have the following cases

$$z(t) = \int_{-\infty}^{\infty} \mathrm{rect}(u)\,\mathrm{rect}(u-t)\,du = \begin{cases} 0 & , t < -1 \\ \int_{-\frac{1}{2}}^{t+\frac{1}{2}} 1\,du = 1+t & , -1 < t < 0 \\ \int_{t-\frac{1}{2}}^{\frac{1}{2}} 1\,du = 1-t & , 0 < t < 1 \\ 0 & , t > 1 \end{cases}$$

which corresponds to $z(t) = \mathrm{triang}(t)$.

3. For the case $x(t) = 1(t)$ and $y(t) = 1(t)$, $y_-(t) = 1(-t)$ we have the following cases

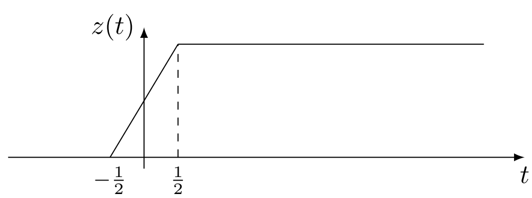
$$z(t) = \int_{-\infty}^{\infty} 1(u)1_-(u-t)\,du = \begin{cases} 0 & , t < 0 \\ \int_0^t 1\,du = t & , t > 0 \end{cases}$$

which corresponds to the ramp $z(t) = t \cdot 1(t)$.

4. For the case $x(t) = 1(t)$ and $y(t) = y_-(t) = \mathrm{rect}(t)$ we have the following cases

$$\begin{aligned} z(t) &= \int_{-\infty}^{\infty} \mathrm{rect}(u)\,\mathrm{rect}(u-t)\,du \\ &= \begin{cases} 0 & , t < -\frac{1}{2} \\ \int_0^{t+\frac{1}{2}} 1\,du = \frac{1}{2}+t & , -\frac{1}{2} < t < \frac{1}{2} \\ \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} 1\,du = 1 & , t > \frac{1}{2} \end{cases} \end{aligned}$$

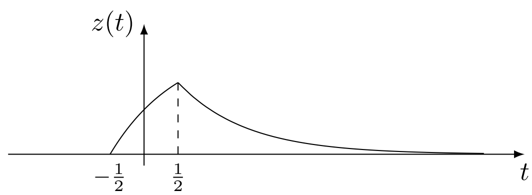
as illustrated in the figure below.



5. For the case $x(t) = e^{-\alpha t}\,1(t)$ and $y(t) = y_-(t) = \mathrm{rect}(t)$ we have the following cases

$$
\begin{aligned}
z(t) &= \int_{-\infty}^{\infty} \mathrm{rect}(u)\,\mathrm{rect}(u-t)\,du \\
&= \begin{cases} 0 & , t < -\frac{1}{2} \\ \int_0^{t+\frac{1}{2}} e^{-\alpha t}\,du = \frac{1-e^{-\alpha(t+\frac{1}{2})}}{\alpha} & , -\frac{1}{2} < t < \frac{1}{2} \\ \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} e^{-\alpha t}\,du = \frac{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}}{\alpha} e^{-\alpha t} & , t > \frac{1}{2} \end{cases}
\end{aligned}
$$

as illustrated in the figure below.



6. In this case, given that $y(t) = y_-(t) = \mathrm{rect}(t/2D)$, we have

$$
\begin{aligned}
z(t) &= \int_{-\infty}^{\infty} A\cos(\omega_0 u)\,\mathrm{rect}\left(\frac{u-t}{2D}\right) du \\
&= \int_{t-D}^{t+D} A\cos(\omega_0 u)\,du \\
&= A\frac{\sin(\omega_0 u)}{\omega_0}\Big|_{t-D}^{t+D} \\
&= A\frac{\sin(\omega_0 t + \omega_0 D) - \sin(\omega_0 t - \omega_0 D)}{\omega_0} \\
&= \frac{2A\sin(\omega_0 D)}{\omega_0}\cos(\omega_0 t)
\end{aligned}
$$

hence the output is still a sinusoid of the same period.

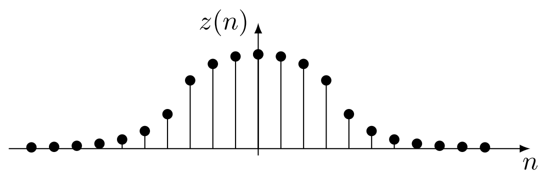
7. In this case we have $y_-(n) = y(n)$ and

$$
y_-(k-n) = \begin{cases} 1 & , n-N \le k \le n+N \\ 0 & , \text{otherwise} \end{cases}
$$

so that we can distinguish three cases

$$
\begin{aligned}
z(n) &= \sum_{k=-\infty}^{\infty} x(k) y_-(k-n) \\
&= \begin{cases} \sum_{k=n-N}^{n+N} a^k & , n+N \leq 0 \\ \sum_{k=n-N}^{0} a^k + \sum_{k=1}^{n+N} a^{-k} & , -N < n < N \\ \sum_{k=n-N}^{n+N} a^{-k} & , n-N \geq 0 \end{cases} \\
&= \begin{cases} \frac{a^{-N}-a^{1+N}}{1-a}\, a^n & , n \leq -N \\ \frac{1-a^{n-N-1}}{1-a^{-1}} + \frac{1-a^{-n-N-1}}{1-a^{-1}} - 1 & , -N < n < N \\ \frac{a^N - a^{-(1+N)}}{1-a^{-1}}\, a^{-n} & , n \geq N \end{cases}
\end{aligned}
$$

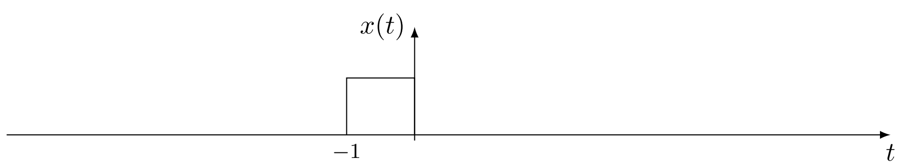
as illustrated in the figure below.

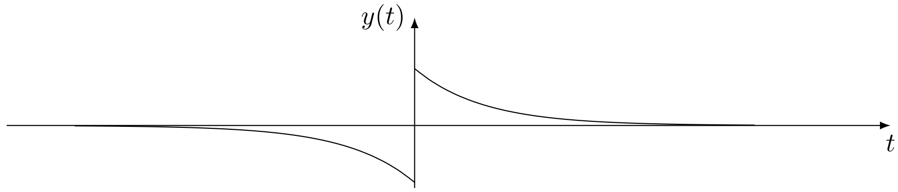


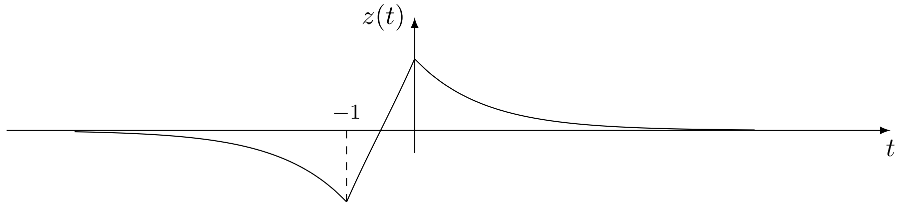
8. In this specific case we have $y_-(t) = -y(t) = -\text{sgn}(t)\, e^{-|t|}$, with a breaking point at zero, hence we obtain three different regions

$$
\begin{aligned}
z(t) &= -\int_{-1}^{0} \text{sgn}(u-t)\, e^{-|u-t|}\, du \\
&= \begin{cases} -\int_{-1}^{0} e^{-(u-t)}\, du & , t < -1 \\ \int_{-1}^{t} e^{(u-t)}\, du - \int_{t}^{0} e^{-(u-t)}\, du & , -1 < t < 0 \\ \int_{-1}^{0} e^{(u-t)}\, du & , t > 0 \end{cases} \\
&= \begin{cases} -(e-1)\, e^t & , t < -1 \\ e^t - e^{-t-1} & , -1 < t < 0 \\ (1-e^{-1})\, e^{-t} & , t > 0 \end{cases}
\end{aligned}
$$

as illustrated in the figure below.







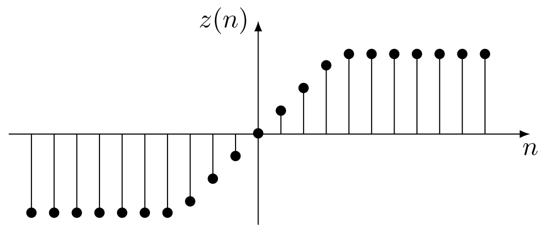
9. In this case it is

$$x(k) = \begin{cases} 1 & , -N \leq k \leq N \\ 0 & , \text{otherwise} \end{cases}$$

and $y_-(k) = -y(k) = -\text{sgn}(k)$, so that we can identify five regions

$$\begin{aligned} z(n) &= -\sum_{k=-N}^{N} \text{sgn}(k-n) \\ &= \begin{cases} -\sum_{k=-N}^{N} 1 = -(1+2N) & , n < -N \\ -\sum_{k=-N+1}^{N} 1 = -2N & , n = -N \\ \sum_{k=-N}^{n-1} 1 - \sum_{k=n+1}^{N} 1 = 2n & , -N < n < N \\ \sum_{k=-N}^{N-1} 1 = 2N & , n = -N \\ \sum_{k=-N}^{N} 1 = 1+2N & , n > N \end{cases} \\ &= \begin{cases} -(1+2N) & , n < -N \\ 2n & , -N \leq n \leq N \\ 1+2N & , n > N \end{cases} \end{aligned}$$

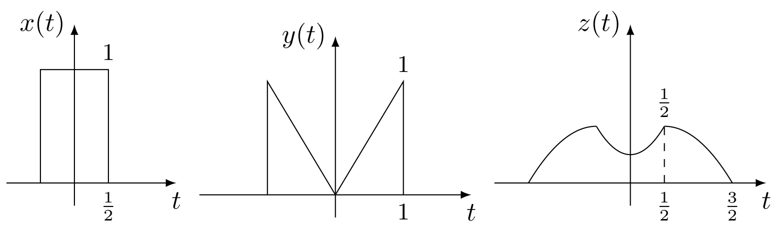
as illustrated in the figure below.



10. In this case it is $y_-(t) = y(t)$ and we have

$$
\begin{aligned}
z(t) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} |u-t| \; \mathrm{rect}(\tfrac{1}{2}(u-t))\, du \\
&= \begin{cases}
\int_{-\frac{1}{2}}^{t+1} (u-t)\, du = \frac{3}{8} - \frac{1}{2}t - \frac{1}{2}t^2 & , t \in (-\frac{3}{2}, -\frac{1}{2}) \\
\int_{t}^{\frac{1}{2}} (u-t)\, du - \int_{-\frac{1}{2}}^{t} (u-t)\, du = \frac{1}{4} + t^2 & , t \in (-\frac{1}{2}, \frac{1}{2}) \\
-\int_{t-1}^{\frac{1}{2}} (u-t)\, du = \frac{3}{8} + \frac{1}{2}t - \frac{1}{2}t^2 & , t \in (\frac{1}{2}, \frac{3}{2}) \\
0 & , \text{otherwise}
\end{cases}
\end{aligned}
$$

as illustrated in the figure below.



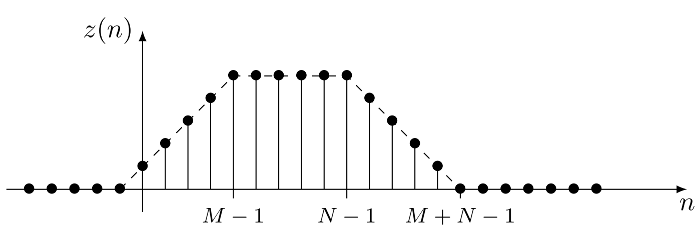
11. We have

$$
x(k) = \begin{cases} 1 & , k \in [0, N) \\ 0 & , \text{otherwise} \end{cases} \qquad y_-(k-n) = \begin{cases} 1 & , k \in (n-M, n] \\ 0 & , \text{otherwise} \end{cases}
$$

so that

$$
\begin{aligned}
z(n) &= \sum_{k=0}^{N-1} y_-(k-n) \\
&= \begin{cases}
\sum_{k=0}^{n} 1 = n+1 & , k \in [0, M) \\
\sum_{k=n-M+1}^{n} 1 = M & , k \in [M, N) \\
\sum_{k=n-M+1}^{N-1} 1 = N+M-1-n & , k \in [N, N+M-1) \\
0 & , \text{otherwise}
\end{cases}
\end{aligned}
$$

as illustrated in the figure below.



As shown in the figure, the result is equivalent to that of a (sampled) isosceles trapezoid, with bases of $M+N$ and $N-M$, and height M, time-shifted to the right by one sample.

12. For the first signal, we have

$$z_1(n) = \sum_{-\infty}^{n-1} 3^k = \sum_{-\infty}^{\infty} 3^k \, 1_0(n-1-k)$$

hence it is the convolution between $x(n) = 3^n$ and $y(n) = 1_0(n-1)$. For the second signal, we similarly have

$$z_2(t) = \int_{t-4}^{t+4} \sin(u)\, du = \int_{-\infty}^{\infty} \sin(u)\, 1(u-(t-4))\, 1((t+4)-u) du$$

hence it is the convolution between $x(t) = \sin(t)$ and $y(t) = 1_-(t-4)\, 1(t+4) = \mathrm{rect}(t/8)$.

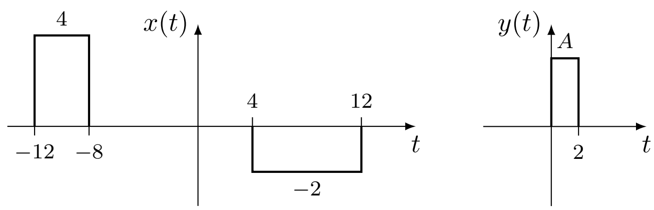
FOUNDATIONS OF SIGNALS AND SYSTEMS
7.5 Solved exercises
Prof. T. Erseghe

Exercises 7.5

Solve the following by using the properties of convolution:

1. prove that the convolution between $x(t) = \text{rect}(T/T_1)$ and $\text{rect}(t/T_2)$, $T_2 \le T_1$, is a trapezoid with bases $T_1 + T_2$ and $T_1 - T_2$, and with height T_2, by expressing rectangles as differences of unit steps, and by exploiting the result $1 * 1(t) = t \cdot 1(t)$;

2. evaluate the convolution between the two signals in figure



   by exploiting the result of the previous exercise;

3. evaluate the convolution between $x(n) = \delta(n) + \frac{1}{2}\delta(n-1)$ and $y(n) = \text{rect}((n-1)/3)$, as well as the convolution between $x(n-3)$ and $y(n+2)$;

4. evaluate the output of a series of two LTI systems with impulse responses $h_1(n) = \sin(8n)$ and $h_2(n) = a^n 1_0(n)$, respectively, by considering an input of the form $x(n) = \delta(n) - a\delta(n-1)$.

Solutions.

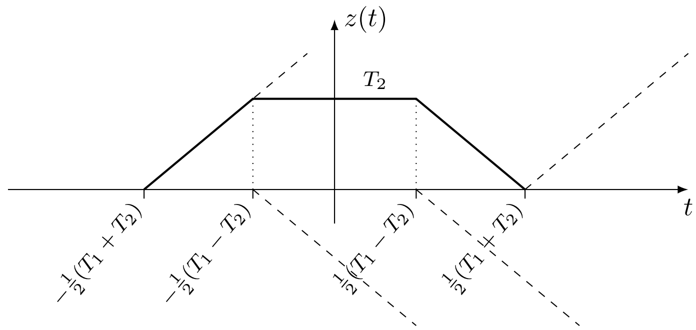
1. We have

$$x(t) = 1(t + \tfrac{1}{2}T_1) - 1(t - \tfrac{1}{2}T_1)\ , \quad y(t) = 1(t + \tfrac{1}{2}T_2) - 1(t - \tfrac{1}{2}T_2)$$

   so that, by linearity and time-shift property, we have

$$\begin{aligned} z(t) &= x * y(t) \\ &= [1(t + \tfrac{1}{2}T_1)] * [1(t + \tfrac{1}{2}T_2)] - [1(t + \tfrac{1}{2}T_1)] * [1(t - \tfrac{1}{2}T_2)] \\ &\quad - [1(t - \tfrac{1}{2}T_1)] * [1(t + \tfrac{1}{2}T_2)] + [1(t - \tfrac{1}{2}T_1)] * [1(t - \tfrac{1}{2}T_2)] \\ &= 1 * 1(t + \tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) - 1 * 1(t + \tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) \\ &\quad - 1 * 1(t - \tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) + 1 * 1(t - \tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) \end{aligned}$$

   whose contributions are illustrated in figure below in dashed lines, together with their sum which readily provides a trapezoid.



As a matter of fact, in the first interval $[-\frac{1}{2}(T_1+T_2), -\frac{1}{2}(T_1-T_2)]$ there is only one contribution active, hence the slope is 1. In the next interval, $[-\frac{1}{2}(T_1-T_2), \frac{1}{2}(T_1-T_2)]$, instead, two contributions are active, where one has positive slope and the second has negative slope, hence the slope is $1-1=0$, and in fact the signal is constant. In the third interval, $[\frac{1}{2}(T_1-T_2), \frac{1}{2}(T_1+T_2)]$, the active contributions are three, with slopes $1-1-1=-1$, and in fact the signal decreases here. In the last interval the signal must be zero because of the property of the extension of the convolution. Incidentally, we can also check the validity of our result through the property of the area, for which we have

$$A_z = (T_1-T_2)T_2 + 2\cdot\frac{1}{2}\left(\frac{1}{2}(T_1+T_2) - \frac{1}{2}(T_1-T_2)\right)T_2 = T_1T_2 = A_xA_y \ ,$$

where the area was derived graphically by summing the sum of the two side triangles to the central rectangle building the trapezoid.

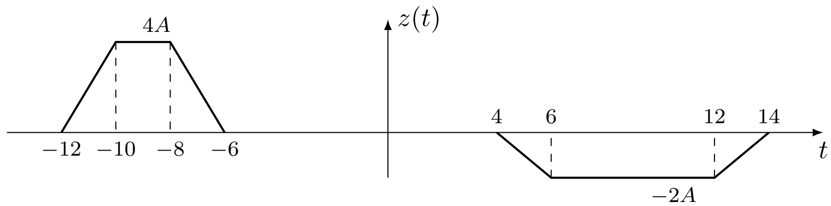
2. By using the notation $\mathrm{rect}_T(t) = \mathrm{rect}(t/T)$ we can express the signals in the form

$$x(t) = 4\,\mathrm{rect}_4(t+10) - 2\,\mathrm{rect}_8(t-8)\ , \qquad y(t) = A\,\mathrm{rect}_2(t-1)\ ,$$

so that by exploiting the linearity and time-shift properties of convolution we have

$$\begin{aligned} z(t) &= x*y(t) \\ &= \Big[4\,\mathrm{rect}_4(t+10) - 2\,\mathrm{rect}_8(t-8)\Big]\cdot\Big[A\,\mathrm{rect}_2(t-1)\Big] \\ &= 4A\,\mathrm{rect}_4(t+10)*\mathrm{rect}_2(t-1) - 2A\,\mathrm{rect}_8(t-8)*\mathrm{rect}_2(t-1) \\ &= 4A\,\mathrm{rect}_4*\mathrm{rect}_2(t+10-1) - 2A\,\mathrm{rect}_8*\underset{2}{\mathrm{rect}}(t-8-1) \\ &= 4A\,\mathrm{rect}_4*\mathrm{rect}_2(t+9) - 2A\,\mathrm{rect}_8*\mathrm{rect}_2(t-9) \end{aligned}$$

where $\mathrm{rect}_4*\mathrm{rect}_2$ is a trapezoid of bases $4+2=6$ and $4-2=2$, and height 2, while $\mathrm{rect}_8*\mathrm{rect}_2$ is a trapezoid of bases $8+2=10$ and $8-2=6$, and height 2. The results is therefore the one illustrated in the figure below.



Note that the property of the area holds also in this case, and in fact

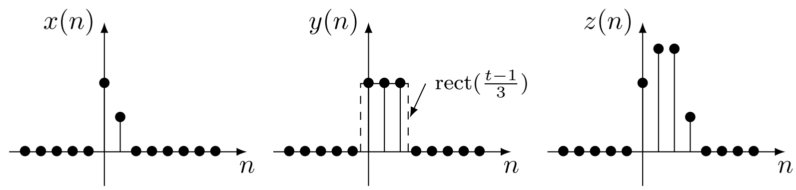
$$A_z = 4 \cdot 4A - 8 \cdot 2A = 0 = A_x A_y = (4 \cdot 4 - 8 \cdot 2)\, 2A$$

and so does the property on the extension $[-12, 12] + [0, 2] \to [-12, 14]$.

3. We preliminarily observe that

$$y(n) = \mathrm{rect}((n-1)/3) = \begin{cases} 1 & , n = 0, 1, 2 \\ 0 & , \text{otherwise} \end{cases}$$

as can be inferred from the graphical illustration below.



Then, by the properties of convolution we have

$$\begin{aligned} z(n) &= x * y(n) \\ &= [\delta(n) + \tfrac{1}{2}\delta(n-1)] * y(n) \\ &= y(n) + \tfrac{1}{2} y(n-1) = \begin{cases} 1 & , n = 0 \\ \frac{3}{2} & , n = 1, 2 \\ \frac{1}{2} & , n = 3 \\ 0 & , \text{otherwise} \end{cases} \end{aligned}$$

whose result is also provided in the figure above. Note that the property on the area of the convolution is verified, since we have

$$A_z = 1 + \tfrac{3}{2} + \tfrac{3}{2} + \tfrac{1}{2} = \tfrac{9}{2} = A_x A_y = 3 \cdot \tfrac{3}{2}\ ,$$

and so is the property on the extension $[0, 1] + [0, 2] \to [0, 3]$. By the time-shift property we also have

$$x(n-3) * y(n+2) = x * y(n-3+2) = z(n-1) = \begin{cases} 1 & , n = 1 \\ \frac{3}{2} & , n = 2, 3 \\ \frac{1}{2} & , n = 4 \\ 0 & , \text{otherwise} \end{cases}$$

4. In a series of two LTI systems, the output of the first system is $z(n) = x * h_1(n)$, and the final output assumes the form

$$y(n) = z * h_2(n) = x * h_1 * h_2(n) \ ,$$

where by commutativity and associativity the convolutions can be taken in any order. In the specific case it is reasonable to approach the calculation in the form

$$y(n) = (x * h_2) * h_1(n) \ ,$$

that is by inverting the two LTI systems. We have

$$\begin{aligned} x * h_2(n) &= [\delta(n) - a\delta(n-1)] * h_2(n) \\ &= h_2(n) - a\, h_2(n-1) \\ &= a^n 1_0(n) - a \cdot a^{n-1} 1_0(n-1) \\ &= a^n \cdot [1_0(n) - 1_0(n-1)] \\ &= a^n\, \delta(n) \\ &= \delta(n) \end{aligned}$$

so that $y(n) = \delta * h_1(n) = h_1(n) = \sin(8n)$. Any other ordering, although leading to the same final result, will involve much cumbersome calculations, hence highlighting the power of commutativity and associativity.

FOUNDATIONS OF SIGNALS AND SYSTEMS
7.6 Homework assignment
Prof. T. Erseghe

Exercises 7.6

Solve the following by using, where needed, the properties of convolution:

1. prove (either in continuous-time or discrete-time) that the convolution between two even signals is even, the convolution between two odd signals is even, and the convolution between an odd and an even signal is odd;
2. evaluate the convolution between $x(t) = \mathrm{sgn}(t)\,\mathrm{rect}(\frac{1}{4}t)$ and the signal $y(t) = A\,\mathrm{rect}(\frac{1}{2}t)$;
3. evaluate the output of a LTI system with impulse response $g(t) = \mathrm{rect}(t)$ when the input is the square wave $x(t) = \mathrm{rep}_2\,\mathrm{rect}(t - \frac{1}{2})$,
4. evaluate the output of a LTI system with impulse response $g(t) = e^{-at}\,1(t)$, $a > 0$, when the input is the square wave $x(t) = \mathrm{rep}_2\,\mathrm{rect}(t)$,
5. evaluate the convolution between $x(n) = \delta(n) - \frac{1}{2}\delta(n-1) - \frac{1}{2}\delta(n+1)$ and $y(n) = a^{-n}\,1_0(-n)$, $|a| < 1$;
6. evaluate the output of a LTI system with impulse response $g(n) = \delta(n-1) - \delta(n+1)$, $a > 0$, when the input is $x(n) = a^{|n|}$, $|a| < 1$;
7. evaluate the output of a LTI system with impulse response $g(n) = \mathrm{sgn}(n)$, when the input is $x(n) = |n|\,1_0(n-3)\,1_0(3-n)$;
8. evaluate the output of a LTI system with impulse response $g(n) = 1_0(n)$, when the input is $x(n) = \mathrm{rect}((n-1)/3)$;
9. evaluate the convolution between $x(n) = 2\delta(n) - \delta(n-1) - \delta(n+1)$ and $y(n) = 1_0(n-1) - 1_0(n-5)$;
10. evaluate the convolution between

$$x(n) = \begin{cases} (\frac{1}{5})^n & , n \in [0,4] \\ 0 & , \text{otherwise} \end{cases} \qquad y(n) = \begin{cases} \cos(\frac{\pi}{10}k) & , k \text{ multiple of } 5 \\ 0 & , \text{otherwise} \end{cases}$$

Solutions.

1. We consider the continuous-time first, for which we write

$$
\begin{aligned}
x * y(-t) &= \int_{-\infty}^{\infty} x(u)y((-t)-u)\,du \\
&= \int_{-\infty}^{\infty} x(-v)y(-t+v)\,dv \\
&= \int_{-\infty}^{\infty} x_-(v)y_-(t-v)\,dv \\
&= x_- * y_-(t)
\end{aligned}
$$

This is enough to prove the results since when both signals are either even or odd we have $x_- * y_- = x * y$ hence it follows that $x * y(-t) = x * y(t)$. When, instead, one signal is even and the other is odd we have $x_- * y_- = -x * y$, so that $x * y(-t) = -x * y(t)$. The counterpart in discrete-time takes the form

$$
\begin{aligned}
x * y(-n) &= \sum_{k-\infty}^{\infty} x(k)y((-n)-k) \\
&= \sum_{m=-\infty}^{\infty} x(-m)y(-n+m) \\
&= \sum_{-\infty}^{\infty} x_-(m)y_-(n-m) \\
&= x_- * y_-(n)
\end{aligned}
$$

and leads to the same result by the same rationale.

2. We have

$$x(t) = u(t-1) - u(t+1)\,, \quad y(t) = A\,u(t)\,, \quad u(t) = \mathrm{rect}(\tfrac{1}{2}t)$$

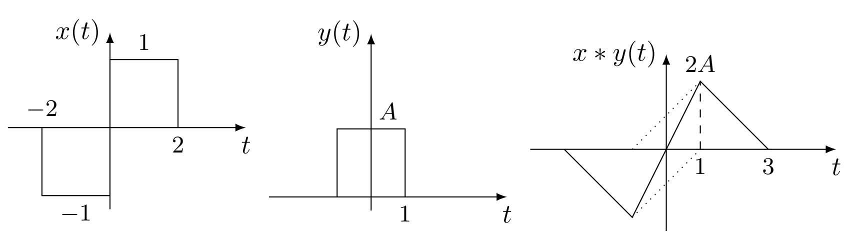
so that by linearity and time-shift property we can write

$$x * y(t) = [u(t-1) - u(t+1)] * [A\,u(t)] = A\,u * u(t-1) - A\,u * u(t+1)$$

with $u * u(t) = 2\,\mathrm{triang}(\frac{1}{2}t)$. Hence, it readily follows that

$$x * y(t) = 2A\,\mathrm{triang}(\tfrac{1}{2}(t-1)) - 2A\,\mathrm{triang}(\tfrac{1}{2}(t+1))$$

as illustrated in the figure below.



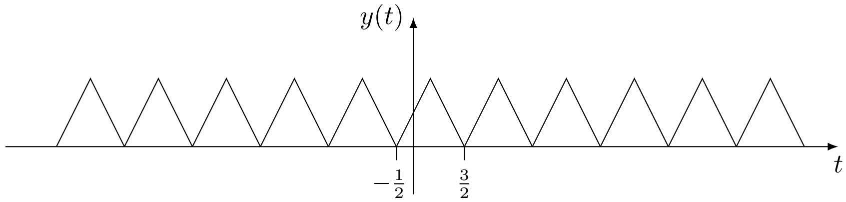
3. By writing the input in the form

$$x(t) = \text{rep}_2 \,\text{rect}(t - \tfrac{1}{2}) = \sum_{k=-\infty}^{\infty} \text{rect}(t - 2k - \tfrac{1}{2})$$

the LTI system output becomes

$$y(t) = x * g(t) = \sum_{k=-\infty}^{\infty} \underbrace{\text{rect} * \text{rect}}_{\text{triang}}(t - 2k - \tfrac{1}{2}) = \text{rep}_2 \,\text{triang}(t - \tfrac{1}{2})$$

where we simply exploited linearity. The result is depicted in the figure below.



4. By writing the input in the form

$$x(t) = \text{rep}_2 \,\text{rect}(t) = \sum_{k=-\infty}^{\infty} \text{rect}(t - 2k)$$

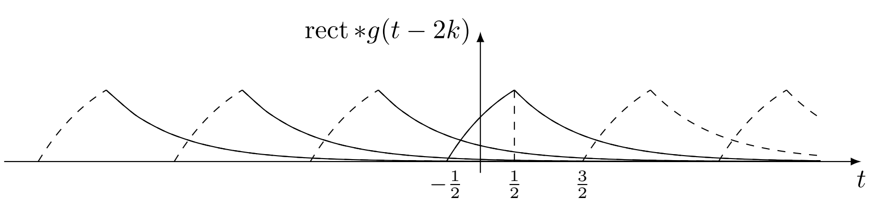
the LTI system output becomes

$$y(t) = x * g(t) = \sum_{k=-\infty}^{\infty} \text{rect} * g(t - 2k) = \text{rep}_2 \,\text{rect} * g(t)$$

where we simply exploited linearity, and where (see Exercise 7.3.5)

$$\text{rect} * g(t) = \begin{cases} 0 & , t < -\frac{1}{2} \\ \frac{1 - e^{-a(t + \frac{1}{2})}}{a} & , -\frac{1}{2} < t < \frac{1}{2} \\ \frac{e^{\frac{a}{2}} - e^{-\frac{a}{2}}}{a} e^{-at} & , t > \frac{1}{2} \end{cases}$$

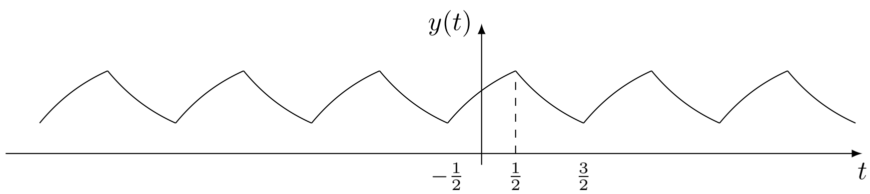
whose periodic repetition segments active in the period $[-\frac{1}{2}, \frac{3}{2}]$ are shown in the figure below.



Hence, in the reference period $[-\frac{1}{2}, \frac{3}{2}]$ the periodic repetition provides

$$\begin{aligned}
y(t) &= \mathrm{rect} * g(t) + \sum_{k=-\infty}^{-1} \frac{e^{\frac{a}{2}} - e^{-\frac{a}{2}}}{a} e^{-a(t-2k)} \\
&= \mathrm{rect} * g(t) + \frac{e^{\frac{a}{2}} - e^{-\frac{a}{2}}}{a} e^{-at} \frac{e^{-2a}}{1 - e^{-2a}} \ , \quad t \in [-\tfrac{1}{2}, \tfrac{3}{2}] \\
&= \begin{cases} \frac{1}{a} - K_a e^{-at} & , -\frac{1}{2} < t < \frac{1}{2}, \text{ where } K_a = \frac{e^{\frac{a}{2}} - e^{-\frac{a}{2}}}{a(e^a - e^{-a})} \\ K_a e^{-a(t-1)} & , \frac{1}{2} < t < \frac{3}{2} \end{cases}
\end{aligned}$$

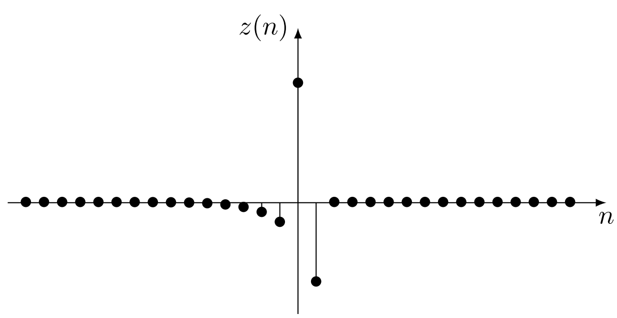
as illustrated in the figure below.



5. We proceed by linearity and time-shift property and write

$$\begin{aligned}
z(n) &= x * y(n) \\
&= [\delta(n) - \tfrac{1}{2}\delta(n-1) - \tfrac{1}{2}\delta(n+1)] * y(n) \\
&= y(n) - \tfrac{1}{2}y(n-1) - \tfrac{1}{2}y(n+1) \\
&= a^{-n}\, 1_0(-n) - \tfrac{1}{2}\, a^{-(n-1)}\, 1_0(-n+1) - \tfrac{1}{2}\, a^{-(n+1)}\, 1_0(-n-1) \\
&= \begin{cases} a^{-n} - \frac{1}{2}\, a^{-(n-1)} - \frac{1}{2}\, a^{-(n+1)} = -\frac{(1-a)^2}{2a}\, a^{-n} & , n < 0 \\ 1 - \frac{1}{2}a + 0 = 1 - \frac{1}{2}a & , n = 0 \\ 0 - \frac{1}{2} + 0 = -\frac{1}{2} & , n = 1 \\ 0 & , \text{otherwise} \end{cases}
\end{aligned}$$

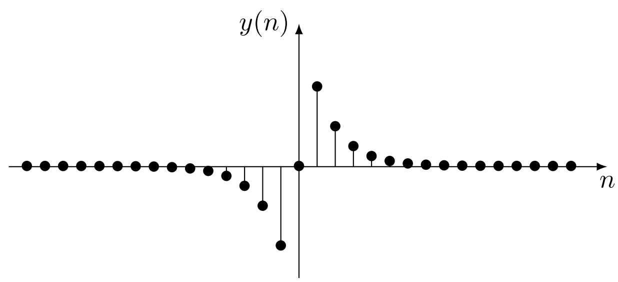
the result being sketched in the figure below for $a = \frac{1}{2}$.



6. By linearity and time-shift properties we have

$$
\begin{aligned}
y(n) &= x * g(n) \\
&= x * [\delta(n-1) - \delta(n+1)] \\
&= x(n-1) - x(n+1) \\
&= a^{|n-1|} - a^{|n+1|} \\
&= \begin{cases} a^{-n+1} - a^{-n-1} = -(a^{-1} - a)\, a^{-n} & , n < 0 \\ a - a = 0 & , n = 0 \\ a^{n-1} - a^{n+1} = (a^{-1} - a)\, a^{n} & , n > 0 \end{cases} \\
&= (a^{-1} - a)\, \mathrm{sgn}(n)\, a^{|n|}
\end{aligned}
$$

the result being sketched in the figure below for $a = \frac{1}{2}$.



7. In this case it is fundamental to observe that

$$x(n) = |n|\, 1_0(n-3)\, 1_0(3-n) = x(n) = |n|\, \delta(n-3) = 3\, \delta(n-3)$$

hence it simply is

$$y(n) = x * g(n) = 3\delta * g(n) = 3\, g(n) = 3\, \mathrm{sgn}(n)\ .$$

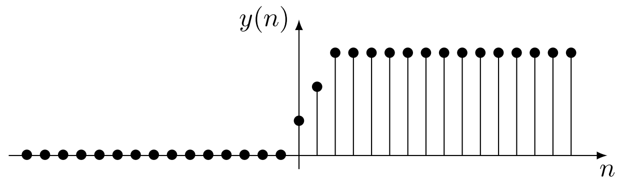
8. Since the input can be written in the form (see signal $y(n)$ in Exercise 7.5.3)

$$x(n) = \mathrm{rect}((n-1)/3) = \delta(n) + \delta(n-1) + \delta(n-2)$$

then by linearity and time-shift properties we have

$$
\begin{aligned}
y(n) &= x * g(n) \\
&= [\delta(n) + \delta(n-1) + \delta(n-2)] * g(n) \\
&= g(n) + g(n-1) + g(n-2) \\
&= 1_0(n) + 1_0(n-1) + 1_0(n-2) \\
&= \begin{cases} 0 & , n < 0 \\ 1 & , n = 0 \\ 2 & , n = 1 \\ 3 & , n > 1 \end{cases}
\end{aligned}
$$

the result being sketched in the figure below.



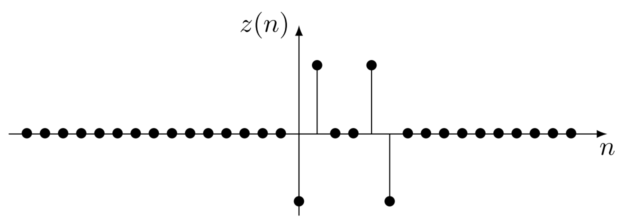
9. Note that $y(n)$ can be written in the form

$$y(n) = \begin{cases} 1 & , n \in [1,4] \\ 0 & , \text{otherwise} \end{cases} = \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4)$$

then by linearity and time-shift properties we have

$$\begin{aligned} z(n) &= x * y(n) \\ &= [2\delta(n) - \delta(n-1) - \delta(n+1)] * y(n) \\ &= 2y(n) - y(n-1) - y(n+1) \\ &= 2\delta(n-1) + 2\delta(n-2) + 2\delta(n-3) + 2\delta(n-4) \\ &\quad - \delta(n-2) - \delta(n-3) - \delta(n-4) - \delta(n-5) \\ &\quad - \delta(n) - \delta(n-1) - \delta(n-2) - \delta(n-3) \\ &= -\delta(n) + \delta(n-1) + \delta(n-4) - \delta(n-5) \end{aligned}$$

as illustrated in the figure below.



10. We first observe that

$$y(n) = \begin{cases} \cos(\frac{\pi}{2}n) & , k = 5n \\ 0 & , \text{otherwise} \end{cases}$$

hence $y(n)$ is periodic of period $N = 20$ with active values $y(0) = 1$ and $y(10) = -1$ in a period, that is

$$y(n) = \text{rep}_{20}\delta(n) - \delta(n-10) \; .$$

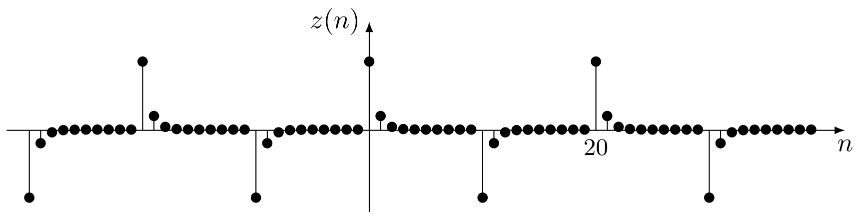
In the convolution with $x(n)$, by linearity and time-shift properties, as already seen many times in the preceding exercises, we have

$$z(n) = x * y(n) = \text{rep}_{20}x(n) * [\delta(n) - \delta(n-10)] = \text{rep}_{20}x(n) - x(n-10)$$

so that, in the period $n \in [0, 20)$, we have

$$z(n) = \begin{cases} (\frac{1}{5})^n & , n \in [0, 5) \\ 0 & , n \in [5, 10) \\ -(\frac{1}{5})^{n-10} & , n \in [10, 15) \\ 0 & , n \in [15, 20) \end{cases}$$

as illustrated in the figure below.



FOUNDATIONS OF SIGNALS AND SYSTEMS
8.2 Solved exercises
Prof. T. Erseghe

Exercises 8.2

Solve the following:

1. evaluate the circular convolution between the signals $x(t) = \text{rep}_3 \, \text{rect}(t-\frac{1}{2})$ and $y(t) = \text{rep}_3 \, \text{rect}(t - \frac{5}{2})$,

2. prove that the circular convolution between $x(n) = \text{rep}_N \, x_0(n)$ and $y(n) = \text{rep}_N \, y_0(n)$, has the form

$$\begin{aligned} z(n) &= x *_{\text{cir}} y(n) \\ &= \text{rep}_N x_0 * y_0(n) \\ &= x * y_0(n) \\ &= x_0 * y(n) \ , \end{aligned}$$

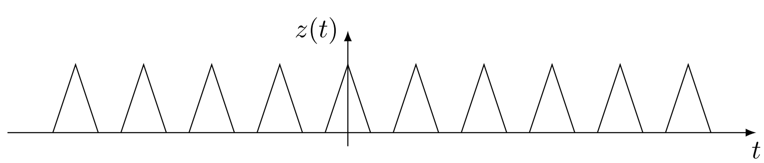
3. evaluate the the circular convolution between the signals $x(t) = 4A \cos(2\pi t) + A$ and $y(t) = \sin(\pi t)$.

Solutions.

1. We have $x_0(t) = \text{rect}(t - \frac{1}{2})$ and $y_0(t) = \text{rect}(t - \frac{5}{2})$, hence the circular convolution provides

$$\begin{aligned} z(t) &= x *_{\text{cir}} y(t) \\ &= \text{rep}_3 x_0 * y_0(t) \\ &= \text{rep}_3 \text{triang}(t - \tfrac{1}{2} - \tfrac{5}{2}) \\ &= \text{rep}_3 \text{triang}(t - 3) \\ &= \text{rep}_3 \text{triang}(t) \end{aligned}$$

as illustrated in the figure below.



2. We follow the same procedure seen for the continuous-time, and write

$$\begin{aligned} z(n) &= x *_{\text{cir}} y(n) \\ &= \sum_{k=0}^{N-1} x(k) y(n-k) \\ &= \sum_{k=0}^{N-1} \left(\sum_{\ell=-\infty}^{\infty} x_0(k-\ell N) \right) y(n-k) \\ &= \sum_{\ell=-\infty}^{\infty} \sum_{k=0}^{N-1} x_0(k-\ell N) y(n-k) \\ &= \sum_{\ell=-\infty}^{\infty} \sum_{m=-\ell N}^{N-1-\ell N} x_0(m) y(n-m-\ell N) \\ &= \sum_{\ell=-\infty}^{\infty} \sum_{m=-\ell N}^{N-1-\ell N} x_0(m) y(n-m) \end{aligned}$$

where we replaced $m = k - \ell N$ and we exploited in the last equivalence the periodicity of $y(n)$. Hence, by observing that the sum in ℓ is combining the sums of m over the entire time-axis, we can write

$$\begin{aligned} z(n) &= x *_{\text{cir}} y(n) \\ &= \sum_{m=-\infty}^{\infty} x_0(m) y(n-m) \\ &= x_0 * y(n) \ , \end{aligned}$$

which proves one of the results. By further expanding y in the form of a periodic repetition we also have

$$\begin{aligned} z(n) &= x *_{\text{cir}} y(n) \\ &= \sum_{m=-\infty}^{\infty} x_0(m) \left(\sum_{\ell=-\infty}^{\infty} y_0(n-m-\ell N) \right) \\ &= \sum_{\ell=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x_0(m) y_0(n-m-\ell N) \right) \\ &= \sum_{\ell=-\infty}^{\infty} x_0 * y_0(n-\ell N) \\ &= \text{rep}_N x_0 * y_0(n) \end{aligned}$$

thus completing the proof.

3. We preliminarily observe that the periodicity of $x(t)$ is $T_x = 1$, and that of $y(t)$ is $T_y = 2$, hence we take the common periodicity $T_p = 2$ in the

circular convolution. In this specific case, the best approach is to apply directly the definition (with $t_0 = 0$) since the signals are not piecewise defined. We obtain

$$\begin{aligned}
z(t) &= x *_{\text{cir}} y(t) \\
&= \int_0^2 x(u)\, y(t-u)\, du \\
&= \int_0^2 [4A\, \cos(2\pi u) + A]\, \sin(\pi(t-u))\, du \\
&= \tfrac{A}{2j} \int_0^2 [2e^{j2\pi u} + 2e^{-j2\pi u} + 1]\, [e^{j\pi(t-u)} - e^{-j\pi(t-u)}]\, du \\
&= \tfrac{A}{2j} \int_0^2 [2e^{j\pi(t+u)} + 2e^{\pi(t-3u)} + e^{j\pi(t-u)} \\
&\qquad\qquad - 2e^{-j\pi(t+u)} - 2e^{-\pi(t-3u)} - e^{-j\pi(t-u)}]\, du \\
&= A \int_0^2 [2\sin(\pi(t+u)) + 2\sin(\pi(t-3u)) + \sin(\pi(t-u))]\, du \\
&= 0
\end{aligned}$$

since the three sinusoids in the integral have periods, respectively, of 2, $\frac{2}{3}$ (i.e., one third of 2), and 2.

FOUNDATIONS OF SIGNALS AND SYSTEMS
8.3 Homework assignment
Prof. T. Erseghe

Exercises 8.3

Solve the following:

1. evaluate the circular convolution between two square waves of duty cycle d_x and d_y, respectively, with $d_x + d_y < 1$, and generic period T_p,

2. evaluate the circular convolution between $x(t) = 4A\cos(2\pi t/T_p) + A$, and a periodic signal $y(t)$, defined in $[-\frac{1}{2}T_p, \frac{1}{2}T_p]$ as $y(t) = 2\cos(\pi t/T_p)$,

3. evaluate the circular convolution between a signal $x(n)$ periodic of period N and defined in the period $[0, N)$ as

$$x(n) = \begin{cases} 1 & , 0 \le n \le \frac{N}{2} - 1 \\ 0 & , \frac{N}{2} \le n < N \end{cases}$$

   and itself (self-convolution),

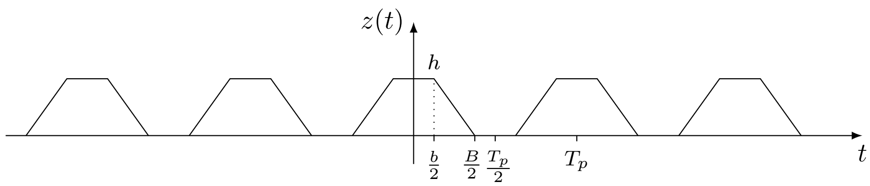
4. identify a periodic signals $x(t)$ whose circular self-convolution provides $z(t) = x * x(t) = \text{rep}_{T_p} \text{triang}(2t/T_p - 1)$.

Solutions.

1. The two square waves can be written as $x(t) = \mathrm{rep}_{T_p} x_0(t)$ and $y(t) = \mathrm{rep}_{T_p} y_0(t)$ where

$$x_0(t) = \mathrm{rect}\left(\frac{t}{d_x T_p}\right) , \qquad y_0(t) = \mathrm{rect}\left(\frac{t}{d_y T_p}\right) ,$$

so that $z(t) = x * y(t) = \mathrm{rep}_{T_p} x_0 * y_0(t)$ where $x_0 * y_0$ is a trapezoid with bases $B = T_p(d_x + d_y)$ and $b = T_p|d_x - d_y|$, and with height $h = T_p \min(d_x, d_y)$, thus providing the periodic signal $z(t)$ illustrated in the figure below.



2. We have $y(t) = \mathrm{rep}_{T_p} y_0(t)$ with

$$y_0(t) = 2\cos(\pi t/T_p)\ \mathrm{rect}(t/T_p) ,$$

so that the circular convolution can be written in the form

$$\begin{aligned}
z(t) &= x *_{\mathrm{cir}} y(t) \\
&= x * y_0(t) \\
&= y_0 * x(t) \\
&= \int_{-\infty}^{\infty} y_0(u) x(t-u)\, du \\
&= \int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} 2\cos(\pi u/T_p)\, [4A\cos(2\pi(t-u)/T_p) + A]\, du \\
&= A\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} [e^{j\pi u/T_p} + e^{-j\pi u/T_p}]\, [2e^{j2\pi(t-u)/T_p} + 2e^{-j2\pi(t-u)/T_p} + 1]\, du \\
&= A\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} [2e^{j\pi(2t-u)/T_p} + 2e^{j\pi(3u-2t)/T_p} + e^{j\pi u/T_p} \\
&\qquad + 2e^{j\pi(2t-3u)/T_p} + 2e^{-j\pi(2t-u)/T_p} + e^{-j\pi u/T_p}]\, du \\
&= A\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} [4\cos(\pi \tfrac{u-2t}{T_p}) + 4\cos(\pi\tfrac{3u-2t}{T_p}) + 2\cos(\pi\tfrac{u}{T_p})]\, du
\end{aligned}$$

By solving the integral, we obtain

$$\begin{aligned}
z(t) &= x *_{\text{cir}} y(t) \\
&= \frac{AT_p}{\pi}\Big[4\sin(\pi\tfrac{u-2t}{T_p}) + \tfrac{4}{3}\sin(\pi\tfrac{3u-2t}{T_p}) + 2\sin(\pi\tfrac{u}{T_p})\Big]_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} \\
&= \frac{AT_p}{\pi}\Big[4\sin(\tfrac{\pi}{2} - \tfrac{2\pi t}{T_p}) + \tfrac{4}{3}\sin(\tfrac{3\pi}{2} - \tfrac{2\pi t}{T_p}) + 2\sin(\tfrac{\pi}{2}) \\
&\qquad - 4\sin(-\tfrac{\pi}{2} - \tfrac{2\pi t}{T_p}) - \tfrac{4}{3}\sin(-\tfrac{3\pi}{2} - \tfrac{2\pi t}{T_p}) - 2\sin(-\tfrac{\pi}{2})\Big] \\
&= \frac{AT_p}{\pi}\Big[8\cos(\tfrac{2\pi t}{T_p}) - \tfrac{8}{3}\cos(\tfrac{2\pi t}{T_p}) + 4\Big] \\
&= \frac{4AT_p}{\pi}\Big[\tfrac{4}{3}\cos(\tfrac{2\pi t}{T_p}) + 1\Big]
\end{aligned}$$

3. We write $x(n) = \text{rep}_n\, x_0(n)$ with

$$x_0(n) = \begin{cases} 1 & , n \in [0, K_x] \\ 0 & , \text{otherwise} \end{cases} \qquad K_x = \begin{cases} \frac{N}{2} - 1 & , N \text{ even} \\ \frac{N}{2} - \frac{3}{2} & , N \text{ odd} \end{cases}$$

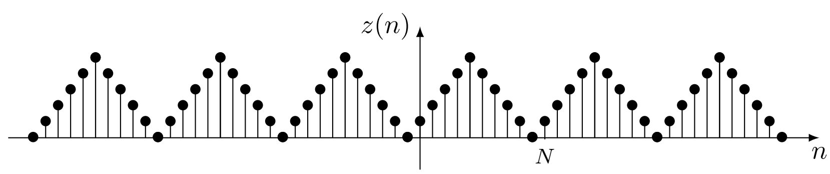
so that $z(n) = x *_{\text{cir}} x(n) = \text{rep}_N\, x_0 * x_0(n)$ with

$$x_0 * x_0(n) = \begin{cases} 1 + n & , n \in [0, K_x] \\ 1 + 2K_x - n & , n \in (K_x, 2K_x] \\ 0 & , \text{otherwise} \end{cases}$$

where

$$2K_x = \begin{cases} N - 2 & , N \text{ even} \\ N - 3 & , N \text{ odd} \end{cases}$$

so that there is no aliasing in the periodic repetition. The resulting signal is illustrated for N even in the figure below (for N odd there would be two zeros in a row separating the periodic repetitions).



4. We look for $x(n) = \text{rep}_n\, u(n)$ such that

$$u * u(t) = \text{triang}\left(\frac{t - \frac{1}{2}T_p}{\frac{1}{2}T_p}\right)$$

which is an isosceles triangle with base T_p, so that we can choose

$$u(t) = \frac{1}{\sqrt{\frac{1}{2}T_p}}\ \text{rect}\left(\frac{t - \frac{1}{4}T_p}{\frac{1}{2}T_p}\right) ,$$

that is a square wave $x(t)$ of duty cycle $\frac{1}{2}$. The result is correctly scaled since we can notice that

$$\text{rect}\left(\frac{t}{\frac{1}{2}T_p}\right) * \text{rect}\left(\frac{t}{\frac{1}{2}T_p}\right) = \frac{1}{2}T_p \, \text{triang}\left(\frac{t}{\frac{1}{2}T_p}\right) \, .$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
8.5 Solved exercises
Prof. T. Erseghe

Exercises 8.5

Solve the following:

1. prove that a necessary and sufficient condition for a discrete-time filter to be BIBO stable is that its impulse response is absolutely summable;
2. discuss the BIBO stability of a filter with impulse response $g(n) = n\ \cos(n\frac{\pi}{4})\, 1_0(n)$;
3. discuss the BIBO stability of a filter with impulse response $g(t) = e^{-t}\ \cos(t)\, 1(t)$.

Solution.

1. We proceed as in the continuous-time case. We first prove that an absolutely summable $g(n)$ identifies a BIBO stable system. We have

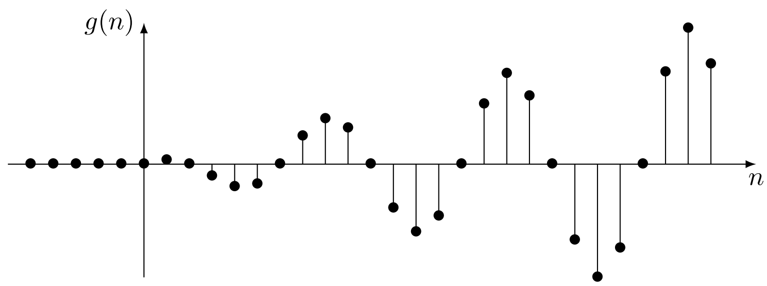
$$\begin{aligned}|y(n)| &= \left|\sum_{k=-\infty}^{\infty} x(k)g(n-k)\right| \\ &\le \sum_{k=-\infty}^{\infty} |x(k)|\cdot|g(n-k)| \\ &< \sum_{k=-\infty}^{\infty} L_x\cdot|g(n-k)| \\ &= L_x \sum_{\ell=-\infty}^{\infty} |g(\ell)| = L_x L_g\end{aligned}$$

which proves BIBO stability. Conversely, we wish to prove that a BIBO stable filter implies an absolutely summable $g(n)$. To do so, we proceed by absurd and assume $L_g = \infty$. We then build a limited signal $x(n) = e^{-j\vartheta(-n)}$ where $g(n) = |g(n)|e^{j\vartheta(n)}$, which is a signal satisfying $|x(n)| = 1$. This limited signal, however produces an output which is not limited at $n = 0$, and in fact

$$\begin{aligned}y(0) &= \sum_{k=-\infty}^{\infty} x(k)g(0-k) \\ &= \sum_{k=-\infty}^{\infty} e^{-j\vartheta(-k)}\cdot|g(-k)|e^{j\vartheta(-k)} \\ &= \sum_{k=-\infty}^{\infty} |g(-k)| = \sum_{\ell=-\infty}^{\infty} |g(\ell)| = L_g\end{aligned}$$

where $L_g = \infty$, which is an absurd proving the result.

2. The causal filter $g(n) = n\,\cos(n\frac{\pi}{4})\,1_0(n)$ is sketched in the figure below



which reveals that the impulse response is not vanishing, and that its values tend to increase (to infinity) as n increases. Hence, it is not absolutely summable and the filter is not BIBO stable.

3. This is a vanishing causal filter for which

$$\begin{aligned} L_g &= \int_{-\infty}^{\infty} |g(t)|\,dt \\ &= \int_{0}^{\infty} e^{-t}\,|\cos(t)|\,dt \\ &\leq \int_{0}^{\infty} e^{-t}\,dt \\ &= 1 \end{aligned}$$

which reveals a BIBO stable system. Note that we are not interested in the true value of L_g, but rather on knowing if L_g is finite or not, which makes it useful also to work with bounds (thus simplifying calculations).

FOUNDATIONS OF SIGNALS AND SYSTEMS
8.6 Homework assignment
Prof. T. Erseghe

Exercises 8.6

For the following systems, state if they are LTI systems (filters). If so, identify their impulse response, and exploit it to assess their memory and BIBO stability properties:

1. the system with input/output relation
$$y(t) = 2\int_{t-2}^{t+2} x(u)e^{u-t}\,du - x(t+2)\ ;$$
2. the system with input/output relation
$$y(t) = 1(t-2)\int_{-1}^{t-2} x(u)\cos(t-u)\,du + 3\,x(t-1)\ ;$$
3. the series of two filter with impulse responses $g_1(n) = \sin(8n)$ and $g_2(n) = a^n\,1_0(n)$, $-1 < a < 1$;
4. the system with input/output relation
$$y(t) = \int_{t-1}^{2t+1} x(u)\,e^{t-u}\,du - x(t-2)\ ;$$
5. the system with input/output relation
$$y(n) = \begin{cases} \displaystyle\sum_{k=-10}^{n-3} x(k)e^{-(n-k)} & n \geq 0 \\ 0 & n < 0; \end{cases}$$
6. the system with input/output relation
$$y(t) = \int_{-\infty}^{t-2} e^{5(t-\tau)}\,x(\tau+2)\,d\tau + 3\,x(t-7)\ ;$$
7. the system with input/output relation
$$y(t) = \begin{cases} 0 & t \leq 2 \\ \cos(t+2)\cdot\displaystyle\int_{-1}^{t-2} x(\tau)d\tau & t > 2\ ; \end{cases}$$
8. the system with input/output relation
$$y(t) = x * g(t)\ , \qquad g(t) = \mathrm{sinc}(8t)\ ;$$

9. the system with input/output relation

$$y(n) = \begin{cases} \sum_{k=-\infty}^{n-3} x(k)e^{-(n-k)} & n \geq 0 \\ 0 & n < 0; \end{cases}$$

Solutions.

1. In this case, we can substitute the extrema of the integration by a rectangular function, to have

$$y(t) = 2\int_{-\infty}^{\infty} x(u)\cdot e^{-(t-u)}\ \mathrm{rect}(\tfrac{1}{4}(t-u))\ du - x * \delta_{-2}(t)\ ,$$

which reveals the system as a convolutional (LTI) system with impulse response

$$g(t) = e^{-t}\ \mathrm{rect}(t/4) + \delta(t+2)\ ,$$

with extension $e(g) = [-2, 2]$, hence it is a dynamic filter (but not causal), which is also BIBO stable since

$$\int_{-\infty}^{\infty} |g(t)|\, dt = \int_{-\infty}^{\infty} g(t)\, dt = \int_{-2}^{2} e^{-t}\, dt + 1 = e^2 - e^{-2} + 1 < \infty\ .$$

The filter is real since $g(t)$ is real valued (and positive).

2. In this case, we can substitute the extrema of the integration by two unit step functions, to have

$$y(t) = 1(t-2)\int_{-\infty}^{\infty} x(u)\, 1(u+1)\cdot\cos(t-u)1(t-u-2)\ du + 3\, x * \delta_1(t)\ ,$$

but the integral cannot take the form of a convolution, hence the system is not a filter.

3. In this case, we need to evaluate the impulse response by discrete-time convolution, that is

$$\begin{aligned}
g(n) &= g_2 * g_1(n) \\
&= \sum_{k=-\infty}^{\infty} a^k\, 1_0(k)\cdot\frac{1}{2j}[e^{j8(n-k)} - e^{-j8(n-k)}] \\
&= \frac{e^{j8n}}{2j}\sum_{k=0}^{\infty}(a\, e^{-j8})^k - \frac{e^{-j8n}}{2j}\sum_{k=0}^{\infty}(a\, e^{j8})^k \\
&= \frac{1}{2j}\frac{e^{j8n}}{1-a\, e^{-j8}} - \frac{1}{2j}\frac{e^{-j8n}}{1-a\, e^{j8}} \\
&= \Im\left[\frac{e^{j8n}}{1-a\, e^{-j8}}\right]
\end{aligned}$$

By defining $b = 1 - a\, e^{-j8} = |b|\, e^{j\varphi_b}$, we have

$$g(n) = \frac{1}{|b|}\ \sin(8n - \varphi_b)$$

which has extension $e(g) = (-\infty, \infty)$, and it is not vanishing hence it is not absolutely summable. The filter is therefore not BIBO stable. It is however real, since $g(n)$ is real-valued.

4. In this case, we can substitute the extrema of the integration by two unit step functions, to have

$$y(t) = \int_{-\infty}^{\infty} x(u)\, e^{t-u}\, 1(u-t+1)\, 1(2t-u-1)\, du - x * \delta_2(t)\ ;$$

where the contribution $1(2t-u-1)$ cannot be expressed as a function of $t-u$, hence the system is not LTI.

5. In this case, we can substitute the extrema of the integration by two unit step functions, to have

$$y(n) = \begin{cases} \sum_{k=-\infty}^{\infty} x(k)\, 1_0(k+10)\, e^{-(n-k)}\, 1_0(n-k-3) & n \geq 0 \\ 0 & n < 0; \end{cases}$$

where the contribution $1_0(k+10)$ multiplying $x(k)$ makes the system not LTI (a product by a waveform is linear but not time-invariant).

6. In this case, we can substitute the upper extrema of the integration by one unit step function, to have

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(u+2)\, e^{5(t-u)}\, 1(t-u-2)\, du + 3\, x * \delta_7(t) \\ &= \int_{-\infty}^{\infty} x(v)\, e^{5(t-v+2)}\, 1(t-v)\, dv + 3\, x * \delta_7(t) \end{aligned}$$

which reveals an LTI system with impulse response

$$g(t) = e^{5(t+2)}\, 1(t) + 3\, \delta(t-7)$$

which has extension $e(g) = [0, \infty)$, hence it is causal. The impulse response is also real-valued, so the filter is real. However, the filter is not BIBO stable since $g(t)$ diverges for $t \to \infty$.

7. In this case, we can substitute the extrema of the integration by two unit step functions, to have

$$y(t) = \cos(t+2) \cdot \begin{cases} 0 & , t \leq 2 \\ \int_{-\infty}^{\infty} x(u)\, 1(u+1)\, 1(t-u-2)\, du & , t > 2 \end{cases}$$

which reveals that the system is not LTI because of the presence of a product $\cos(t+2)$ (linear but not time-invariant), and also because of the contribution $1(u+1)$ multiplying $x(u)$ (linear but not time-invariant).

8. In this case the system is evidently LTI with impulse response $g(t) = \mathrm{sinc}(8t)$, it is also dynamic and real (since the impulse response is real-valued). The system is, however, not BIBO stable since the sinc function is not absolutely integrable. In fact, we have

$$L_g = \int_{-\infty}^{\infty} |\mathrm{sinc}(8t)|\, dt = \tfrac{1}{8} \int_{-\infty}^{\infty} |\mathrm{sinc}(u)|\, du = \tfrac{1}{8} L_{\mathrm{sinc}}$$

but

$$
\begin{aligned}
L_{\text{sinc}} &= 2\int_0^{\infty} \left|\frac{\sin(\pi u)}{\pi u}\right| du \\
&= 2\sum_{n=0}^{\infty}\int_n^{n+1} \left|\frac{\sin(\pi u)}{\pi u}\right| du \\
&\geq 2\sum_{n=0}^{\infty}\int_n^{n+1} \left|\frac{\sin(\pi u)}{\pi (n+1)}\right| du \\
&= 2\sum_{n=0}^{\infty}\frac{1}{\pi(n+1)}\int_0^1 \sin(\pi u)\, du \\
&= \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n+1} \\
&= \infty
\end{aligned}
$$

where we exploited the inequality $1/u \geq 1/(n+1)$ for $u \in [n, n+1]$, and where the divergence is ensured by the divergence of the harmonic series.

9. In this case, we can substitute the upper extremum of the integration by a unit step function, to have

$$
y(n) = \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} x(k)\, e^{-(n-k)}\, 1(n-k-3) & n \geq 0 \\ 0 & n < 0; \end{cases}
$$

which reveals a convolution with $g(n) = e^{-n} \cdot 1(n-3)$ (casual and BIBO stable) in the upper part, that is we have $y(n) = 1_0(n) \cdot x * g(n)$, where the multiplication by the unit step makes the system not LTI.

FOUNDATIONS OF SIGNALS AND SYSTEMS
9.3 Solved exercises
Prof. T. Erseghe

Exercises 9.3

Solve the following MatLab problem:

1. Plot the signal $s_1(t) = \tanh(t)$, as well as its time-shifted versions, $s_2(t) = \tanh(t-b)$ and $s_3(t) = \tanh(t+b)$ with $b = 3$, in the same plot in the time range $[-10, 10]$.

2. Plot the one-sided exponential $s(t) = e^{-1}\,1(t)$ in the range $t \in [-1, 10]$.

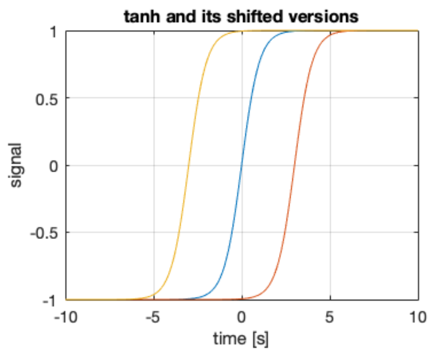
3. Draw the (periodic) square-wave of period $T_p = 3$ and duty cycle $d = .3$ in the interval $t \in [-4, 5]$.

Solution.

1. The code can be pretty simple, the fundamental aspect being that of correctly choosing a sampling spacing sufficiently small to capture the hyperbolic tangent shape. In the code example this is set to 0.1. The code can then read as follows

```
close all
clear all
clc

t = -10:.1:10;
b = 3;
s1 = tanh(t);
s2 = tanh(t-b);
s3 = tanh(t+b);
figure
plot(t,s1,t,s2,t,s3)
grid on
xlabel('time [s]')
ylabel('signal')
title('tanh and its shifted versions')

% printing figure in png format
set(gcf,'PaperUnits','inches','PaperPosition',[0 0
    4 3])
print -dpng ex9_3_1.png -r100
```

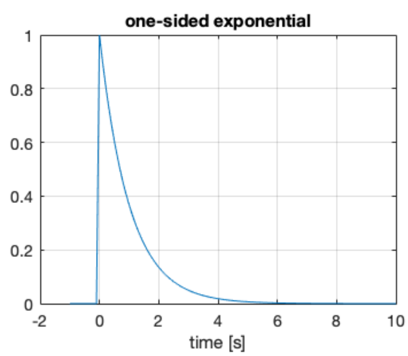
where the last part of the code is used to export the plot in png format. This provides the following output



2. In this case we can use the vector $t \geq 0$ to set to 0 values at negative times, to have the following result.

```
t = -1:.1:10;
s = (t>=0).*exp(-t);
figure
plot(t,s)
grid on
xlabel('time [s]')
title('one-sided exponential')
```

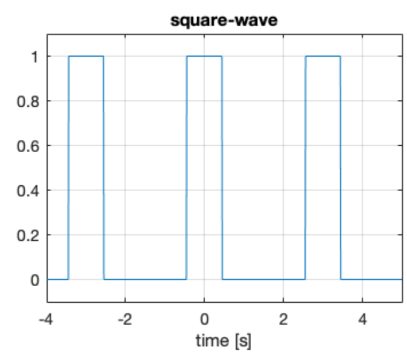
where the last part of the code is used to export the plot in png format. This provides the following output



3. In this case we define a function for the square wave depending on parameters T_p (period) and d (duty cycle). This builds on the definition of rect, which is defined according to the absolute value $|t|$. In the definition of the square wave, note how time is first reported to the period by $t_1 = t/T_p$ (mod 1), then we exploit the fact that in the reference period the signal behaves as $\text{rect}(t/d) + \text{rect}((t-1)/d)$.

```
t = -4:.01:5;
s = square_wave(t,3,.3);
figure
```

```
plot(t,s)
grid on
xlabel('time [s]')
title('square-wave')
axis([xlim -.1 1.1])

function s = square_wave(t,Tp,d)
t1 = mod(t/Tp,1);
s = rect(t1/d) + rect((t1-1)/d);
end

function s = rect(t)
s = (abs(t)<.5)+.5*(abs(t)==.5);
end
```



FOUNDATIONS OF SIGNALS AND SYSTEMS
9.4 Homework assignment
Prof. T. Erseghe

Exercises 9.4

Solve the following MatLab problems:

1. Plot the signal $s(t) = \tanh(t)$ together with its time-shifted and scaled versions $\tanh(at)$, $\tanh(t/a)$, $\tanh(at - b)$, $\tanh(at + b)$, $\tanh((t - b)/a)$, $\tanh((t+b)/a)$ in the same plot in the time range $[-10, 10]$, by using $a = 2$ and $b = 6$.

2. Plot the signal $x(t) = \tanh(t)$ together with its time-reversed and shifted versions $y_u(t) = x(u - t)$ with u an integer in the range $[-9, 10]$. Make sure that each couple (x, y_u) is plotted on a different area of a 4×5 grid, and that the time span of each plot is $[-10, 10]$. You will need to check how a for cycle works to solve the exercise.

3. Consider the signals

$$x(t) = \cos(2\pi t + \tfrac{\pi}{2}) \ , \quad y(t) = \sin(\omega_0 t + \tfrac{\pi}{3}) \ ,$$

and their sum $z(t) = x(t) + y(t)$. Plot the three signals on two separate subplots, one for $\omega_0 = \pi$ and one for $\omega_0 = 2$. Are the signals all periodic? Why? Use MAtLab functions cos() and sin() for defining the signals.

4. Consider the complex exponential

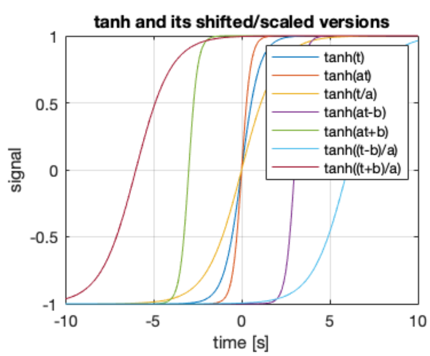
$$s(t) = 100\, e^{(-1+j2\pi)\, t}\, 1(t) \ ,$$

by representing, in four separate subplots its real and imaginary parts, its absolute value, and its phase. Use MatLab functions real(), imag(), abs(), and angle().

Solutions.

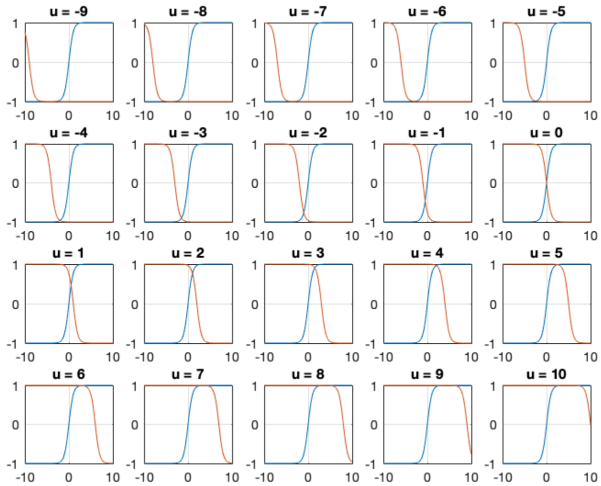
1. The code can mimic that of Exercise 9.3.1, as follows

```
t = -10:.1:10;
a = 2;
b = 6;
figure
plot(t,tanh(t),... % <-- this continues the code in
    the next line
     t,tanh(a*t),t,tanh(t/a),...
     t,tanh(a*t-b),t,tanh(a*t+b),...
     t,tanh((t-b)/a),t,tanh((t+b)/a))
grid on
xlabel('time [s]')
ylabel('signal')
legend('tanh(t)','tanh(at)','tanh(t/a)',...
    'tanh(at-b)','tanh(at+b)',...
    'tanh((t-b)/a)','tanh((t+b)/a)')
title('tanh and its shifted/scaled versions')
```



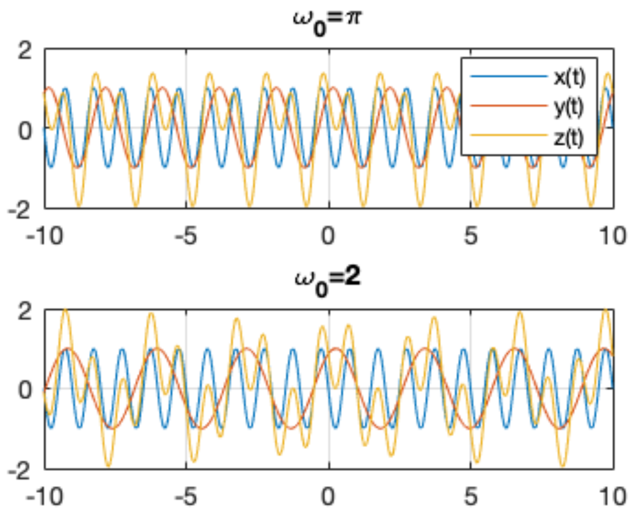
2. In this case, since there are many subplots active, we can skip insering xlabel and ylabel, and we can solve the dependence on integer u through a for cycle. Note how the subplot position is here set to $u + 10$, ranging from 1 to 20. Note also how we insert the value of u in the string title through the map num2str.

```
t = -10:.1:10;
figure
for u = -9:10
    subplot(4,5,u+10)
    plot(t,tanh(t),t,tanh(u-t))
    grid on
    title(['u = ' num2str(u)])
end
```



3. The key point of this exercise is to correctly choose the time span and the time samples, here set to $[-10, 10]$ and .01, respectively. We also note that all signals are periodic except for $z(t)$ when $\omega_0 = 2$. Sinusoids are periodic by construction, but their sum is only in case the pulsations are in rational relation, which is true for 2π and π, but not for 2π and 2. Observe also how we can write ω_0 and π in the title by exploiting the standard LaTeX format.

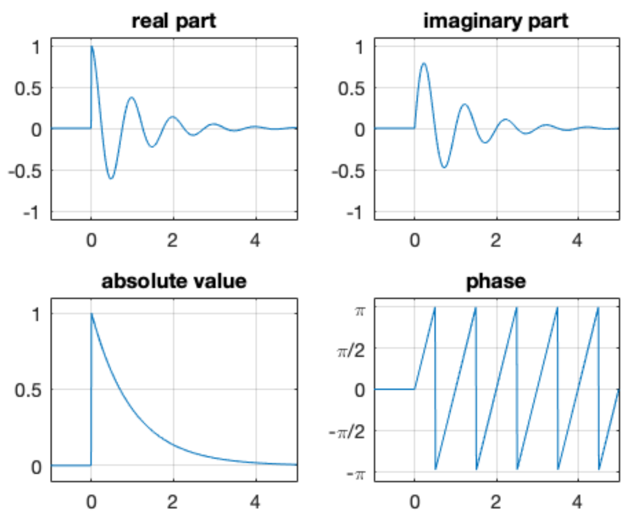
```
t = -10:.01:10;
x = cos(2*pi*t+pi/2);
y1 = sin(pi*t+pi/3);
y2 = sin(2*t+pi/3);
figure
subplot(2,1,1)
plot(t,x,t,y1,t,x+y1)
grid on
title('\omega_0=\pi')
legend('x(t)','y(t)','z(t)')
subplot(2,1,2)
plot(t,x,t,y2,t,x+y2)
grid on
title('\omega_0=2')
```



4. The key point of this exercise is to correctly choose the time span and the time samples, here set to $[-1, 5]$ and .01, respectively. We also note that in the code we control the active area of each plot through the function axis(), and we also force the grid in the plot of the phase to appear on the values set by yticks() with labels set by yticklabels(). Observe how function angle() reports the phase in the symmetric interval $[-\pi, \pi]$.

```
t = -1:.01:5;
s = (t>=0).*exp((-1+1i*2*pi)*t);
figure
subplot(2,2,1)
plot(t,real(s))
grid on
axis([-1 5 -1.1 1.1])
title('real part')
subplot(2,2,2)
plot(t,imag(s))
grid on
axis([-1 5 -1.1 1.1])
title('imaginary part')
subplot(2,2,3)
plot(t,abs(s))
grid on
axis([-1 5 -.1 1.1])
title('absolute value')
subplot(2,2,4)
plot(t,angle(s))
grid on
```

```
yticks([-pi,-pi/2,0,pi/2,pi])
yticklabels({'-\pi','-\pi/2','0','\pi/2','\pi'})
axis([-1 5 -3.5 3.5])
title('phase')
```



FOUNDATIONS OF SIGNALS AND SYSTEMS
10.2 Solved exercises
Prof. T. Erseghe

Exercises 10.2

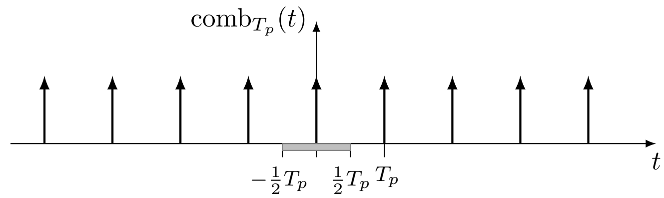
Prove that the following Fourier series pairs are correct by either forward or backward relation:

1. $s(t) = \mathrm{comb}_{T_p}(t) = \mathrm{rep}_{T_p}\,\delta(t)$ and $S_k = \frac{1}{T_p}$,
2. $s(t) = 1$ and $S_k = \delta(k)$,
3. $s(t) = \mathrm{rep}_{T_p}\,\mathrm{rect}(t/dT_p)$ and $S_k = d\,\mathrm{sinc}(kd)$, for $0 < d < 1$,
4. $s(t) = M\,\mathrm{sinc}_M(Mt/T_p) = \frac{\sin(\pi Mt/T_p)}{\sin(\pi t/T_p)}$, $M = 1{+}2N$, and $S_k = \mathrm{rect}(k/M)$,
5. $s(t) = \cos(n\omega_0 t + \varphi_0)$ and $S_k = \frac{1}{2}e^{j\varphi_0}\delta(k-n) + \frac{1}{2}e^{-j\varphi_0}\delta(k+n)$.

These are all fundamental Fourier pairs that must be kept in mind!!!

Solutions.

1. We evaluate the Fourier coefficients by applying the forward relation to the comb, to have
$$\begin{aligned} S_k &= \frac{1}{T_p}\int_{t_0}^{t_0+T_p} \mathrm{rep}_{T_p}\delta(t)\,e^{-jk\omega_0 t}\,dt \\ &= \frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} \mathrm{rep}_{T_p}\delta(t)\,e^{-jk\omega_0 t}\,dt \\ &= \frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} \delta(t)\,e^{-jk\omega_0 t}\,dt = \frac{1}{T_p}e^{-jk\omega_0\cdot 0} = \frac{1}{T_p}\ , \end{aligned}$$
where we used $t_0 = -\frac{1}{2}T_p$ so that the integration range $[-\frac{1}{2}T_p, \frac{1}{2}T_p]$ only includes one delta of the comb, as illustrated in the figure below.

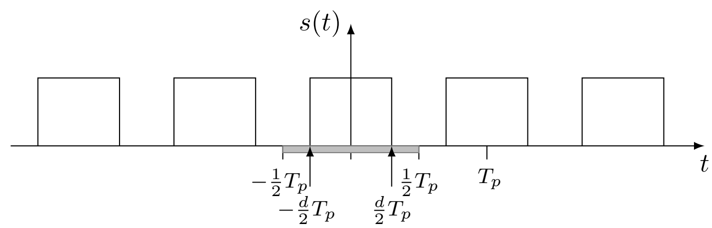


2. In this case we evaluate the periodic signal by applying the backward relation (Fourier series) to the Kronecker delta, to have
$$s(t) = \sum_{k=-\infty}^{\infty} \delta(k)\,e^{jk\omega_0 t} = e^{j0\cdot\omega_0 t} = 1\ .$$

3. We evaluate the Fourier coefficients by applying the forward relation to the square wave, to have

$$
\begin{aligned}
S_k &= \frac{1}{T_p}\int_{t_0}^{t_0+T_p} \mathrm{rep}_{T_p}\,\mathrm{rect}(\tfrac{t}{dT_p})\, e^{-jk\omega_0 t}\; dt \\
&= \frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} \mathrm{rep}_{T_p}\,\mathrm{rect}(\tfrac{t}{dT_p})\, e^{-jk\omega_0 t}\; dt \\
&= \frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} \mathrm{rect}(\tfrac{t}{dT_p})\, e^{-jk\omega_0 t}\; dt \\
&= \frac{1}{T_p}\int_{-\frac{d}{2}T_p}^{\frac{d}{2}T_p} e^{-jk\omega_0 t}\; dt
\end{aligned}
$$

where we used $t_0 = -\frac{1}{2}T_p$ so that the integration range $[-\frac{1}{2}T_p, \frac{1}{2}T_p]$ only includes the rectangle centred at 0, as illustrated in the figure below.



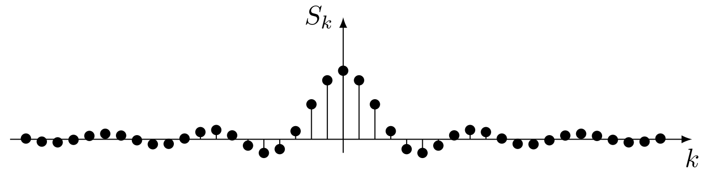
By solving the integral we have

$$
\begin{aligned}
S_k &= \begin{cases} \dfrac{dT_p}{T_p} & , k = 0 \\ \dfrac{1}{T_p}\dfrac{e^{-jk\omega_0 t}}{-jk\omega_0}\Big|_{-\frac{d}{2}T_p}^{\frac{d}{2}T_p} & , k \neq 0 \end{cases} \\
&= \begin{cases} d & , k = 0 \\ \dfrac{e^{jk\omega_0 \frac{d}{2}T_p} - e^{-jk\omega_0 \frac{d}{2}T_p}}{jk\omega_0 T_p} = \dfrac{e^{jkd\pi} - e^{-jkd\pi}}{2j\,k\pi} = \dfrac{\sin(kd\pi)}{k\pi} & , k \neq 0 \end{cases}
\end{aligned}
$$

where we exploited $\omega_0 T_p = 2\pi$. Overall, the result can be compactly expressed as

$$S_k = d\ \mathrm{sinc}(kd)\ ,$$

which encompasses both the cases $k = 0$ and $k \neq 0$, since by definition it is $\mathrm{sinc}(0) = 1$.



4. In this case we evaluate the periodic signal by applying the backward relation (Fourier series) to

$$S_k = \mathrm{rect}(\tfrac{k}{1+2N}) = \begin{cases} 1 & , k \in [-N, N] \\ 0 & , \text{otherwise} \end{cases}$$

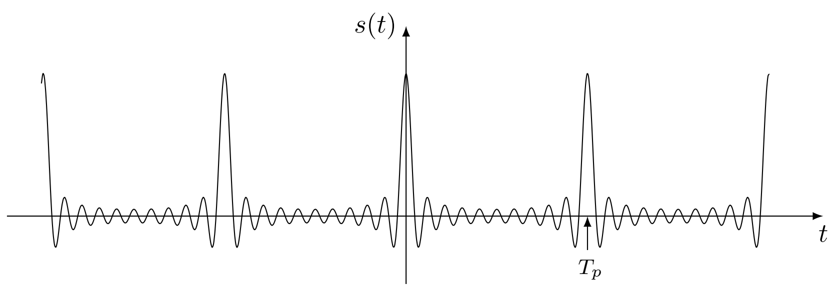
so that

$$\begin{aligned} s(t) &= \sum_{k=-N}^{N} e^{jk\omega_0 t} \\ &= \sum_{m=0}^{2N} e^{j(m-N)\omega_0 t} \\ &= e^{-jN\omega_0 t} \sum_{m=0}^{2N} (e^{j\omega_0 t})^m \\ &= e^{-jN\omega_0 t} \frac{1 - e^{j(1+2N)\omega_0 t}}{1 - e^{j\omega_0 t}} \\ &= \frac{e^{-jN\omega_0 t} - e^{j(1+N)\omega_0 t}}{1 - e^{j\omega_0 t}} \cdot \frac{e^{-j\frac{1}{2}\omega_0 t}}{e^{-j\frac{1}{2}\omega_0 t}} \\ &= \frac{e^{-j(\frac{1}{2}+N)\omega_0 t} - e^{j(\frac{1}{2}+N)\omega_0 t}}{e^{-j\frac{1}{2}\omega_0 t} - e^{j\frac{1}{2}\omega_0 t}} \\ &= \frac{\sin((\frac{1}{2}+N)\omega_0 t)}{\sin(\frac{1}{2}\omega_0 t)} = \frac{\sin(\frac{1}{2}M\omega_0 t)}{\sin(\frac{1}{2}\omega_0 t)} \end{aligned}$$

which is a periodic sinc function. By exploiting $\omega_0 = 2\pi/T_p$, we finally have

$$s(t) = \frac{\sin(\pi M t/T_p)}{\sin(\pi t/T_p)} = M \mathrm{sinc}_M(\tfrac{Mt}{T_p}) \ .$$

The signal is illustrated in the figure below for $N = 10$.



5. We evaluate the periodic signal by applying the backward relation (Fourier series) to have

$$\begin{aligned} s(t) &= \sum_{k=-\infty}^{\infty} \left[\frac{1}{2} e^{j\varphi_0} \delta(k-n) + \frac{1}{2} e^{-j\varphi_0} \delta(k+n)\right] e^{jk\omega_0 t} \\ &= \frac{1}{2} e^{j\varphi_0}\, e^{j(n\omega_0 t} + \frac{1}{2} e^{-j\varphi_0}\, e^{-jn\omega_0 t} \\ &= \frac{1}{2}\, e^{jn\omega_0 t+\varphi_0)} + \frac{1}{2}\, e^{-j(n\omega_0 t+\varphi_0)} \\ &= \cos(n\omega_0 t + \varphi_0) \end{aligned}$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
10.3 Homework assignment
Prof. T. Erseghe

Exercises 10.3

Solve the following:

1. evaluate the Fourier coefficients of $x(t) = e^{jm\omega_0 t}$,
2. evaluate the Fourier coefficients of $x(t) = \mathrm{rep}_{T_p}\,\delta(t - t_1)$,
3. evaluate the Fourier coefficients of $x(t) = 3 - \sin(2t) + 4\cos(2t) + 2\cos(6t - \frac{\pi}{4})$,
4. evaluate the Fourier coefficients of $x(t) = |\cos(2\pi f_0 t)|$,
5. evaluate the Fourier coefficients of $x(t) = \mathrm{rep}_{T_p}\,\mathrm{triang}(2t/T_p)$,
6. evaluate the Fourier coefficients of $x(t) = \sin(\pi t) + \cos^2(\frac{2}{3}\pi t)$,
7. evaluate the Fourier coefficients of $x(t) = [\cos(2\pi f_0 t)]^+$, where $[x]^+ = x \cdot 1(x)$ is the positive part operator.

Solutions.

1. We evaluate the Fourier coefficients by applying the forward relation, to have

$$\begin{aligned} S_k &= \frac{1}{T_p}\int_0^{T_p} e^{jm\omega_0 t}\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{T_p}\int_0^{T_p} e^{j(m-k)\omega_0 t}\, dt \\ &= \begin{cases} \frac{1}{T_p}\, T_p & , k = m \\ 0 & , k \neq m \end{cases} \end{aligned}$$

where we exploited the fact that, for $k \neq m$ the complex exponential $e^{j(m-k)\omega_0 t}$ has period $T_p/|m-k|$, and its integration over $[0, T_p]$ (i.e., over $|m-k| \geq 1$ periods) is zero. We can equivalently write

$$S_k = \frac{1}{T_p}\, \delta(k-m)\ .$$

2. We evaluate the Fourier coefficients by applying the forward relation, to have

$$\begin{aligned} S_k &= \frac{1}{T_p}\int_{t_0}^{t_0+T_p} \mathrm{rep}_{T_p}\delta(t-t_1)\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{T_p}\int_{t_1-\frac{1}{2}T_p}^{t_1+\frac{1}{2}T_p} \mathrm{rep}_{T_p}\delta(t-t_1)\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{T_p}\int_{t_1-\frac{1}{2}T_p}^{t_1+\frac{1}{2}T_p} \delta(t-t_1)\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{T_p}\, e^{-jk\omega_0 t_1}\ , \end{aligned}$$

where we choose the period $[t_1 - \frac{1}{2}T_p, t_1 + \frac{1}{2}T_p]$ in such aa way that the only active delta in the integration of $\mathrm{rep}_{T_p}\delta(t-t_1)$ is $\delta(t-t_1)$.

3. In this case, the signal is already written in the form of a Fourier series, as one can appreciate by expanding the sinusoids through Euler's identity, to have

$$\begin{aligned} x(t) &= 3 - \sin(2t) + 4\cos(2t) + 2\cos(6t - \frac{\pi}{4}) \\ &= 3 + \tfrac{j}{2}e^{j2t} - \tfrac{j}{2}e^{-j2t} + 2e^{j2t} + 2e^{-j2t} + e^{j(6t-\frac{\pi}{4})} + e^{-j(6t-\frac{\pi}{4})} \\ &= 3 + (2+\tfrac{j}{2})\, e^{j2t} + (2-\tfrac{j}{2})\, e^{-j2t} + \tfrac{1-j}{\sqrt{2}}\, e^{j6t} + \tfrac{1+j}{\sqrt{2}}\, e^{-j6t} \end{aligned}$$

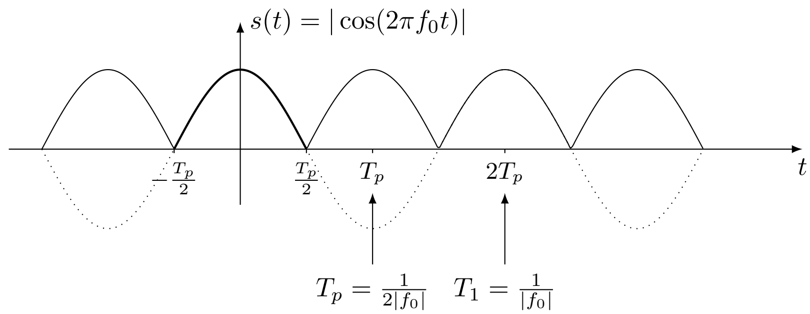
where the active pulsations are 0, ± 2 and ± 6, hence we can set $\omega_0 = 2$ (that is $T_p = \pi$) and write the signal as

$$x(t) = 3 + (2+\tfrac{j}{2})\, e^{j\omega_0 t} + (2-\tfrac{j}{2})\, e^{-j\omega_0 t} + \tfrac{1-j}{\sqrt{2}}\, e^{j3\omega_0 t} + \tfrac{1+j}{\sqrt{2}}\, e^{-j3\omega_0 t}$$

so that

$$X_k = \begin{cases} 3 & , k = 0 \\ 2 + \frac{j}{2} & , k = 1 \\ 2 - \frac{j}{2} & , k = -1 \\ \frac{1-j}{\sqrt{2}} & , k = 3 \\ \frac{1+j}{\sqrt{2}} & , k = -3 \\ 0 & , \text{otherwise} \end{cases}$$

4. The signal $x(t) = |\cos(2\pi f_0 t)|$ has period $T_p = 1/(2f_0)$, so that $\omega_0 = 2\pi/T_p = 4\pi f_0$. We evaluate the Fourier coefficients by applying the forward relation, and integration over the period $[-\frac{1}{2}T_p, \frac{1}{2}T_p]$ where $|\cos(2\pi f_0 t)| = \cos(2\pi f_0 t)$, as illustrated in the figure below.



We obtain

$$\begin{aligned} S_k &= \frac{1}{T_p} \int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} |\cos(2\pi f_0 t)|\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{T_p} \int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} \cos(2\pi f_0 t)\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{2T_p} \int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} [e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}]\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{2T_p} \int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} [e^{j(2\pi f_0 - k\omega_0)t} + e^{-j(2\pi f_0 + k\omega_0)t}]\, dt \\ &= \frac{e^{j(2\pi f_0 - k\omega_0)t}}{j2T_p(2\pi f_0 - k\omega_0)} - \frac{e^{-j(2\pi f_0 + k\omega_0)t}}{j2T_p(2\pi f_0 + k\omega_0)} \Bigg|_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} \end{aligned}$$

We now solve the expression as a function of ω_0 by recalling that $2\pi f_0 =$

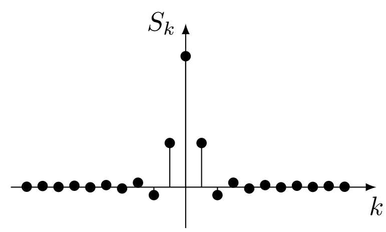
$\frac{1}{2}\omega_0$ and $T_p\omega_0 = 2\pi$, to have

$$\begin{aligned}
S_k &= \left.\frac{e^{j(\frac{1}{2}-k)u}}{j4\pi(\frac{1}{2}-k)} - \frac{e^{-j(\frac{1}{2}+k)u}}{j4\pi(\frac{1}{2}+k)}\right|_{-\pi}^{\pi} \\
&= \frac{e^{j(\frac{1}{2}-k)\pi} - e^{-j(\frac{1}{2}-k)\pi}}{j4\pi(\frac{1}{2}-k)} - \frac{e^{-j(\frac{1}{2}+k)\pi} - e^{j(\frac{1}{2}+k)\pi}}{j4\pi(\frac{1}{2}+k)} \\
&= \frac{\sin(\pi(\frac{1}{2}-k))}{2\pi(\frac{1}{2}-k)} + \frac{\sin(\pi(\frac{1}{2}+k))}{2\pi(\frac{1}{2}+k)} \\
&= \tfrac{1}{2}\ \mathrm{sinc}(\tfrac{1}{2}-k) + \tfrac{1}{2}\ \mathrm{sinc}(\tfrac{1}{2}+k) \\
&= \tfrac{1}{2}\ \mathrm{sinc}(k-\tfrac{1}{2}) + \tfrac{1}{2}\ \mathrm{sinc}(k+\tfrac{1}{2})
\end{aligned}$$

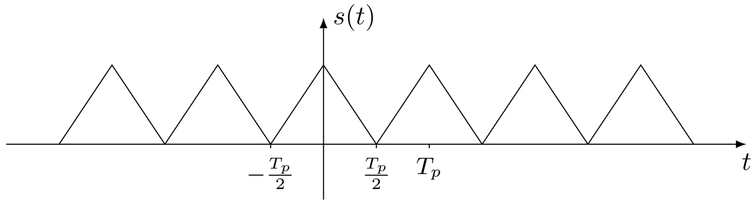
We can further express the signal in compact form by expanding the sinc functions, to obtain

$$\begin{aligned}
S_k &= \frac{\sin(\pi k - \frac{\pi}{2})}{(2k-1)\pi} + \frac{\sin(\pi k + \frac{\pi}{2})}{(2k+1)\pi} \\
&= -\frac{\cos(\pi k)}{(2k-1)\pi} + \frac{\cos(\pi k)}{(2k+1)\pi} \\
&= \left[-\frac{1}{(2k-1)\pi} + \frac{1}{(2k+1)\pi}\right](-1)^k \\
&= \frac{2\,(-1)^k}{(1-4k^2)\pi}
\end{aligned}$$

The resulting signal is depicted in the picture below.



5. The signal is the periodic repetition of a triangle (with no aliasing), as illustrated in the figure below.



Therefore, for $k \neq 0$ we have

$$
\begin{aligned}
S_k &= \frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} \text{triang}(2t/T_p)\, e^{-jk\omega_0 t}\, dt \\
&= \frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{0} (1+\tfrac{2t}{T_p})\, e^{-jk\omega_0 t}\, dt + \frac{1}{T_p}\int_{0}^{\frac{1}{2}T_p} (1-\tfrac{2t}{T_p})\, e^{-jk\omega_0 t}\, dt \\
&= \frac{2}{T_p^2}\int_{-\frac{1}{2}T_p}^{0} t\, e^{-jk\omega_0 t}\, dt - \frac{2}{T_p^2}\int_{0}^{\frac{1}{2}T_p} t\, e^{-jk\omega_0 t}\, dt + \frac{1}{T_p}\int_{-\frac{1}{2}T_p}^{\frac{1}{2}T_p} e^{-jk\omega_0 t}\, dt \\
&= \left.\frac{2\,(1+jk\omega_0 t)\, e^{-jk\omega_0 t}}{(k\omega_0 T_p)^2}\right|_{-\frac{1}{2}T_p}^{0} - \left.\frac{2\,(1+jk\omega_0 t)\, e^{-jk\omega_0 t}}{(k\omega_0 T_p)^2}\right|_{0}^{\frac{1}{2}T_p} + 0 \\
&= \frac{4 - (2+jk\omega_0 T_p)\, e^{-j\frac{k}{2}\omega_0 T_p} - (2-jk\omega_0 T_p)\, e^{j\frac{k}{2}\omega_0 T_p}}{(k\omega_0 T_p)^2}
\end{aligned}
$$

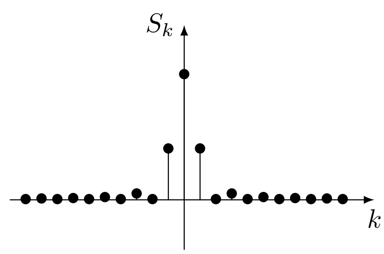
but, since $\omega_0 T_p = 2\pi$, the result for $k \neq 0$ can be simplified in the form

$$
\begin{aligned}
S_k &= \frac{4 - (2+j2\pi k)\, e^{-jk\pi} - (2-j2\pi k)\, e^{jk\pi}}{(2\pi k)^2} \\
&= \frac{1-(-1)^k}{(\pi k)^2}
\end{aligned}
$$

since $e^{\pm jk\pi} = (-1)^k$. For $k = 0$ it simply is $S_0 = m_x = \frac{1}{2}$, so that

$$
S_k = \begin{cases} \frac{1}{2} & , k = 0 \\ \dfrac{1-(-1)^k}{(\pi k)^2} & , k \neq 0 \end{cases}
$$

The resulting signal is depicted in the picture below.



6. This is another example where the signal is can be easly written in the form of a Fourier series, that is

$$
\begin{aligned}
x(t) &= \sin(\pi t) + \cos^2(\tfrac{2}{3}\pi t) \\
&= \sin(\pi t) + \tfrac{1}{2} + \tfrac{1}{2}\cos(\tfrac{4}{3}\pi t) \\
&= \tfrac{1}{2j}e^{j\pi t} - \tfrac{1}{2j}e^{-j\pi t} + \tfrac{1}{2} + \tfrac{1}{2}e^{j\frac{4}{3}\pi t} + \tfrac{1}{2}e^{-j\frac{4}{3}\pi t}
\end{aligned}
$$

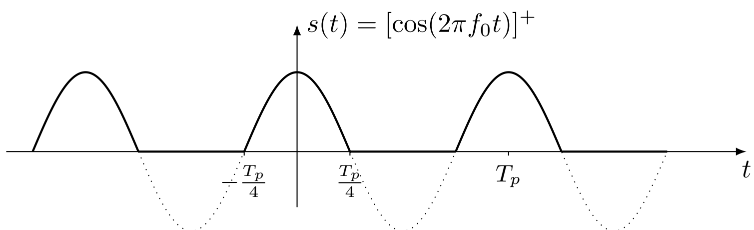
where the active pulsations are 0, $\pm\pi$ and $\pm\frac{4}{3}\pi$, so that $\omega_0 = \frac{\pi}{3}$, that is $T_p = 6$. Therefore, we have

$$x(t) = \tfrac{1}{2j}e^{j3\omega_0 t} - \tfrac{1}{2j}e^{-j3\omega_0 t} + \tfrac{1}{2} + \tfrac{1}{2}e^{j4\omega_0 t} + \tfrac{1}{2}e^{-j4\omega_0 t}$$

and therefore

$$X_k = \begin{cases} \frac{1}{2} & , k = 0, \pm 4 \\ -\frac{j}{2} & , k = 3 \\ \frac{j}{2} & , k - 3 \\ 0 & , \text{otherwise} \end{cases}$$

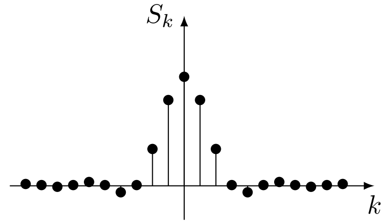
7. The signal $x(t) = [\cos(2\pi f_0 t)]^+$ has period $T_p = 1/f_0$, so that $\omega_0 = 2\pi/T_p = 2\pi f_0$. We evaluate the Fourier coefficients by applying the forward relation, and integration over the period $[-\frac{1}{2}T_p, \frac{1}{2}T_p]$ where the signal is active only in $[-\frac{1}{4}T_p, \frac{1}{4}T_p]$ its value being $\cos(2\pi f_0 t)$, as illustrated in the figure below.



Therefore, by recalling that $2\pi f_0 = \omega_0$ and $\omega_0 T_p = 2\pi$, we have

$$\begin{aligned} S_k &= \frac{1}{T_p}\int_{-\frac{1}{4}T_p}^{\frac{1}{4}T_p} \cos(\omega_0 t)\, e^{-jk\omega_0 t}\, dt \\ &= \frac{1}{2T_p}\int_{-\frac{1}{4}T_p}^{\frac{1}{4}T_p} [e^{-j(k-1)\omega_0 t} + e^{-j(k+1)\omega_0 t}]\, dt \\ &= \frac{e^{-j(k-1)\omega_0 t}}{-j(k-1)2\omega_0 T_p} + \frac{e^{-j(k+1)\omega_0 t}}{-j(k+1)2\omega_0 T_p}\bigg|_{-\frac{1}{4}T_p}^{\frac{1}{4}T_p} \\ &= \frac{e^{-j(k-1)u}}{-j4(k-1)\pi} + \frac{e^{-j(k+1)u}}{-j4(k+1)\pi}\bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{\sin((k-1)\frac{\pi}{2})}{2(k-1)\pi} + \frac{\sin((k+1)\frac{\pi}{2})}{2(k+1)\pi} \\ &= \tfrac{1}{4}\,\mathrm{sinc}(\tfrac{1}{2}(k-1)) + \tfrac{1}{4}\,\mathrm{sinc}(\tfrac{1}{2}(k+1)) \end{aligned}$$

which is also valid at $k = \pm 1$. The resulting signal is depicted in the picture below.

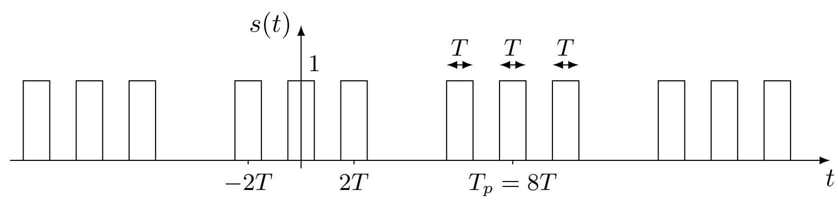

S_k
k

FOUNDATIONS OF SIGNALS AND SYSTEMS
10.5 Solved exercises
Prof. T. Erseghe

Exercises 10.5

Solve the following by exploiting the properties of the Fourier series:

1. prove that any real-valued continuous-time period signal of period T_p can be expressed through the (real-valued) trigonometric series
$$s(t) = S_0 + \sum_{k=1}^{\infty} 2|S_k| \cos(k\omega_0 t + \varphi_k) \ , \quad S_k = |S_k|\, e^{j\varphi_k} \ ,$$
where S_k are its Fourier coefficients and $\omega_0 = 2\pi/T_p$;
2. evaluate the Fourier coefficients of $s(t) = e^{jm\omega_0 t}$ by exploiting the Fourier couple $x(t) = 1$, $X_k = \delta(k)$;
3. evaluate the Fourier coefficients of $s(t) = \mathrm{rep}_{T_p} \mathrm{triang}(2t/T_p)$;
4. evaluate the Fourier coefficients of

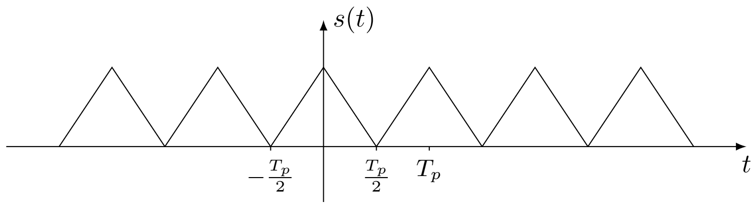


Solutions.

1. Being the signal real-valued, we exploit the Hermitian symmetry $S_k = S^*_{-k}$ of the Fourier coefficients, that is $S_{-k} = S^*_k = |S_k| e^{-j\varphi_k}$ which also reveals that S_0 (the mean value) is real-valued. Hence, we can write the Fourier series in the form
$$\begin{aligned}
s(t) &= \sum_{k=-\infty}^{-1} S_k\, e^{jk\omega_0 t} + S_0 + \sum_{k=1}^{\infty} S_k\, e^{jk\omega_0 t} \\
&= S_0 + \sum_{k=1}^{\infty} [S_k\, e^{jk\omega_0 t} + S_{-k}\, e^{-jk\omega_0 t}] \\
&= S_0 + \sum_{k=1}^{\infty} |S_k| [e^{j\varphi_k}\, e^{jk\omega_0 t} + e^{-j\varphi_k}\, e^{-jk\omega_0 t}] \\
&= S_0 + \sum_{k=1}^{\infty} |S_k| [e^{j(k\omega_0 t + \varphi_k)} + e^{-j(k\omega_0 t + \varphi_k)}] \\
&= S_0 + \sum_{k=1}^{\infty} 2|S_k| \cos(k\omega_0 t + \varphi_k) \ .
\end{aligned}$$

2. Since $s(t) = e^{jm\omega_0 t} = e^{jm\omega_0 t} \cdot 1 = e^{jm\omega_0 t} \cdot x(t)$, we can exploit the modulation property to state that $S_k = X_{k-m} = \delta(k - m)$.

3. The signal is the periodic repetition of a triangle (with no aliasing), as illustrated in the figure below.



Since a triangle is the convolution between two identical rectangles, we can recognise $s(t)$ to be the self-circular-convolution $s(t) = u *_{\text{cir}} u(t)$ between two square waves $u(t)$ with duty cycle $d = \frac{1}{2}$ (so that the duty cycle of the triangle becomes 1). Specifically, we can set

$$u(t) = \operatorname*{rep}_{T_p} \frac{1}{\sqrt{dT_p}} \operatorname{rect}\left(\frac{t}{dT_p}\right) , \quad d = \tfrac{1}{2} ,$$

a choice ensuring that the triangle wave has height 1. The Fourier coefficients of $u(t)$ are, straightforwardly,

$$U_k = \frac{1}{\sqrt{dT_p}} d \operatorname{sinc}(kd) ,$$

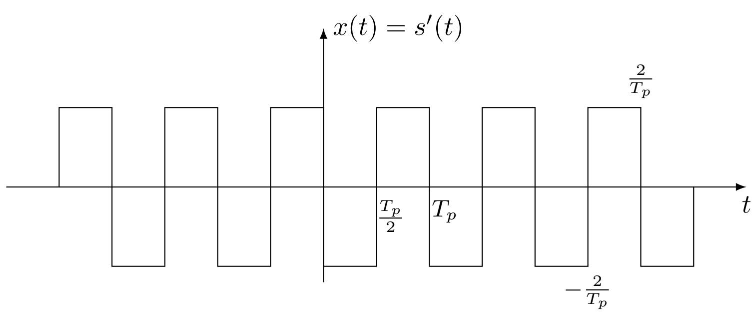
and from the circular convolution property we also have

$$S_k = T_p U_k^2 = d \operatorname{sinc}^2(kd) = \tfrac{1}{2}\operatorname{sinc}^2(\tfrac{k}{2}) .$$

This result corresponds to the outcome of Exercise 10.3.5, since

$$S_k = \begin{cases} \frac{1}{2} & , k = 0 \\ 0 & , k \neq 0 \text{ and even} \\ \frac{2\sin^2(k\frac{\pi}{2})}{(\pi k)^2} = \frac{2}{(\pi k)^2} & , k \text{ odd} \end{cases}$$

As an alternative way, we could have exploited the derivative $x(t) = s'(t)$ illustrated in the figure below



evidencing that

$$x(t) = s'(t) = \text{rep}_{T_p} \tfrac{2}{T_p} \, \text{rect}\left(\tfrac{t+\frac{1}{4}T_p}{\frac{1}{2}T_p}\right) - \tfrac{2}{T_p} \, \text{rect}\left(\tfrac{t-\frac{1}{4}T_p}{\frac{1}{2}T_p}\right)$$

whose Fourier coefficients are (by exploiting the result for a square wave plus the time-shift property)

$$\begin{aligned} X_k &= \frac{\text{sinc}(\frac{1}{2}k)}{T_p} \left[e^{jk\omega_0 \frac{1}{4}T_p} - e^{-jk\omega_0 \frac{1}{4}T_p}\right] \\ &= \frac{\text{sinc}(\frac{1}{2}k)}{T_p} \left[e^{jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}}\right] \\ &= \frac{2j \sin(k\frac{\pi}{2}) \, \text{sinc}(\frac{1}{2}k)}{T_p} \ , \end{aligned}$$

where we also exploited the fact that $\omega_0 T_p = 2\pi$. By then inverting the derivative rule, we have

$$S_k = \begin{cases} m_x = \frac{1}{2} & , \, k = 0 \\ \dfrac{X_k}{jk\omega_0} = \dfrac{\sin(k\frac{\pi}{2}) \, \text{sinc}(\frac{1}{2}k)}{k\pi} = \frac{1}{2} \, \text{sinc}^2(\frac{1}{2}k) & , \, k \neq 0 \end{cases}$$

which, again, provides the same result. Possibly, in this specific case, the circular convolution approach is the preferred one.

4. The signal $s(t)$ can be recognised to be the composition of three time-shifted square waves of duty cycle $\frac{1}{8}$. Specifically, by denoting the reference square waves of duty cycle $\frac{1}{8}$ as

$$u(t) = \text{rep}_{T_p} \, \text{rect}\left(\tfrac{t}{dT_p}\right) \ , \qquad U_k = d \, \text{sinc}(kd) \ , \quad d = \tfrac{1}{8} \ ,$$

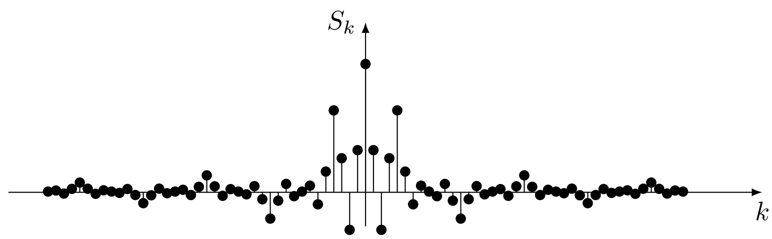
where $T_p = 8T$, we can write

$$s(t) = u(t) + u(t - 2T) + u(t + 2T) \ ,$$

so that by use of the time-shift property we obtain

$$\begin{aligned} S_k &= U_k + U_k \, e^{-jk\omega_0 2T} + U_k \, e^{jk\omega_0 2T} \\ &= U_k[1 + 2\cos(k\omega_0 2T)] \\ &= U_k[1 + 2\cos(k\tfrac{\pi}{2})] \\ &= \tfrac{1}{8} \, \text{sinc}(\tfrac{k}{8}) \, [1 + 2\cos(k\tfrac{\pi}{2})] \end{aligned}$$

where we exploited $\omega_0 = 2\pi/T_p = \pi/(4T)$. The resulting signal is depicted in the picture below.



Note that $s(t)$ is real and even, and therefore also S_k is real and even.

As an alternative solution, we could have noted that $s(t)$ is *almost* a square wave of duty cycle $\frac{1}{2}$ and period $2T$. Specifically, by denoting the square wave of duty cycle $\frac{1}{2}$ as

$$v(t) = \mathrm{rep}_{2T}\,\mathrm{rect}\left(\tfrac{t}{T}\right) \;, \qquad V_k = \tfrac{1}{2}\;\mathrm{sinc}(\tfrac{k}{2}) \;,$$

we have

$$s(t) = v(t) - u(t - 4T) \;,$$

whose Fourier coefficients are

$$S_k = \tilde{V}_k - U_k\, e^{-jk\omega_0 4T} \;, \qquad \tilde{V}_k = \begin{cases} V_m & , k = 4m \\ 0 & , \text{otherwise} \end{cases}$$

where the Fourier coefficients $\tilde{V}_k$ are those relating to the period $T_p = 8T$, which is 4 times the minimum period of $v(t)$. By recalling $\omega_0 = 2\pi/T_p = \pi/(4T)$, we further have

$$\begin{aligned} S_k &= \tilde{V}_k - U_k\,(-1)^k \\ &= \tfrac{1}{8}\;\mathrm{sinc}(\tfrac{k}{8})\,(-1)^{k+1} + \begin{cases} \tfrac{1}{2}\;\mathrm{sinc}(\tfrac{k}{8}) & , k = 4m \\ 0 & , \text{otherwise} \end{cases} \\ &= \tfrac{1}{8}\;\mathrm{sinc}(\tfrac{k}{8}) \cdot \begin{cases} 3 & , k = 4m \\ (-1)^{k+1} & , \text{otherwise} \end{cases} \end{aligned}$$

which corresponds to the result previously found. Note, however, that the de-periodisation property might be tricky to be correctly used.

FOUNDATIONS OF SIGNALS AND SYSTEMS
10.6 Homework assignment
Prof. T. Erseghe

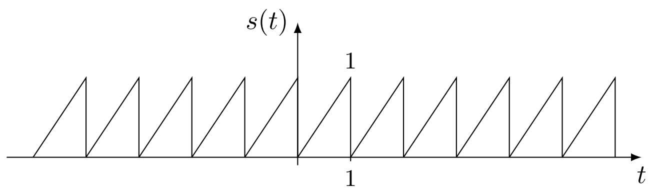
Exercises 10.6

Solve the following by exploiting the properties of the Fourier series:

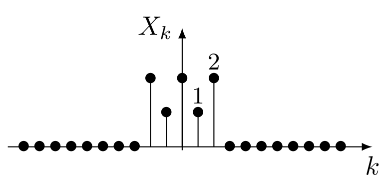
1. evaluate the Fourier coefficients of $s(t) = \mathrm{rep}_{T_p}\,\delta(t - t_1)$ by exploiting the Fourier couple $x(t) = \mathrm{rep}_{T_p}\,\delta(t)$, $X_k = \frac{1}{T_p}$;

2. evaluate the Fourier coefficients of $s(t) = [\sin(2\pi f_0 t)]^+$, where $[x]^+ = x \cdot 1(x)$ is the positive part operator, by exploiting the product property;

3. evaluate the Fourier coefficients of the saw-tooth waveform



4. evaluate the Fourier coefficients of $s(t) = x(t)\cos(10\omega_0 t)$ with



5. evaluate mean value and power of the signal $s(t) = \frac{3}{5}\sin(\pi t)/\sin(\frac{\pi}{5}t)$;

6. identify the analytic expression of a signal $s(t)$ which is real and odd, has period $T_p = 2$, has Fourier coefficients $S_k = 0$ for $|k| > 1$, and has power $P_s = 1$;

7. identify the analytic expression of a signal $s(t)$ which is real and even, has period $T_p = 2$, has Fourier coefficients $S_k = 0$ for $|k| > 2$, has mean value $m_s = -1$ and power $P_s = 11$, and such that the signal $x(t) = s'(t)$ has Fourier coefficients for $k = \pm 2$ of the form $X_2 = j2\pi$ and $X_{-2} = -j2\pi$;

8. which of the following signals are real, and which are even?

$$s_1(t) = \sum_{k=0}^{100} e^{j2\pi \frac{k}{50} t}$$

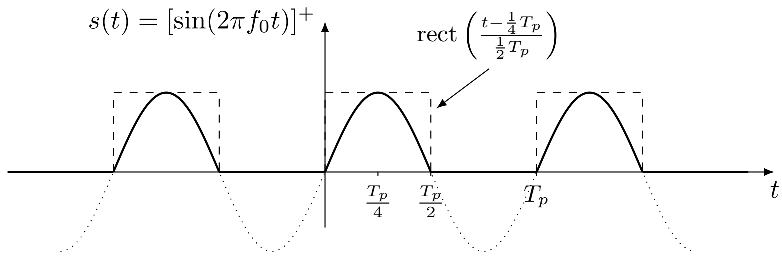
$$s_2(t) = \sum_{k=-100}^{100} \cos(k\pi)\, e^{j2\pi \frac{k}{50} t}$$

$$s_3(t) = \sum_{k=-100}^{100} j\sin(k\frac{\pi}{2})\, e^{j2\pi \frac{k}{50} t}$$

Solutions.

1. Since $s(t) = x(t-t_1)$ by the time-shift property we have $S_k = X_k\, e^{-jk\omega_0 t_1} = \frac{1}{T_p}\, e^{-jk\omega_0 t_1}$.

2. The signal $x(t) = [\sin(2\pi f_0 t)]^+$ has period $T_p = 1/f_0$, so that $\omega_0 = 2\pi/T_p = 2\pi f_0$, and is illustrated in the figure below.



As one can appreciate from the figure it can be written as the product $s(t) = x(t)y(t)$ between

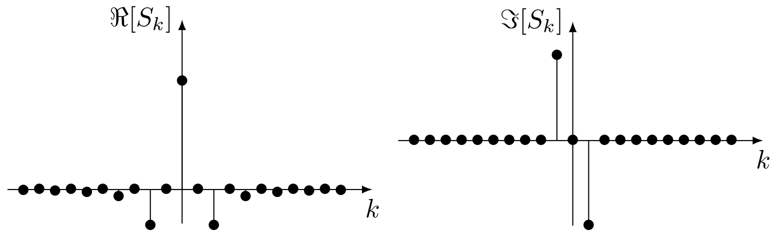
$$x(t) = \sin(\omega_0 t)$$
$$y(t) = \text{rep}_{T_p}\, \text{rect}\left(\frac{t - \frac{1}{4}T_p}{\frac{1}{2}T_p}\right)$$

$$\begin{aligned} X_k &= \tfrac{1}{2j}\delta(k-1) - \tfrac{1}{2j}\delta(k+1) \\ Y_k &= \tfrac{1}{2}\,\text{sinc}(\tfrac{k}{2})e^{-jk\omega_0 \frac{1}{4}T_p} \\ &= \tfrac{1}{2}\,\text{sinc}(\tfrac{k}{2})e^{-jk\frac{\pi}{2}} \\ &= \tfrac{1}{2}\,\text{sinc}(\tfrac{k}{2})\,(-j)^k \end{aligned}$$

where we exploited the known results of the Fourier coefficients for a sinusoid and the square wave, plus the time-shift rule. Hence, the Fourier coefficients of $s(t)$ are straightforwardly derived by exploiting the property that a product in time involves a convolution in the Fourier domain, to have

$$\begin{aligned} S_k &= X_k * Y_k \\ &= \sum_{m=-\infty}^{\infty} X_m Y_{k-m} \\ &= \sum_{m=-\infty}^{\infty} [\tfrac{1}{2j}\delta(m-1) - \tfrac{1}{2j}\delta(m+1)]\, \tfrac{1}{2}\,\text{sinc}(\tfrac{1}{2}(k-m))\,(-j)^{k-m} \\ &= \tfrac{1}{4j}\,\text{sinc}(\tfrac{1}{2}(k-1))\,(-j)^{k-1} - \tfrac{1}{4j}\,\text{sinc}(\tfrac{1}{2}(k+1))\,(-j)^{k+1} \\ &= \left[\tfrac{1}{4}\,\text{sinc}(\tfrac{1}{2}(k-1)) + \tfrac{1}{4}\,\text{sinc}(\tfrac{1}{2}(k+1))\right]\,(-j)^k \end{aligned}$$

where we use the sifting property of the Kronecker delta. The resulting signal is depicted, separately in its real and imaginary parts, in the picture below, where we note how (thanks to the Hermitian symmetry) the real part is even symmetric and the imaginary part odd symmetric.

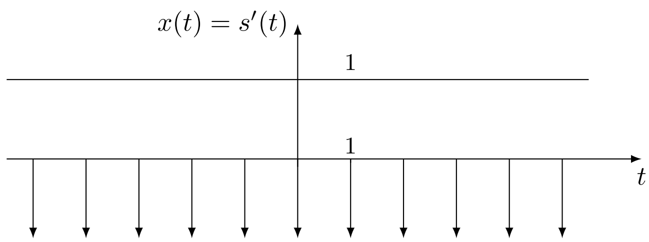


Incidentally note that $z(t) = [\cos(2\pi f_0 t)]^+ = s(t + \frac{1}{4}T_p)$, so that

$$Z_k = S_k\, e^{jk\omega_0 \frac{1}{4}T_p} = S_k\,(j)^k = \tfrac{1}{4}\,\mathrm{sinc}(\tfrac{1}{2}(k-1)) + \tfrac{1}{4}\,\mathrm{sinc}(\tfrac{1}{2}(k+1))$$

which corresponds to the outcome of Exercise 10.3.7.

3. For the saw-tooth waveform, which is a linear-piecewise function, the easiest approach is that of the derivative $x(t) = s'(t)$, illustrated in the figure below



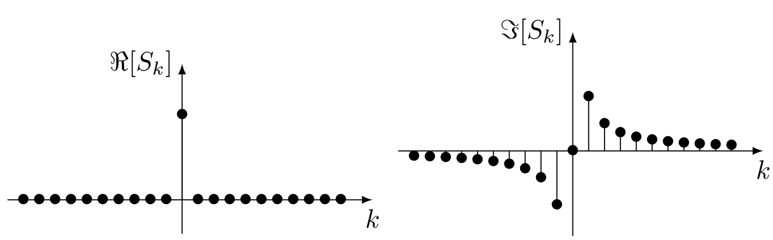
for which we have

$$x(t) = s'(t) = 1 - \underset{T_p}{\mathrm{rep}}\,\delta(t)\ , \qquad X_k = \delta(k) - \tfrac{1}{T_p} = \delta(k) - 1\ ,$$

since the period is $T_p = 1$ (hence $\omega_0 = 2\pi$). By inverting the derivative we find

$$S_k = \begin{cases} m_x = \frac{1}{2} & ,\ k = 0 \\ \dfrac{X_k}{jk\omega_0} = \dfrac{-1}{j2\pi k} = \dfrac{j}{2\pi k} & ,\ k \neq 0 \end{cases}$$

The resulting signal is depicted, separately in its real and imaginary parts, in the picture below, where we note how (thanks to the Hermitian symmetry) the real part is even symmetric and the imaginary part odd symmetric.



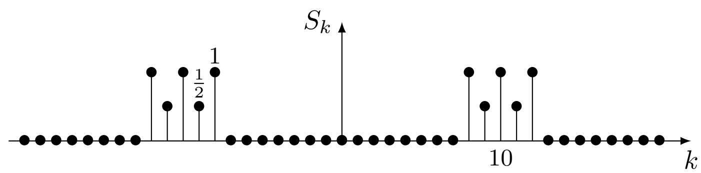
4. In this case we can exploit the multiplication property $s(t) = x(t)y(t)$ giving $S_k = X_k * Y_k$, where

$$y(t) = \cos(10\omega_0 t) , \qquad Y_k = \tfrac{1}{2}\delta(k-10) + \tfrac{1}{2}\delta(k+10) ,$$

so that

$$\begin{aligned} S_k &= X_k * Y_k \\ &= \sum_{m=-\infty}^{\infty} Y_m X_{k-m} \\ &= \sum_{m=-\infty}^{\infty} \left[\tfrac{1}{2}\delta(m-10) + \tfrac{1}{2}\delta(m+10)\right] X_{k-m} \\ &= \tfrac{1}{2} X_{k-10} + \tfrac{1}{2} X_{k+10} \end{aligned}$$

providing the result displayed in the figure below.



5. Evaluating mean value and power by integration is impossible since a primitive can be hardly identified. Therefore, we proceed by inspection of the Fourier coefficients. We observe that the signal can be written in the form

$$s(t) = 3\,\mathrm{sinc}_5(t) ,$$

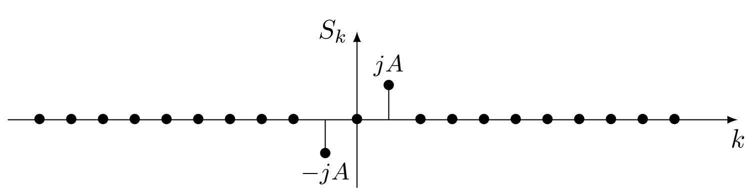
which is a signal of period $T_p = 5$, and from standard Fourier couples (see also Exercise 10.2.3) it is

$$S_k = \tfrac{3}{5}\ \mathrm{rect}\left(\tfrac{k}{5}\right) = \begin{cases} \tfrac{3}{5} & , k \in [-2, 2] \\ 0 & , \text{otherwise} \end{cases}$$

Therefore, we have

$$m_s = S_0 = \tfrac{3}{5} , \quad P_s = E_S = \sum_k |S_k|^2 = 5 \cdot |\tfrac{3}{5}|^2 = \tfrac{9}{5} .$$

6. Being the signal real and odd, its Fourier coefficients are Hermitian and odd, that is imaginary and odd. Since it is $S_k = 0$ for $|k| > 1$, only two coefficients are active (it is $S_0 = 0$ because of the odd symmetry), as illustrated in figure where the odd and imaginary symmetry is evidenced.



Hence, we have

$$s(t) = jA\,e^{j\omega_0 t} - jA\,e^{-j\omega_0 t} = -2A\sin(\omega_0 t) = -2A\sin(\pi t)\ .$$

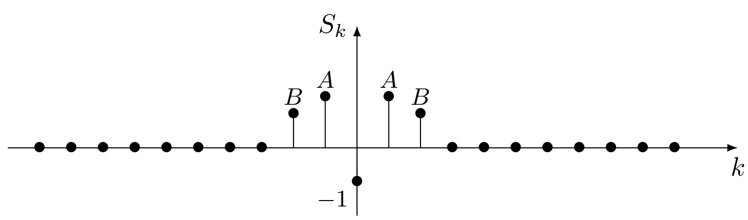
where we exploited the fact that the period is $T_p = 2$ to set $\omega_0 = 2\pi/T_p = \pi$. The value of A can be finally identified through the power, top have

$$P_s = \tfrac{1}{2}(2A)^2 = 2A^2 = 1$$

providing $A^2 = \frac{1}{2}$ and $A = \pm 1/\sqrt{2}$, so that

$$s(t) = \pm\sqrt{2}\,\sin(\pi t)\ .$$

7. Being the signal real and even, its Fourier coefficients are Hermitian and even, that is real and even. Since it is $S_k = 0$ for $|k| > 2$, and we also know that $S_0 = m_x = -1$, then the Fourier coefficients are as displayed in the figure below.



The value of $S_2 = B$ can be inferred from the Fourier coefficients of the derivative, that is

$$X_2 = S_2 \cdot j2\omega_0 = j2\pi\,B = j2\pi \quad\longrightarrow\quad B = 1$$

where we exploited the fact that the period is $T_p = 2$ to set $\omega_0 = 2\pi/T_p = \pi$. The value of A can be instead inferred from the power, providing

$$P_s = 1 + 2B^2 + 2A^2 = 3 + 2A^2 = 11\ ,$$

evidencing that $A^2 = 4$, that is $A = \pm 2$. As a consequence, we have

$$\begin{aligned} s(t) &= -1 + A\,e^{j\pi t} + A\,e^{-j\pi t} + B\,e^{j2\pi t} + B\,e^{-j2\pi t} \\ &= -1 + 2A\cos(\pi t) + 2B\cos(2\pi t) \\ &= -1 \pm 4\cos(\pi t) + 2\cos(2\pi t) \end{aligned}$$

8. Signal s_1 is neither real nor even, since its Fourier coefficients

$$S_k = \begin{cases} 1 & ,\ k \in [0, 100] \\ 0 & ,\ \text{otherwise} \end{cases}$$

do not have any symmetry with respect to the reversal of index k. For signal s_2, instead, we have

$$S_k = \begin{cases} \cos(k\pi) & , k \in [-100, 100] \\ 0 & , \text{otherwise} \end{cases}$$

where $\cos(k\pi)$ is real and even, hence the Fourier coefficients are real and even, and so is the signal by the properties of symmetries. For signal s_3, finally, we have

$$S_k = \begin{cases} j\sin(k\frac{\pi}{2}) & , k \in [-100, 100] \\ 0 & , \text{otherwise} \end{cases}$$

where $j\sin(k\frac{\pi}{2})$ is imaginary and odd, hence the Fourier coefficients are imaginary and odd, which corresponds to a signal that is real and odd in the time domain. In fact real and odd in the time domain implies Hermitian and odd in the Fourier domain, this being equivalent to imaginary and odd.

FOUNDATIONS OF SIGNALS AND SYSTEMS
11.2 Solved exercises
Prof. T. Erseghe

Exercises 11.2

Solve the following MatLab problems:

1. Consider the signals

$$x(n) = \begin{cases} -1 & n=-1 \\ 3 & n=0 \\ -5 & n=1 \\ 2 & n=2 \\ 0 & \text{otherwise} \end{cases} \qquad g(n) = \begin{cases} 1 & n=0 \\ 2 & n=1 \\ -1 & n=2 \\ 0 & \text{otherwise} \end{cases}$$

   Plot the signals as well as their convolution $y(n) = x * g(n)$ in different subplots.

2. Evaluate numerically the convolution between the signals $x(t) = g(t) = \text{rect}(t-1)$ and check that the result is $y(t) = x * g(t) = \text{triang}(t-2)$. Choose a very small sampling spacing T to get an accurate result.

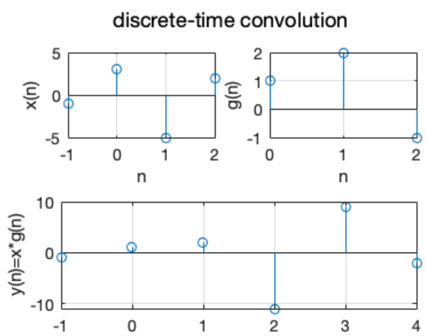
Solutions.

1. In the code we first define samples of x and g together with their sample time, then y is obtained by convolution and its sample times are built using the extension of convolution by using the starting elements $nx(0) + ng(0)$ and the ending elements $nx(end) + ng(end)$ of the arrays (here end is a keyword indicating the last element).

```
x = [-1,3,-5,2];
nx = -1:2;
g = [1,2,-1];
ng = 0:2;
y = conv(x,g);
ny = nx(1)+ng(1):nx(end)+ng(end);

figure
subplot(2,2,1)
stem(nx,x)
grid
xlabel('n')
ylabel('x(n)')
subplot(2,2,2)
stem(ng,g)
grid
xlabel('n')
```

```
ylabel('g(n)')
subplot(2,1,2)
stem(ny,y)
grid
xlabel('n')
ylabel('y(n)=x*g(n)')
sgtitle('discrete-time convolution')
```

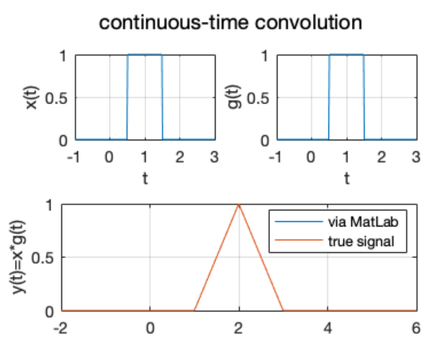
We also note how sgtitle draws a title for the entire figure.



2. We set $T = 0.01$ and the range for both x and g (which are equal) to $[-1, 3]$. The numerical convolution conv is multiplied by T to obtain a correct approximation of the continuous-time convolution, and the extension of signals is set by the rule on the extension of convolution, as in the previous exercise. The true convolution, here named y_2, is calculated via a triang function defined at the end of the script, in a way similar to the rect function. Note the perfect accordance in the lower plot. You can try to modify the value of T, e.g., to $T = 0.1$, to see how some errors arise.

```
T = 0.01;
tx = -1:T:3;
x = rect(tx-1);
tg = -1:T:3;
g = rect(tg-1);
y = T*conv(x,g);
ty = tx(1)+tg(1):T:tx(end)+tg(end);
y2 = triang(ty-2);

figure
subplot(2,2,1)
plot(tx,x)
grid
xlabel('t')
ylabel('x(t)')
subplot(2,2,2)
plot(tg,g)
```

```
grid
xlabel('t')
ylabel('g(t)')
subplot(2,1,2)
plot(ty,y,ty,y2)
grid
xlabel('t')
ylabel('y(t)=x*g(t)')
legend('via MatLab','true signal')
sgtitle('continuous-time convolution')

function s = triang(t)
s = (abs(t)<1).*(1-abs(t));
end

function s = rect(t)
s = (abs(t)<.5)+.5*(abs(t)==.5);
end
```



FOUNDATIONS OF SIGNALS AND SYSTEMS
11.3 Homework assignment
Prof. T. Erseghe

Exercises 11.3

Solve the following MatLab problems:

1. Evaluate numerically the convolution between the signals $x(t) = g(t) = \mathrm{sinc}(t)$ and check that the result is $y(t) = x * g(t) = \mathrm{sinc}(t)$. Choose a very small sampling spacing T to get an accurate result.

2. Insuline secretion rates can be measured by the hormone's C-peptide levels. Let $x(t)$ be the C-peptide pancreatic secretion, normalized by volume and measured in [pmol/L/min], and $y(t)$ the plasma C-peptide concentration, measured in [pmol/L]. These can be related through a filter $y(t) = x * g(t)$ with impulse response

$$g(t) = \left(0.76\, e^{-at} + 0.24\, e^{-bt}\right) 1(t) \ ,$$

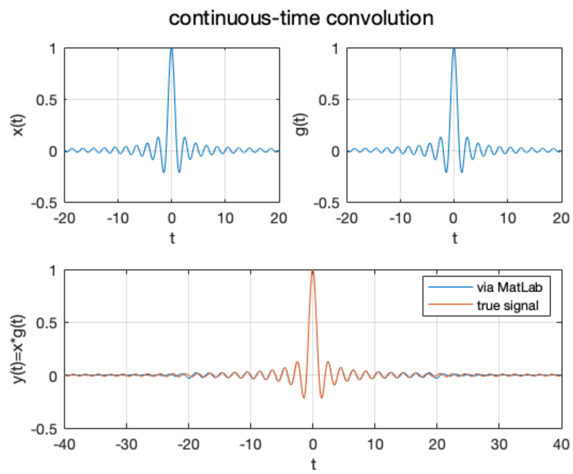
where $a = 0.14$ [min^{-1}], and $b = 0.02$ [min^{-1}]. Compute numerically the concentrration level $y(t)$ when the secretion is the one available in file ex11_3_2.mat, sampled by $T = 1\,\mathrm{min}$ in the range $[0, 420]\,\mathrm{min}$. How does the result change for $b = 0.2$ and $b = 0.002$? Provide a multiple plot of x, g, and y that compares the three cases.

Solutions.

1. The code can mimic that of Exercise 11.2.2, as follows, where we used the time span $[-20, 20]$ for the sinc.

```
T = 0.01;
tx = -20:T:20;
x = sinc(tx);
tg = -20:T:20;
g = sinc(tg);
y = T*conv(x,g);
ty = tx(1)+tg(1):T:tx(end)+tg(end);
y2 = sinc(ty);

figure
subplot(2,2,1)
plot(tx,x)
grid
xlabel('t')
ylabel('x(t)')
subplot(2,2,2)
plot(tg,g)
grid
xlabel('t')
ylabel('g(t)')
subplot(2,1,2)
plot(ty,y,ty,y2)
grid
xlabel('t')
ylabel('y(t)=x*g(t)')
legend('via MatLab','true signal')
sgtitle('continuous-time convolution')
```

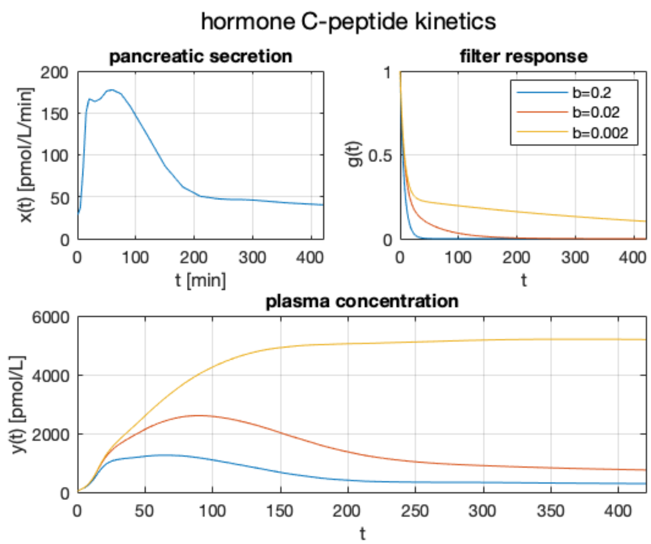
   Note how the result is accurate only in the range $[-10, 10]$, due to the fact that, although decaying, the sinc function is a slowly decaying function which is not zero outside the considered interval.



2. This exercise repeats the standard aspects of numerical convolution, with the usual product of conv by T. Note that we selected for $g(t)$ the same interval as $x(t)$, namely $[0, 420]$ min, and that we showed the convolution result only in this range since, outside the range, the convolution values assume $x(t) = 0$, which is not the case (the ending level of $x(t)$ is about 50, as one can infer from the first plot).

```
load('ex11_3_2.mat') % defines T, x, tx
tg = 0:T:420; % we use the same range of x
g1 = .76*exp(-.14*tg)+.24*exp(-0.2*tg);
y1 = T*conv(x,g1);
ty = tx(1)+tg(1):T:tx(end)+tg(end);
g2 = .76*exp(-.14*tg)+.24*exp(-0.02*tg);
y2 = T*conv(x,g2);
g3 = .76*exp(-.14*tg)+.24*exp(-0.002*tg);
y3 = T*conv(x,g3);

figure
subplot(2,2,1)
plot(tx,x)
grid
axis([0 420 ylim])
xlabel('t [min]')
ylabel('x(t) [pmol/L/min]')
title('pancreatic secretion')
subplot(2,2,2)
plot(tg,g1,tg,g2,tg,g3)
grid
axis([0 420 ylim])
xlabel('t')
```

```
ylabel('g(t)')
legend('b=0.2','b=0.02','b=0.002')
title('filter response')
subplot(2,1,2)
plot(ty,y1,ty,y2,ty,y3)
grid
axis([0 420 ylim])
xlabel('t')
ylabel('y(t) [pmol/L]')
title('plasma concentration')
sgtitle('hormone C-peptide kinetics')
```



FOUNDATIONS OF SIGNALS AND SYSTEMS
11.5 Solved exercises
Prof. T. Erseghe

Exercises 11.5

Solve the following MatLab problems:

1. Consider a rectangular wave of period $T_p = 5$ and duty cycle $d = \frac{1}{2}$, and its truncated Fourier series

$$s_N(t) = \sum_{k=-N}^{N} a_k \, e^{jk\omega_0 t} \ , \quad \omega_0 = \frac{2\pi}{T_p} \ , \ a_k = d \ \mathrm{sinc}(kd) \ .$$

In the same plot, show how the truncated series approximates the square wave for $N = 5, 10, 20, 50, 100, 200$, and observe the Gibbs phenomenon in the range $[0, \frac{1}{2}T_p]$. Use a very small sampling spacing T for the representation in MatLab.

2. Consider again a rectangular wave $s(t)$ of period $T_p = 5$ and duty cycle $d = \frac{1}{2}$ and its truncated Fourier series

$$s_N(t) = \sum_{k=-N}^{N} a_k \, e^{jk\omega_0 t} \ , \quad \omega_0 = \frac{2\pi}{T_p} \ ,$$

where the Fourier coefficients are now approximated via the numerical integration

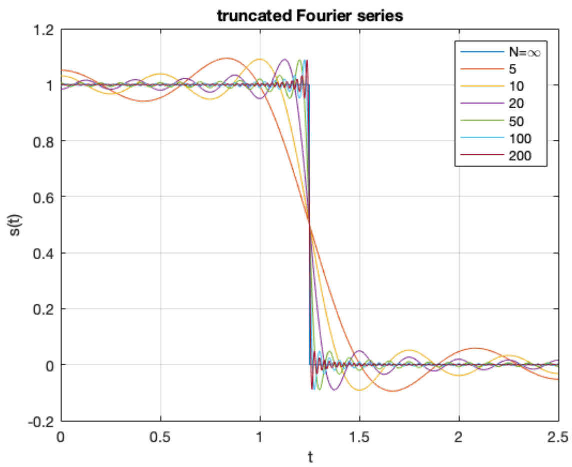
$$a_k = \frac{1}{T_p} \int_0^{T_p} s(t) e^{-jk\omega_0 t} dt \simeq b_k = \frac{1}{T_p} \cdot T \sum_{n=0}^{M-1} s(nT) \, e^{-jk\omega_0 nT} \ ,$$

for $T = T_p/M$ and a large M indicating the number of samples in the period. Compare, for $N = 100$, the different output obtained by the true coefficients a_k and the approximated coefficients b_k, using $M = 200, 500, 1000$.

Solutions.

1. In the code we first define constants and set the sampling spacing to a very small value to fully capture the Gibbs phenomenon. The different truncated series are obtained by a loop in N. Inside this loop, the truncated series is computed by first initializing a vector to zero values, and by then adding the contribution of each single Fourier coefficient, which is performed through a cycle in k. Before plotting, the imaginary part is removed, since this only accounts for numerical errors. The plots superposition is obtained by freezing the figure through an hold command.

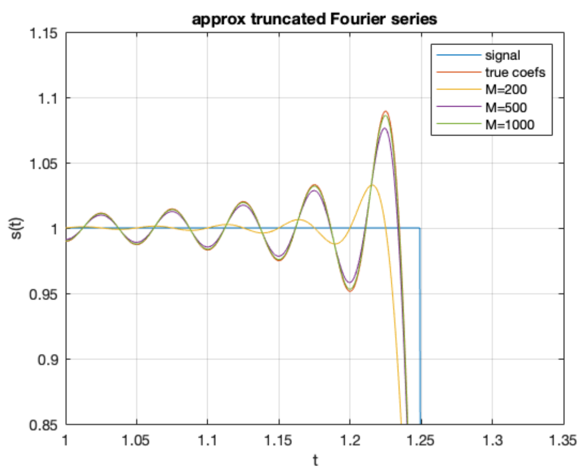
```matlab
Tp = 5; % period
d = .5; % duty cycle
om0 = 2*pi/Tp; % omega0
T = 0.001; % sampling spacing

t = 0:T:Tp/2;
s = square_wave(t,Tp,d);
figure
plot(t,s)
grid
axis([0 Tp/2 -.2 1.2])
xlabel('t')
ylabel('s(t)')
hold on

for N = [5,10,20,50,100,200]
    tfs = zeros(size(t)); % truncated Fourier
       series
    for k = -N:N % cicle on coefficients
        ak = d*sinc(k*d); % Fourier coefficient
        tfs = tfs +  ak*exp(1i*k*om0*t);
    end
    tfs = real(tfs); % prevent numerical errors
    plot(t,tfs)
end
legend('N=\infty','5','10','20','50','100','200')
title('truncated Fourier series')

function s = square_wave(t,Tp,d)
t1 = mod(t/Tp,1);
s = rect(t1/d) + rect((t1-1)/d);
end

function s = rect(t)
s = (abs(t)<.5)+.5*(abs(t)==.5);
end
```



2. This exercise repeats the previous one in its first part, then re-evaluates the coefficients by numerical integration trough a cycle on M where samples nT and $s(nT)$ are first stored, then used for calculating the approximate coefficients through a sum. The plot is zooming on the signal part that better evidences details, to appreciate how only $M = 1000$ is able to closely match the true result.

```
Tp = 5; % period
d = .5; % duty cycle
om0 = 2*pi/Tp; % omega0
T = 0.001; % sampling spacing
N = 100; % number of Fourier coefficients

t = 0:T:Tp/2;
s = square_wave(t,Tp,d);
figure
plot(t,s)
grid
axis([1 1.3 .85 1.15])
xlabel('t')
ylabel('s(t)')
hold on

% true coefficients
tfs = zeros(size(t)); % truncated Fourier series
for k = -N:N % cicle on coefficients
    ak = d*sinc(k*d); % Fourier coefficient
    tfs = tfs +  ak*exp(1i*k*om0*t);
end
tfs = real(tfs); % prevent numerical errors
```

```
plot(t,tfs)

% numerical coefficients
for M = [200,500,1000]
    tfs = zeros(size(t)); % truncated Fourier
        series
    nT = (0:M-1)*Tp/M;
    snT = square_wave(nT,Tp,d);
    for k = -N:N % cicle on coefficients
        bk = sum(snT.*exp(-1i*k*om0*nT))/M;
        tfs = tfs +  bk*exp(1i*k*om0*t);
    end
    tfs = real(tfs); % prevent numerical errors
    plot(t,tfs)
end
legend('signal','true coefs','M=200','M=500','M
   =1000')
title('approx truncated Fourier series')

function s = square_wave(t,Tp,d)
t1 = mod(t/Tp,1);
s = rect(t1/d) + rect((t1-1)/d);
end

function s = rect(t)
s = (abs(t)<.5)+.5*(abs(t)==.5);
end
```



FOUNDATIONS OF SIGNALS AND SYSTEMS
11.6 Homework assignment
Prof. T. Erseghe

Exercises 11.6

Solve the following MatLab problems:

1. Consider the triangular wave

$$s(t) = \operatorname*{rep}_{T_p} \operatorname{triang}(\tfrac{t}{dT_p}) \ ,$$

period $T_p = 5$ and duty cycle $d = \frac{1}{2}$, and its truncated Fourier series

$$s_N(t) = \sum_{k=-N}^{N} a_k \, e^{jk\omega_0 t} \ , \quad \omega_0 = \frac{2\pi}{T_p} \ , \ a_k = d \operatorname{sinc}^2(kd) \ .$$

In the same plot, show how the truncated series approximates the square wave for $N = 5, 10, 20, 50, 100, 200$ in the range $[0, \frac{1}{2}T_p]$. Can we see the Gibbs phenomenon? Use a very small sampling spacing T for the representation in MatLab.

2. Consider a periodic signal of period $T_p = 3$, defined in a period as

$$s(t) = \begin{cases} t & 0 < t < 1 \\ 1 & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$

so that in MatLab we can have

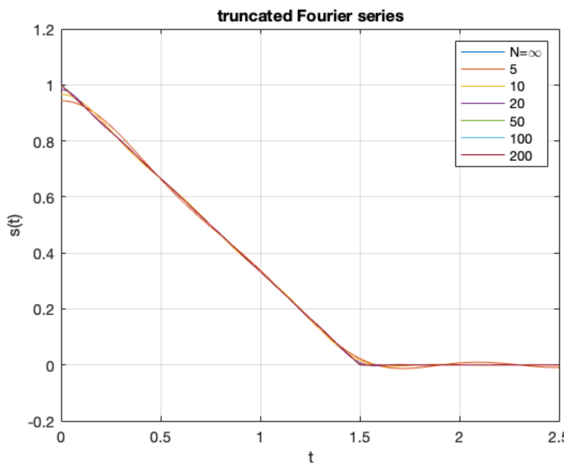
```
function s = signal(t)
t1 = mod(t,3);
s = t1.*(t1<1) + (t1>=1).*(t1<2);
end
```

Evaluate its Fourier coefficients by resorting to numerical integration, as in Exercise 11.5.2, with $M = 10^3$ and 10^4, and show the corresponding truncated Fourier series for $N = 100$. Is the Gibbs phenomenon visible? Where?

Solutions.

1. The code can mimic that of Exercise 11.5.1, but since the triangular wave has no discontinuities, no Gibbs phenomenon can be observed.

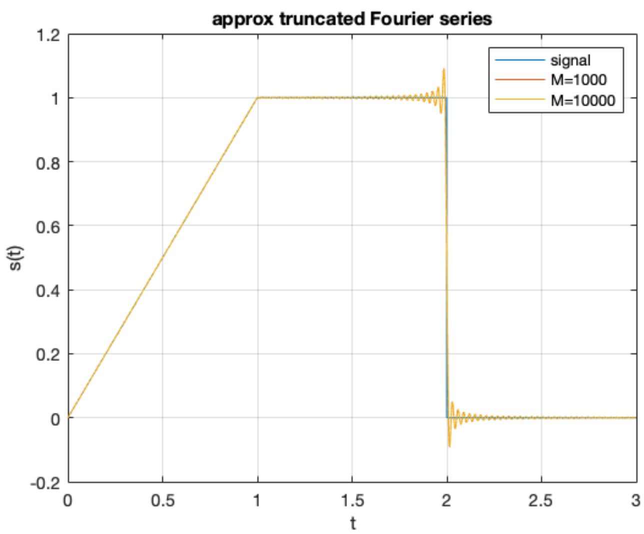
```
Tp = 5; % period
d = .3; % duty cycle
om0 = 2*pi/Tp; % omega0
T = 0.001; % sampling spacing

t = 0:T:Tp/2;
s = triang_wave(t,Tp,d);
figure
plot(t,s)
grid
axis([0 Tp/2 -.2 1.2])
xlabel('t')
ylabel('s(t)')
hold on

for N = [5,10,20,50,100,200]
    tfs = zeros(size(t)); % truncated Fourier
       series
    for k = -N:N % cicle on coefficients
        ak = d*sinc(k*d)^2; % Fourier coefficient
        tfs = tfs +  ak*exp(1i*k*om0*t);
    end
    tfs = real(tfs); % prevent numerical errors
    plot(t,tfs)
end
legend('N=\infty','5','10','20','50','100','200')
title('truncated Fourier series')

function s = triang_wave(t,Tp,d)
t1 = mod(t/Tp,1);
s = triang(t1/d) + triang((t1-1)/d);
end

function s = triang(t)
s = (abs(t)<1).*(1-abs(t));
end
```



2. This exercise repeats tExercise 11.5.2 by substituting the signal values. Note that with $M = 10^3$ we already obtain a very reliable result, showing that the Gibbs phenomenon only appears at discontinuities.

```
Tp = 3; % period
om0 = 2*pi/Tp; % omega0
T = 0.001; % sampling spacing
N = 100; % number of Fourier coefficients

t = 0:T:Tp;
s = signal(t);
figure
plot(t,s)
grid
xlabel('t')
ylabel('s(t)')
hold on

% numerical coefficients
for M = [1e3,1e4]
    tfs = zeros(size(t)); % truncated Fourier
       series
    nT = (0:M-1)*Tp/M;
    snT = signal(nT);
    for k = -N:N % cicle on coefficients
        bk = sum(snT.*exp(-1i*k*om0*nT))/M;
        tfs = tfs +  bk*exp(1i*k*om0*t);
    end
    tfs = real(tfs); % prevent numerical errors
    plot(t,tfs)
```

```
end
legend('signal','M=1000','M=10000')
title('approx truncated Fourier series')
```



FOUNDATIONS OF SIGNALS AND SYSTEMS
12.2 Solved exercises
Prof. T. Erseghe

Exercises 12.2

Prove that the following discrete Fourier transform pairs are correct by either forward or backward relation:

1. $s(n) = 1$ and $S_k = \text{rep}_N \delta(k)$,
2. $s(n) = \text{rep}_N \delta(n)$ and $S_k = \frac{1}{N}$,
3. $s(n) = \text{rep}_N \, \text{rect}(\frac{n}{M})$, $N > M = 1 + 2K$, and $S_k = \frac{M}{N} \, \text{sinc}_M(\frac{Mk}{N})$,
4. $s(n) = M \, \text{sinc}_M(\frac{M}{N} n)$ and $S_k = \text{rep}_N \, \text{rect}(\frac{k}{M})$, for $N > M = 1 + 2K$,
5. $s(n) = \cos(m \frac{2\pi}{N} n + \varphi_0)$ and
$$S_k = \tfrac{1}{2} e^{j\varphi_0} \text{comb}_N(k - m) + \tfrac{1}{2} e^{-j\varphi_0} \text{comb}_N(k + m) \ .$$

Solutions.

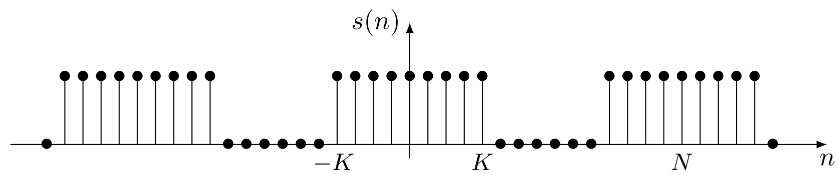
1. In this case we evaluate the periodic signal by applying the backward relation (DFT series) to the comb, to have
$$\begin{aligned} s(n) &= \sum_{k=0}^{N-1} \text{rep}_N \delta(k) \, e^{jk\frac{2\pi}{N} n} \\ &= \sum_{k=0}^{N-1} \delta(k) \, e^{jk\frac{2\pi}{N} n} \\ &= e^{j0 \cdot \frac{2\pi}{N} n} = 1 \ , \end{aligned}$$
where we exploited the fact that the only active element of the comb in $[0, N)$ is $\delta(k)$.

2. We evaluate the DFT coefficients by applying the forward relation to the comb, to have
$$\begin{aligned} S_k &= \frac{1}{N} \sum_{n=0}^{N-1} \text{rep}_N \delta(n) \, e^{-jk\frac{2\pi}{N} n} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \delta(n) \, e^{-jk\frac{2\pi}{N} n} \\ &= \frac{1}{N} e^{-jk\frac{2\pi}{N} \cdot 0} = \frac{1}{N} \ , \end{aligned}$$
where we exploited the fact that the only active element of the comb in $[0, N)$ is $\delta(n)$.

3. We evaluate the DFT coefficients by applying the forward relation to the square wave. We observe that

$$\mathrm{rect}(\tfrac{n}{M}) = \begin{cases} 1 & , n \in [-K, K] \\ 0, & \text{otherwise} \end{cases}$$

and that, thanks to $N > M = 1 + 2K$ there is no aliasing in the periodic repetition, as illustrated in the figure below.



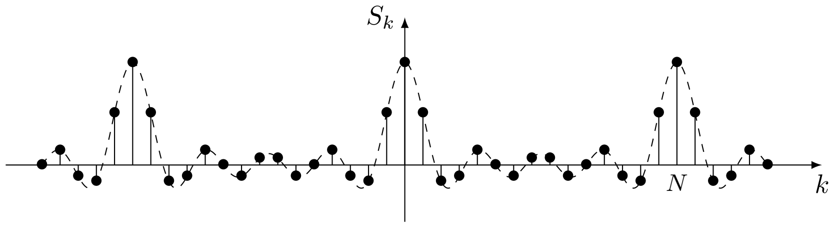
Hence, we can choose the period $[-K, N-1-K]$ as a reference, to include only the rectangle centred at 0. We have

$$\begin{aligned}
S_k &= \frac{1}{N} \sum_{n=-K}^{N-1-K} \mathrm{rep}_N\, \mathrm{rect}\left(\tfrac{n}{M}\right) e^{jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{n=-K}^{K} e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{m=0}^{2K} e^{-jk\frac{2\pi}{N}(m-K)} \\
&= \frac{e^{jk\frac{2\pi}{N}K}}{N} \sum_{m=0}^{2K} (e^{-jk\frac{2\pi}{N}})^m \\
&= \frac{e^{jk\frac{2\pi}{N}K}}{N} \frac{1 - e^{-jk\frac{2\pi}{N}(1+2K)}}{1 - e^{-jk\frac{2\pi}{N}}} \\
&= \frac{1}{N} \frac{e^{jk\frac{2\pi}{N}(\frac{1}{2}+K)} - e^{-jk\frac{2\pi}{N}(\frac{1}{2}+K)}}{e^{jk\frac{2\pi}{N}\frac{1}{2}} - e^{-jk\frac{2\pi}{N}\frac{1}{2}}} \\
&= \frac{1}{N} \frac{\sin(k\frac{2\pi}{N}(\frac{1}{2}+K))}{\sin(k\frac{2\pi}{N}\frac{1}{2})} \\
&= \frac{1}{N} \frac{\sin(\frac{M}{N}k\pi)}{\sin(\frac{1}{N}k\pi)}
\end{aligned}$$

The resulting signal, which is a sampled periodic sinc of the form

$$S_k = \tfrac{M}{N}\, \mathrm{sinc}_M(\tfrac{M}{N}k)\,, \qquad \mathrm{sinc}_M(x) = \frac{\sin(\pi x)}{M \sin(\pi \frac{x}{M})}\,,$$

is illustrated in the figure below.



4. In this case we could either evaluate the periodic signal by applying the backward relation (DFT series), or, more simply, exploit the symmetry rule from the result of the previous exercise that we write in the form

$$x(n) = N \operatorname{rep}_N \operatorname{rect}\left(\tfrac{n}{M}\right)$$
$$X_k = M \operatorname{sinc}_M(\tfrac{M}{N}k)$$

to have

$$s(n) = X_n = M \operatorname{sinc}_M(\tfrac{M}{N}n)$$
$$S_k = \frac{1}{N}x(-k) = \operatorname{rep}_N \operatorname{rect}\left(\tfrac{k}{M}\right)$$

5. We evaluate the periodic signal by applying the backward relation (DFT series) to have

$$\begin{aligned} s(n) = & \sum_{k=n_0}^{n_0+N-1} \left[\tfrac{1}{2}e^{j\varphi_0}\operatorname{rep}_N\delta(k-m) + \tfrac{1}{2}e^{-j\varphi_0}\operatorname{rep}_N\delta(k+m)\right] e^{jk\frac{2\pi}{N}n} \\ = & \sum_{k=m}^{m+N-1} \tfrac{1}{2}e^{j\varphi_0}\delta(k-m)\, e^{jk\frac{2\pi}{N}n} + \sum_{k=-m}^{-m+N-1} \tfrac{1}{2}e^{-j\varphi_0}\delta(k+m)\Big]\, e^{jk\frac{2\pi}{N}n} \\ = & \tfrac{1}{2}e^{j\varphi_0}\, e^{jm\frac{2\pi}{N}n} + \tfrac{1}{2}e^{-j\varphi_0}\, e^{-jm\frac{2\pi}{N}n} \\ = & \cos(m\tfrac{2\pi}{N}n + \varphi_0) \end{aligned}$$

where we used two different values of n_0, namely $n_0 = \pm m$, for the two comb functions.

FOUNDATIONS OF SIGNALS AND SYSTEMS
12.3 Homework assignment
Prof. T. Erseghe

Exercises 12.3

Solve the following:

1. evaluate the DFT coefficients of $x(n) = e^{jm\frac{2\pi}{N}n}$,

2. evaluate the DFT coefficients of $x(n) = \mathrm{rep}_N\, \delta(n - n_1)$,

3. evaluate the DFT coefficients of

$$x(n) = 3 - \sin(\tfrac{2}{5}\pi n) + \cos(\tfrac{4}{5}\pi n) + 2\cos(\tfrac{1}{5}\pi n - \tfrac{\pi}{4}) \ ,$$

4. evaluate the DFT coefficients of $x(n) = |\cos(\frac{2}{M}\pi n)|$, with M multiple of 4, by taking care of choosing a proper period for the sum,

5. evaluate the DFT coefficients of $x(n) = \cos(\frac{2}{7}\pi n) + \sin^2(\frac{3}{7}\pi n)$.

Solutions.

1. We observe that the signal is already in DFT series form, with only one coefficient active, the one for $k = m$, that is

$$S_k = \text{rep}_N \delta(k - m) \ .$$

2. We evaluate the DFT coefficients by applying the forward relation, to have

$$\begin{aligned} S_k &= \frac{1}{N} \sum_{n_0}^{n_0+N-1} \text{rep}_N \delta(n - n1)\, e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{N} \sum_{n_1}^{n_1+N-1} \delta(n - n1)\, e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{N}\, e^{-jk\frac{2\pi}{N}n_1} \end{aligned}$$

where we choose the period $[n_1, n_1 + N)$ in such a way that the only active delta in the integration of $\text{rep}_N \delta(n - n_1)$ is $\delta(n - n_1)$.

3. In this case, the signal is already written in the form of a DFT series, as one can appreciate by expanding the sinusoids through Euler's identity, to have

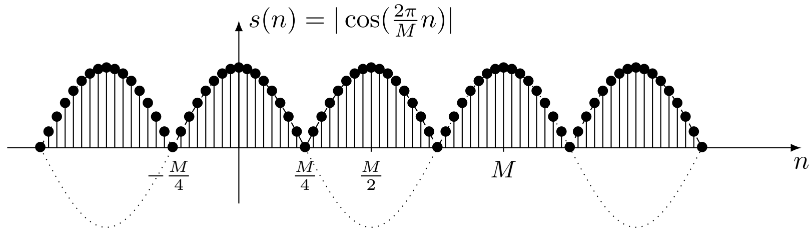
$$\begin{aligned} x(t) &= 3 - \sin(\tfrac{2}{5}\pi n) + \cos(\tfrac{4}{5}\pi n) + 2\cos(\tfrac{1}{5}\pi n - \tfrac{\pi}{4}) \\ &= 3 - \tfrac{1}{2j} e^{2j\frac{2\pi}{10}n} + \tfrac{1}{2j} e^{-j2\frac{2\pi}{10}n} + \tfrac{1}{2} e^{j4\frac{2\pi}{10}n} + \tfrac{1}{2} e^{-j4\frac{2\pi}{10}n} \\ &\qquad + e^{j(\frac{2\pi}{10}n - \frac{\pi}{4})} + e^{-j(\frac{2\pi}{10}n - \frac{\pi}{4})} \end{aligned}$$

where the complex exponentials have been already written in a form that reveals that the period is $N = 10$. Hence, by inspection we have

$$X_k = \begin{cases} 3 & , k = 0 \pmod N \\ e^{-j\frac{\pi}{4}} & , k = 1 \pmod N \\ e^{j\frac{\pi}{4}} & , k = -1 \pmod N \\ \frac{j}{2} & , k = 2 \pmod N \\ -\frac{j}{2} & , k = -2 \pmod N \\ \frac{1}{2} & , k = \pm 4 \pmod N \\ 0 & , \text{otherwise} \end{cases}$$

where we incidentally observe the presence of an Hermitian symmetry (since the signal is real-valued).

4. The signal $x(t) = |\cos(\frac{2\pi}{M}n)|$ for M even has period $N = \frac{M}{2}$, since this provides a shift by π. We evaluate the DFT coefficients by applying the forward relation, and integration over the period $[\frac{N}{2}, \frac{N}{2}) = [\frac{M}{4}, \frac{M}{4})$ where $|\cos(\frac{2\pi}{M}n)| = \cos(\frac{2\pi}{M}n)$, as illustrated in the figure below.



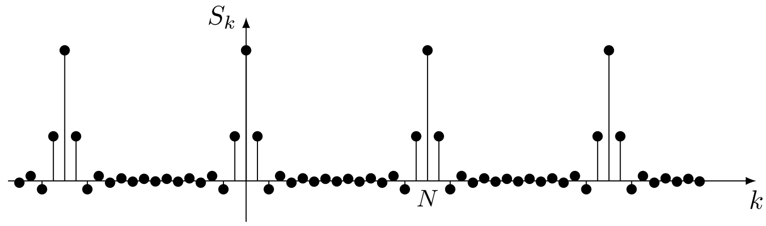
We obtain

$$\begin{aligned}
S_k &= \frac{1}{N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} |\cos(\tfrac{2\pi}{M}n)|\, e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \cos(\tfrac{2\pi}{M}n)\, e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \cos(\tfrac{2\pi}{M}n)\, e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{2N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} [e^{j\frac{2\pi}{M}n} + e^{-j\frac{2\pi}{M}n}]\, e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{2N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} e^{-j(k-\frac{1}{2})\frac{2\pi}{N}n} + e^{-j(k+\frac{1}{2})\frac{2\pi}{N}n}\,.
\end{aligned}$$

where in the second line we exploited the fact that $\cos(\frac{2\pi}{M}n)$ is zero for $n = \frac{N}{2}$. We now solve the expression by resorting to the truncated geometric series, to have

$$\begin{aligned}
S_k &= \frac{1}{2N}\sum_{m=0}^{N} e^{-j(k-\frac{1}{2})\frac{2\pi}{N}(m-\frac{N}{2})} + e^{-j(k+\frac{1}{2})\frac{2\pi}{N}(m-\frac{N}{2})} \\
&= \frac{e^{j(k-\frac{1}{2})\pi}}{2N}\sum_{m=0}^{N}[e^{-j(k-\frac{1}{2})\frac{2\pi}{N}}]^m + \frac{e^{j(k+\frac{1}{2})\pi}}{2N}\sum_{m=0}^{N}[e^{-j(k+\frac{1}{2})\frac{2\pi}{N}}]^m \\
&= \frac{e^{j(k-\frac{1}{2})\pi}}{2N}\cdot\frac{1-e^{-j(k-\frac{1}{2})(2\pi+\frac{2\pi}{N})}}{1-e^{-j(k-\frac{1}{2})\frac{2\pi}{N}}} + \frac{e^{j(k+\frac{1}{2})\pi}}{2N}\cdot\frac{1-e^{-j(k+\frac{1}{2})(2\pi+\frac{2\pi}{N})}}{1-e^{-j(k+\frac{1}{2})\frac{2\pi}{N}}} \\
&= \frac{\sin((k-\frac{1}{2})(\pi+\frac{\pi}{N}))}{2N\,\sin((k-\frac{1}{2})\frac{\pi}{N})} + \frac{\sin((k+\frac{1}{2})(\pi+\frac{\pi}{N}))}{2N\,\sin((k+\frac{1}{2})\frac{\pi}{N})}
\end{aligned}$$

The resulting signal is depicted in the picture below.



With a little effort we can show how the signal is related to the periodic sinc function. The trick is to expand the numerator by exploiting the trigonometric identity $\sin(a+b) = \sin a \cos b + \cos a \sin b$, the fact that $\cos((k-\frac{1}{2})\pi) = 0$, and the definition of sinc_N, to have

$$\begin{aligned} S_k &= \frac{\sin((k-\frac{1}{2})\pi)\ \cos((k-\frac{1}{2})\frac{\pi}{N})}{2N\ \sin((k-\frac{1}{2})\frac{\pi}{N})} + \frac{\sin((k+\frac{1}{2})\pi)\ \cos((k+\frac{1}{2})\frac{\pi}{N})}{2N\ \sin((k+\frac{1}{2})\frac{\pi}{N})} \\ &= \tfrac{1}{2}\,\text{sinc}_N(k-\tfrac{1}{2})\ \cos((k-\tfrac{1}{2})\tfrac{\pi}{N}) + \tfrac{1}{2}\,\text{sinc}_N(k+\tfrac{1}{2})\ \cos((k+\tfrac{1}{2})\tfrac{\pi}{N}) \end{aligned}$$

5. This is another example where the signal is can be easly written in the form of a Fourier series, that is

$$\begin{aligned} x(t) &= \cos(\tfrac{2\pi}{7}n) + \sin^2(\tfrac{3\pi}{7}n) \\ &= \sin(\tfrac{2\pi}{7}n) + \tfrac{1}{2} - \tfrac{1}{2}\cos(\tfrac{6\pi}{7}n) \\ &= \sin(\tfrac{2\pi}{7}n) + \tfrac{1}{2} - \tfrac{1}{2}\cos(3\tfrac{2\pi}{7}n) \\ &= \tfrac{1}{2j}e^{j\frac{2\pi}{7}n} - \tfrac{1}{2j}e^{-j\frac{2\pi}{7}n} + \tfrac{1}{2} - \tfrac{1}{4}e^{j3\frac{2\pi}{7}n} - \tfrac{1}{4}e^{-j3\frac{2\pi}{7}n} \end{aligned}$$

which reveals that the period is $N = 7$ and that the DFT coefficients are

$$S_k = \begin{cases} \frac{1}{2} & , k = 0 \pmod N \\ \frac{1}{2j} & , k = 1 \pmod N \\ -\frac{1}{2j} & , k = -1 \pmod N \\ -\frac{1}{4} & , k = \pm 3 \pmod N \\ 0 & , \text{otherwise} \end{cases}$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
12.5 Solved exercises
Prof. T. Erseghe

Exercises 12.5

Solve the following by exploiting, where useful, the properties of the DFT:

1. prove the time-shift property $x(n-n_1) \to X_k e^{-jk\frac{2\pi}{N}n_1}$;

2. prove the circular convolution property $x *_{\text{cir}} y(n) \to N X_k Y_k$;

3. prove that any real-valued discrete-time periodic signal of period N can be expressed through the trigonometric (and real-valued) series

$$s(n) = S_0 + \sum_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} 2|S_k| \cos(k\tfrac{2\pi}{N}n + \varphi_k) + \begin{cases} S_{\frac{N}{2}}\,(-1)^n & , N \text{ even} \\ 0 & , N \text{ odd} \end{cases}$$

   where $S_k = |S_k| e^{-j\varphi_k}$ are its DFT coefficients;

4. evaluate the DFT coefficients of $s(n) = e^{jm\frac{2\pi}{N}n}$ by exploiting the Fourier couple $x(n) = 1$, $X_k = \text{rep}_N\, \delta(k)$;

5. prove, by exploiting the increment property, that the DFT coefficients of the sampled square wave

$$s(n) = \begin{cases} 1 & , n \in [0, M) \pmod N \\ 0 & , \text{otherwise} \end{cases}$$

   for $N > M$ can be expressed in the form

$$S_k = \tfrac{M}{N}\, \text{sinc}_M(\tfrac{M}{N}k)\, e^{-j\frac{M-1}{N}k\pi} \ .$$

Solutions.

1. For the time-shift property, we investigate the DFT of $y(n) = x(n-n_1)$, providing

$$\begin{aligned} Y_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n-n_1) e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{N}\sum_{m=-n_1}^{N-1-n_1} x(m) e^{-jk\frac{2\pi}{N}(m+n_1)} \\ &= X_k e^{-jk\frac{2\pi}{N}n_1} \end{aligned}$$

   where we denoted $m = n - n_1$ and where we note that the sum range $[-n_1, N-1-n_1]$ is one period, hence the final result.

2. For the circular convolution property, we investigate the DFT of $z(n) = x *_{\text{cir}} y(n)$, providing

$$
\begin{aligned}
Z_k &= \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{\ell=0}^{N-1} x(n-\ell)y(\ell)\right) e^{-jk\frac{2\pi}{N}n} \\
&= \sum_{\ell=0}^{N-1} y(\ell)\left(\frac{1}{N}\sum_{n=0}^{N-1} x(n-\ell)e^{-jk\frac{2\pi}{N}n}\right) \\
&= \sum_{\ell=0}^{N-1} y(\ell)\, X_k\, e^{-jk\frac{2\pi}{N}\ell} = N X_k Y_k
\end{aligned}
$$

where we swapped the order of the two summations in the second equivalence, and exploited the time-shift rule in the third equivalence.

3. Being the signal real-valued, we exploit the Hermitian symmetry $S_k = S^*_{-k}$ of the DFT coefficients, that is $S_{-k} = S^*_k = |S_k|e^{-j\varphi_k}$ which also reveals that S_0 (the mean value) is real-valued. When N is even, also $S_{\frac{N}{2}}$ is real-valued, since $S_{\frac{N}{2}} = S_{-\frac{N}{2}}$ by periodicity and $S_{\frac{N}{2}} = S^*_{-\frac{N}{2}}$ by the hermitian property. Hence, for N even we can write the Fourier series in the form

$$
\begin{aligned}
s(n) &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} S_k\, e^{jk\frac{2\pi}{N}n} \\
&= \sum_{k=-(\frac{N}{2}-1)}^{-1} S_k\, e^{jk\frac{2\pi}{N}n} + S_0 + \sum_{k=1}^{\frac{N}{2}-1} S_k\, e^{jk\frac{2\pi}{N}n} + S_{\frac{N}{2}}\, e^{j\frac{N}{2}\frac{2\pi}{N}n} \\
&= S_0 + S_{\frac{N}{2}}\,(-1)^n + \sum_{k=1}^{\frac{N}{2}-1}[S_k\, e^{jk\frac{2\pi}{N}n} + S_{-k}\, e^{-jk\frac{2\pi}{N}n}] \\
&= S_0 + S_{\frac{N}{2}}\,(-1)^n + \sum_{k=1}^{\frac{N}{2}-1}|S_k|[e^{j\varphi_k}\, e^{jk\frac{2\pi}{N}n} + e^{-j\varphi_k}\, e^{-jk\frac{2\pi}{N}n}] \\
&= S_0 + S_{\frac{N}{2}}\,(-1)^n + \sum_{k=1}^{\frac{N}{2}-1} 2|S_k|\,\cos(k\tfrac{2\pi}{N}n + \varphi_k)\ .
\end{aligned}
$$

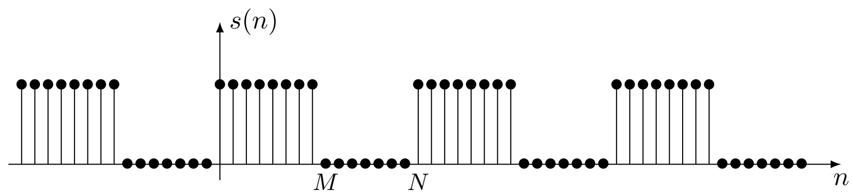
For N odd, instead, we have

$$
\begin{aligned}
s(n) &= \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} S_k\, e^{jk\frac{2\pi}{N}n} \\
&= \sum_{k=-\frac{N-1}{2}}^{-1} S_k\, e^{jk\frac{2\pi}{N}n} + S_0 + \sum_{k=1}^{\frac{N-1}{2}} S_k\, e^{jk\frac{2\pi}{N}n} \\
&= S_0 + \sum_{k=1}^{\frac{N-1}{2}} [S_k\, e^{jk\frac{2\pi}{N}n} + S_{-k}\, e^{-jk\frac{2\pi}{N}n}] \\
&= S_0 + \sum_{k=1}^{\frac{N-1}{2}} |S_k| [e^{j\varphi_k}\, e^{jk\frac{2\pi}{N}n} + e^{-j\varphi_k}\, e^{-jk\frac{2\pi}{N}n}] \\
&= S_0 + \sum_{k=1}^{\frac{N-1}{2}} 2|S_k|\, \cos(k\tfrac{2\pi}{N}n + \varphi_k)\ .
\end{aligned}
$$

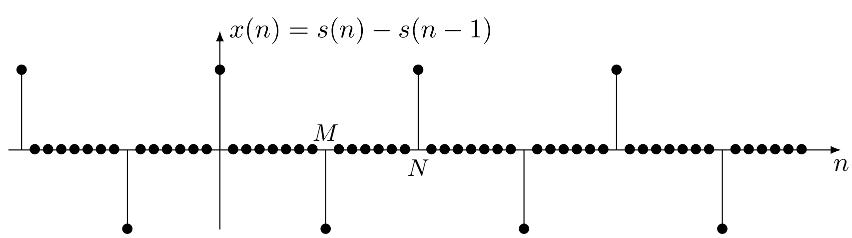
Altogether the two results provide the result compactly expressed in the exercise text.

4. Since $s(n) = e^{jm\frac{2\pi}{N}n} = e^{jm\frac{2\pi}{N}n} \cdot 1 = e^{jm\frac{2\pi}{N}n} \cdot x(t)$, we can exploit the modulation property to state that $S_k = X_{k-m} = \mathrm{rep}_N\, \delta(k-m)$.

5. The square wave is illustrated in the figure below.



In this case we can exploit the increment signal $x(n) = s(n) - s(n-1)$



which is simply

$$x(n) = \text{rep}_N \delta(n) - \text{rep}_N \delta(n - M)$$

hence it is straightforward calculating its DFT coefficients by standard rules, to have

$$X_k = \tfrac{1}{N}(1 - e^{-jk\frac{2\pi}{N}M})$$

and by inversion of the increment rule we obtain

$$\begin{aligned} S_k &= \begin{cases} \frac{X_k}{1-e^{-jk\frac{2\pi}{N}}} & , k \neq 0 \pmod N \\ m_s & , k = 0 \pmod N \end{cases} \\ &= \begin{cases} \frac{1-e^{-jk\frac{2\pi}{N}M}}{N(1-e^{-jk\frac{2\pi}{N}})} & , k \neq 0 \pmod N \\ \frac{M}{N} & , k = 0 \pmod N \end{cases} \end{aligned}$$

We also observe that

$$\frac{1 - e^{-jk\frac{2\pi}{N}M}}{N(1 - e^{-jk\frac{2\pi}{N}})} \cdot \frac{e^{jk\frac{\pi}{N}(M+1)}}{e^{jk\frac{\pi}{N}(M+1)}} = \frac{\sin(\frac{M}{N}k\pi)}{N\,\sin(\frac{1}{N}k\pi)}\,\frac{e^{jk\frac{\pi}{N}}}{e^{jk\frac{\pi}{N}M}}$$

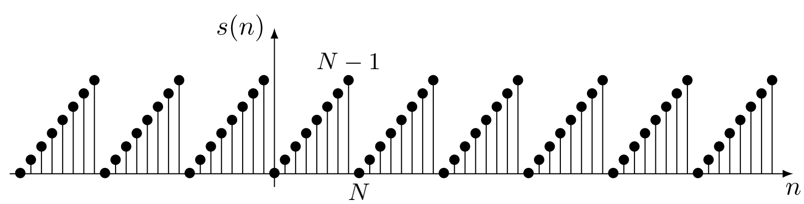
so that the result can be written in the form given in the exercise text by use of the periodic sinc function $\text{sinc}_M(x)$.

FOUNDATIONS OF SIGNALS AND SYSTEMS
12.6 Solved exercises
Prof. T. Erseghe

Exercises 12.6

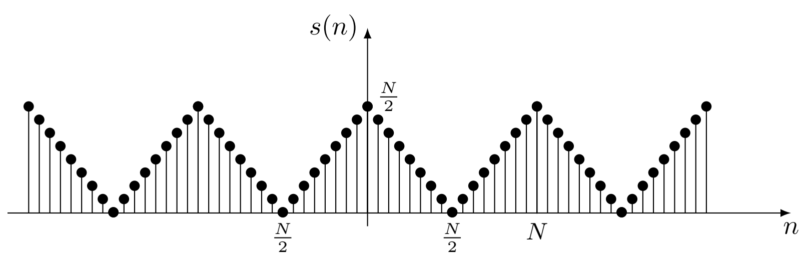
Solve the following by exploiting the properties of the Fourier series:

1. prove the modulation property $x(n)\, e^{jm\frac{2\pi}{N}n} \to X_{k-m}$;
2. prove the product property $x(n)y(n) \to X *_{\text{cir}} Y_k$;
3. evaluate the DFT coefficients of $s(n) = \text{rep}_N\, \delta(n - n_1)$ by exploiting the Fourier couple $x(n) = \text{rep}_N\, \delta(n)$, $X_k = \frac{1}{N}$;
4. evaluate the DFT coefficients of the saw-tooth waveform



by exploiting the increment property;

5. evaluate the DFT coefficients of the triangular waveform



by exploiting the circular convolution property and the DFT pair of Exercise 12.5.5, and by considering N even;

6. evaluate the DFT coefficients of $s(n) = [\sin(2\pi \frac{n}{N})]^+$, where $[x]^+ = x \cdot 1(x)$ is the positive part operator, by exploiting the product property and the DFT pair of Exercise 12.5.5.

Solutions.

1. For the modulation property, we investigate the DFT coefficients of $y(n) = x(n)\,e^{jm\frac{2\pi}{N}n}$, providing

$$\begin{aligned} Y_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{jm\frac{2\pi}{N}n}e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j(k-m)\frac{2\pi}{N}n} \\ &= X_{k-m}\ . \end{aligned}$$

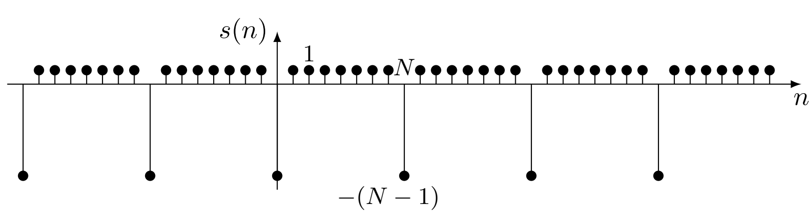
2. For the product property, we investigate the DFT coefficients of $z(n) = x(n)\,y(n)$, providing

$$\begin{aligned} Z_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)y(n)e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} X_m\,e^{jm\frac{2\pi}{N}n}\right) y(n)e^{-jk\frac{2\pi}{N}n} \\ &= \sum_{m=0}^{N-1} X_m\,\left(\frac{1}{N}\sum_{n=0}^{N-1} y(n)\,e^{-j(k-m)\frac{2\pi}{N}n}\right) \\ &= \sum_{m=0}^{N-1} X_m Y_{k-m} \\ &= X *_{\text{circ}} Y_k \end{aligned}$$

   where in the second equivalence we expressed $x(n)$ through its DFT series, and in the third equivalence we simply changed the order of summations.

3. Since $s(n) = x(n-n_1)$ by the time-shift property we have $S_k = X_k\,e^{-jk\frac{2\pi}{N}n_1} = \frac{1}{N}\,e^{-jk\frac{2\pi}{N}n_1}$.

4. In a period $[0, N)$, the saw-tooth takes the expression $s(n) = n$. In this case it is easier to investigate the increment signal $x(n) = s(n) - s(n-1)$ providing



so that we can write

$$x(n) = 1 - N\,\mathrm{rep}_N \delta(n)\ , \qquad X_k = \mathrm{rep}_N \delta(k) - 1\ .$$

By inversion of the increment rule we obtain

$$S_k = \begin{cases} \dfrac{X_k}{1 - e^{-jk\frac{2\pi}{N}}} = \dfrac{-1}{1 - e^{-jk\frac{2\pi}{N}}} & , k \neq 0 \pmod N \\ m_s = \frac{N-1}{2} & , k = 0 \pmod N \end{cases}$$

5. The square wave of Exercise 12.5.5 for $M = \frac{N}{2}$ assumes the form

$$z(n) = \begin{cases} 1 & , n \in [0, \frac{N}{2} - 1] \\ 0 & , \text{otherwise} \end{cases} \qquad Z_k = \tfrac{1}{2}\,\mathrm{sinc}_{\frac{N}{2}}\left(\tfrac{1}{2}k\right) e^{-j(\frac{1}{2} - \frac{1}{N})k\pi}$$

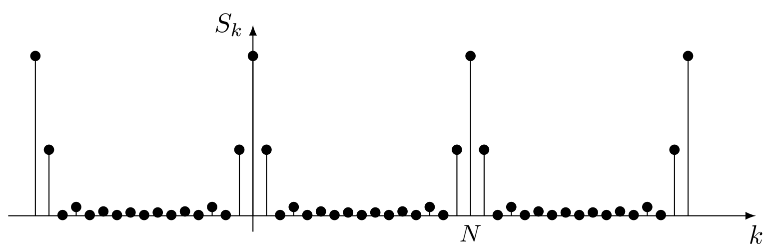
its self-circular-convolution $z *_{\mathrm{cir}} z$ being a triangular wave active, in the reference period $[0, N)$, in $[0, 2(\frac{N}{2} - 1)] = [0, N-2]$, so that we can write

$$s(n) = z *_{\mathrm{cir}} z(n + \tfrac{N}{2} - 1)$$

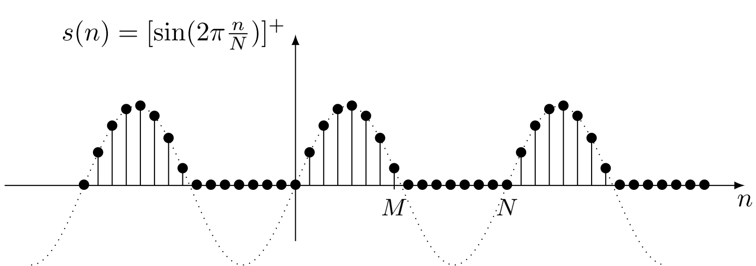
so that by the DFT properties of circular convolution and time-shift we obtain

$$\begin{aligned} S_n &= N\, Z_k^2\, e^{jk\frac{2\pi}{N}(\frac{N}{2} - 1)} \\ &= N\, Z_k^2\, e^{jk2\pi(\frac{1}{2} - \frac{1}{N})} \\ &= \tfrac{N}{4}\,\mathrm{sinc}^2_{\frac{N}{2}}\left(\tfrac{1}{2}k\right) \end{aligned}$$

The DFT is illustrated in the figure below for $N = 16$.



6. The signal $s(n) = [\sin(2\pi\frac{n}{N})]^+$ has period N, and is illustrated in the figure below.



As one can appreciate from the figure it can be written as the product $s(n) = x(n)y(n)$ between the full sinusoid

$$x(n) = \sin(2\pi \tfrac{n}{N})\ , \qquad X_k = \tfrac{1}{2j}\mathrm{rep}_N\delta(k-1) - \tfrac{1}{2j}\mathrm{rep}_N\delta(k+1)$$

and the square wave

$$y(n) = \begin{cases} 1 & , n \in [1, M] \pmod N \\ 0 & , \text{otherwise} \end{cases}\ , \qquad M = \begin{cases} \frac{N}{2} - 1 & , N \text{ even} \\ \frac{N-1}{2} & , N \text{ odd} \end{cases}$$

which, is equivalent, apart for a time-shift of $n_1 = 1$ to the square wave of Exercise 12.5.5, namely we have

$$y(n) = z(n-1)\ , \qquad Z_k = \tfrac{M}{N}\,\mathrm{sinc}_M(\tfrac{M}{N}k)\, e^{-j\frac{M-1}{N}k\pi}$$

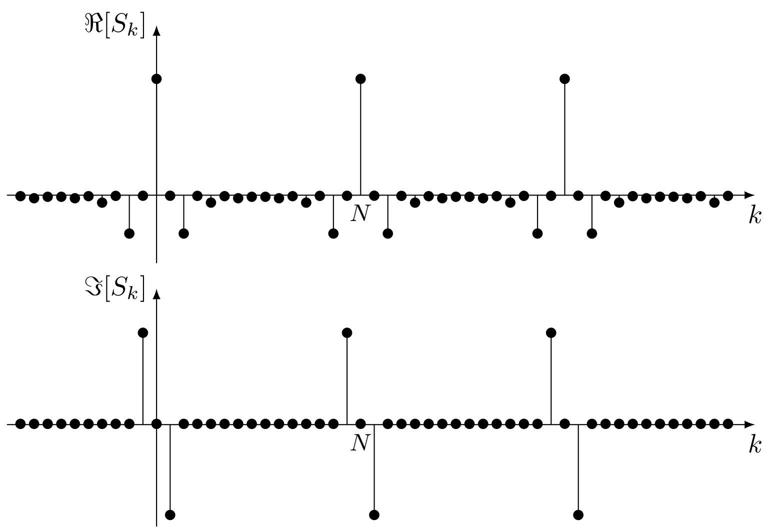
Therefore, by the time-shift property, it has DFT coefficients of the form

$$Y_k = Z_k\, e^{-jk\frac{2\pi}{N}} = \tfrac{M}{N}\,\mathrm{sinc}_M(\tfrac{M}{N}k)\, e^{-j\frac{M+1}{N}k\pi}$$

Hence, the DFT coefficients of $s(n)$ are straightforwardly derived by exploiting the property that a product in time involves a circular convolution in the DFT domain, to have

$$\begin{aligned} S_k &= X_k * Y_k \\ &= [\tfrac{1}{2j}\mathrm{rep}_N\delta(k-1) - \tfrac{1}{2j}\mathrm{rep}_N\delta(k+1)] * Y_k \\ &= \tfrac{1}{2j}Y_{k-1} - \tfrac{1}{2j}Y_{k+1} \\ &= \tfrac{M}{2Nj}\left[\mathrm{sinc}_M(\tfrac{M}{N}(k-1))\, e^{-j\frac{M+1}{N}(k-1)\pi} - \mathrm{sinc}_M(\tfrac{M}{N}(k+1))\, e^{-j\frac{M+1}{N}(k+1)\pi}\right] \end{aligned}$$

where we used the sifting property of the comb. The result is illustrated in the figure below for $N = 15$ and $M = 7$. Note the Hermitian symmetry: even symmetric real part and odd symmetric imaginary part.



We add some considerations that are not required in the solution of the exercise, but are useful to understand the structure of the DFT coefficients. We note that the imaginary part seems to be zero, except for $k = \pm 1 \pmod N$. This can be appreciated by extracting the even and odd signals parts, that is

$$s_e(n) = \tfrac{1}{2}[s(n) + s(-n)] = \tfrac{1}{2}\,|\sin(2\pi \tfrac{n}{N})|$$
$$s_o(n) = \tfrac{1}{2}[s(n) - s(-n)] = \tfrac{1}{2}\,\sin(2\pi \tfrac{n}{N})\ ,$$

whose DFT coefficients are, respectively, $S_{e,k} = \Re[S_k]$ and $S_{o,k} = j\Im[S_k]$ thanks to symmetry properties. This reveals that the imaginary part satisfies

$$\Im[S_k] = -jS_{o,k} = -\tfrac{1}{4}\mathrm{rep}_N\delta(k-1) + \tfrac{1}{4}\mathrm{rep}_N\delta(k+1)\ ,$$

which is the exact effect we observe in the figure. For the real part, instead, the DFT coefficients must, for even N, where $|\sin(2\pi\frac{n}{N})|$ is periodic of period $\frac{N}{2}$ (and so is s_e), be zero valued for odd k, as stated by the de-periodisation property. For odd N, instead, no sub-periodisation is active, and therefore this property does not hold (be aware that the figure is partly misleading since the values for k odd are small but non zero, but in any case the value at $N = 15$ is active!). These properties can, with some effort, be appreciated also from the (rather involved) expression of S_k, as we explain in the following.

We first investigate the imaginary part which we write in the form

$$\Im[S_k] = \tfrac{1}{4N}[f(k+1) - f(k-1)]\ ,$$

where we used function

$$f(x) = 2M\mathrm{sinc}_M(\tfrac{M}{N}x)\,\cos(\tfrac{M+1}{N}x\pi)\ ,$$

for integer x. By working on the expression of $f(x)$, we obtain

$$\begin{aligned} f(x) &= \frac{2\sin(\frac{M}{N}x\pi)\,\cos(\frac{M+1}{N}x\pi)}{\sin(\frac{1}{N}x\pi)} \\ &= \frac{\sin(\frac{2M+1}{N}x\pi) - \sin(\frac{1}{N}x\pi)}{\sin(\frac{1}{N}x\pi)} = \frac{\sin(\frac{2M+1}{N}x\pi)}{\sin(\frac{1}{N}x\pi)} - 1 \end{aligned}$$

where we exploited the trigonometric equivalence $2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$. We then distinguish between even and odd N. For odd N we have $1 + 2M = N$, hence

$$\begin{aligned} f(x) &= \frac{\sin(x\pi)}{\sin(\frac{1}{N}x\pi)} - 1 \\ &= -1 + \begin{cases} 0 & ,\ x \neq 0 \pmod N \\ \frac{N\cos(x\pi)}{\cos(\frac{1}{N}x\pi)} = N & ,\ x = 0 \pmod N \end{cases} \\ &= N\,\mathrm{rep}_N\delta(x) - 1\ , \end{aligned}$$

where for $x = 0 \pmod N$ we first exploited de l'Hospital rule, then the fact that N is odd. For even N, instead it is $1 + 2M = N - 1$, hence

$$\begin{aligned}
f(x) &= \frac{\sin(\frac{N-1}{N}x\pi)}{\sin(\frac{1}{N}x\pi)} - 1 \\
&= \frac{\sin(x\pi)\cos(\frac{1}{N}x\pi) - \cos(x\pi)\sin(\frac{1}{N}x\pi)}{\sin(\frac{1}{N}x\pi)} - 1 \\
&= -\cos(x\pi) - 1 + \frac{\sin(x\pi)}{\sin(\frac{1}{N}x\pi)}\cos(\frac{1}{N}x\pi) \\
&= -\cos(x\pi) - 1 + \begin{cases} 0 & , x \neq 0 \pmod N \\ \cos(\frac{1}{N}x\pi) \cdot \frac{N\cos(x\pi)}{\cos(\frac{1}{N}x\pi)} = N & , x = 0 \pmod N \end{cases} \\
&= N\,\mathrm{rep}_N\delta(x) - 1 - \cos(x\pi) \ ,
\end{aligned}$$

where we used the trigonometric identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, the result previously seen for N odd, and the fact that now N is even. By exploiting the above results, we finally have

$$\begin{aligned}
\Im[S_k] = {} & \tfrac{1}{4}\,\mathrm{rep}_N\delta(k+1) - \tfrac{1}{4}\,\mathrm{rep}_N\delta(k-1) \\
& + \begin{cases} 0 & , N \text{ odd} \\ \frac{1}{4N}\cos((k-1)\pi) - \frac{1}{4N}\cos((k+1)\pi) = 0 & , N \text{ even} \end{cases}
\end{aligned}$$

which proves the correctness of the expression of S_k.

For the real part, instead, it is

$$\Re[S_k] = \tfrac{1}{4N}\left[g(k+1) - g(k-1)\right] \ ,$$

where we used function

$$g(x) = 2M\mathrm{sinc}_M(\tfrac{M}{N}x)\,\sin(\tfrac{M+1}{N}x\pi) \ ,$$

for integer x. By working on the expression of $g(x)$, we obtain

$$g(x) = \frac{2\sin(\frac{M}{N}x\pi)\,\cos(\frac{M+1}{N}x\pi)}{\sin(\frac{1}{N}x\pi)} = \frac{\cos(\frac{1}{N}x\pi) - \cos(\frac{2M+1}{N}x\pi)}{\sin(\frac{1}{N}x\pi)}$$

where we exploited the trigonometric equivalence $2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha+\beta)$. We then distinguish between even and odd N. For odd N we have $1 + 2M = N$, hence

$$\begin{aligned}
g(x) &= \frac{\cos(\frac{1}{N}x\pi) - \cos(x\pi)}{\sin(\frac{1}{N}x\pi)} \\
&= \begin{cases} \frac{\cos(\frac{1}{N}x\pi) - \cos(x\pi)}{\sin(\frac{1}{N}x\pi)} & , x \neq 0 \pmod N \\ 0 & , x = 0 \pmod N \end{cases}
\end{aligned}$$

For even N, instead it is $1+2M=N-1$, hence

$$\begin{aligned} g(x) &= \frac{\cos(\frac{1}{N}x\pi) - \cos(\frac{N-1}{N}x\pi)}{\sin(\frac{1}{N}x\pi)} \\ &= \frac{\cos(\frac{1}{N}x\pi) - \cos(x\pi)\cos(\frac{1}{N}x\pi) - \sin(x\pi)\sin(\frac{1}{N}x\pi)}{\sin(\frac{1}{N}x\pi)} \\ &= \frac{\cos(\frac{1}{N}x\pi)\,[1-\cos(x\pi)]}{\sin(\frac{1}{N}x\pi)} \\ &= \begin{cases} 0 & , x \neq 0 \pmod{N},\ x \text{ even} \\ 2\cot(\frac{1}{N}x\pi) & , x \neq 0 \pmod{N},\ x \text{ odd} \\ 0 & , x = 0 \pmod{N} \end{cases} \\ &= \begin{cases} 0 & , x \text{ even} \\ 2\cot(\frac{1}{N}x\pi) & , x \text{ odd} \end{cases} \end{aligned}$$

where we used $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, and the fact that N is now even. Being $g(x) = 0$ for x even implies $\Re[S_k] = 0$ for k odd, as desired.

FOUNDATIONS OF SIGNALS AND SYSTEMS
13.2 Solved exercises
Prof. T. Erseghe

Exercises 13.2

Prove that the following Fourier transform pairs are correct by either forward or backward relation, or by the symmetry rule:

1. $s(t) = \delta(t)$ and $S(j\omega) = 1$,
2. $s(t) = 1$ and $S(j\omega) = 2\pi\,\delta(\omega)$,
3. $s(t) = \mathrm{rect}(t)$ and $S(j\omega) = \mathrm{sinc}(\frac{\omega}{2\pi})$,
4. $s(t) = \mathrm{sinc}(t)$ and $S(j\omega) = \mathrm{rect}(\frac{\omega}{2\pi})$,
5. $s(t) = \cos(\omega_1 t + \varphi_1)$ and
$$S(j\omega) = \pi\, e^{j\varphi_1}\delta(\omega - \omega_1) + \pi\, e^{-j\varphi_1}\delta(\omega + \omega_1)\ ;$$
6. $s(t) = e^{-at}\,1(t)$, $a > 0$ and $S(j\omega) = \frac{1}{a+j\omega}$.

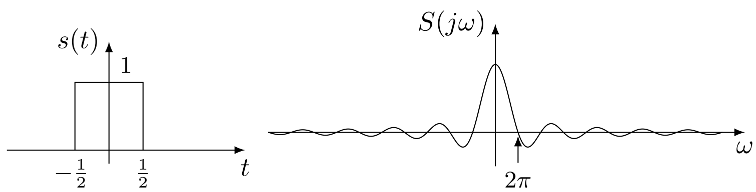
Solutions.

1. In this case we apply the forward relation (Fourier transform) to the delta, to have
$$S(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = e^{-j\omega\cdot 0} = 1\ ,$$
where we exploited the sifting property of the delta.
2. In this case we apply the inverse relation (inverse Fourier transform) to the delta, to have
$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega)\, e^{j\omega t}\, d\omega = e^{j0\cdot t} = 1$$
where we exploited the sifting property of the delta. Alternatively, we could have used the symmetry rule starting from the couple $x(t) = \delta(t)$ and $X(j\omega) = 1$ of the previous exercise, to have
$$s(t) = X(jt) = 1\ , \qquad S(j\omega) = 2\pi\, x(-\omega) = 2\pi\,\delta(\omega)\ .$$

3. In this case we apply the forward relation (Fourier transform) to the rectangle, to have

$$\begin{aligned}
S(j\omega) &= \int_{-\infty}^{\infty} \mathrm{rect}(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t}\, dt \\
&= \begin{cases} 1 & , \omega = 0 \\ \left. \dfrac{e^{-j\omega t}}{-j\omega} \right|_{-\frac{1}{2}}^{\frac{1}{2}} = \dfrac{e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega}}{j\omega} = \dfrac{2\sin(\frac{1}{2}\omega)}{\omega} & , \omega \neq 0 \end{cases} \\
&= \mathrm{sinc}(\tfrac{\omega}{2\pi})
\end{aligned}$$

The Fourier couple is illustrated in the figure below.



4. In this case we can apply the inverse relation (inverse Fourier transform) to the rect, to have

$$\begin{aligned}
s(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{rect}(\frac{\omega}{2\pi})\, e^{j\omega t}\, d\omega \\
&= \int_{-\infty}^{\infty} \mathrm{rect}(u)\, e^{j2\pi u t}\, du \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j2\pi u t}\, du \\
&= \begin{cases} 1 & , t = 0 \\ \left. \dfrac{e^{j2\pi u t}}{j2\pi t} \right|_{-\frac{1}{2}}^{\frac{1}{2}} = \dfrac{e^{j\pi t} - e^{-j\pi t}}{j2\pi t} = \dfrac{\sin(\pi t)}{\pi t} & , t \neq 0 \end{cases} \\
&= \mathrm{sinc}(t)\ .
\end{aligned}$$

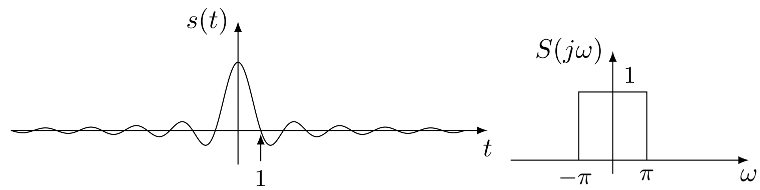
Alternatively, we could have used the symmetry rule starting from the couple $x(t) = \mathrm{rect}(t)$ and $X(j\omega) = \mathrm{sinc}(\frac{\omega}{2\pi})$ of the previous exercise, to have

$$y(t) = X(jt) = \mathrm{sinc}(\tfrac{t}{2\pi})\ , \qquad Y(j\omega) = 2\pi\ \mathrm{rect}(-\omega) = 2\pi\ \mathrm{rect}(\omega)\ .$$

and successively the scale property to obtain the correct couple

$$s(t) = y(2\pi\, t) = \mathrm{sinc}(t)\ , \qquad S(j\omega) = \tfrac{1}{2\pi}\, Y(\tfrac{\omega}{2\pi}) = \mathrm{rect}(\tfrac{\omega}{2\pi})\ .$$

The two approaches are in practice equivalent, since the symmetry rule is exploiting the fact that the inverse Fourier transform (although not exactly in the same form) has already been calculated. The Fourier couple is illustrated in the figure below.



5. In this case we can apply the inverse relation (inverse Fourier transform), to have

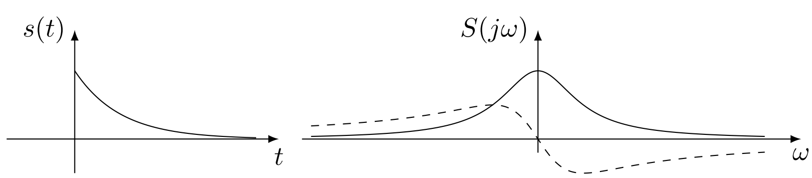
$$
\begin{aligned}
s(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\pi\, e^{j\varphi_1} \delta(\omega - \omega_1) + \pi\, e^{-j\varphi_1} \delta(\omega + \omega_1)\ . \right] e^{j\omega t}\, d\omega \\
&= \tfrac{1}{2}\, e^{j\varphi_1}\, e^{j\omega_1 t} + \tfrac{1}{2}\, e^{-j\varphi_1}\, e^{-j\omega_1 t} \\
&= \tfrac{1}{2}\, e^{j(\omega_1 t + \varphi_1)} + \tfrac{1}{2}\, e^{-j(\omega_1 t + \varphi_1)} \\
&= \cos(\omega_1 t + \varphi_1)
\end{aligned}
$$

where we used the sifting property of the delta.

6. In this case we apply the forward relation (Fourier transform), to obtain

$$
\begin{aligned}
S(j\omega) &= \int_{-\infty}^{\infty} e^{-at}\, 1(t)\, e^{-j\omega t}\, dt \\
&= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt \\
&= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt \\
&= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_{0}^{\infty} \\
&= \frac{1}{a + j\omega} \\
&= \frac{a - j\omega}{a^2 + \omega^2}
\end{aligned}
$$

providing the pair illustrated in the figure below.



FOUNDATIONS OF SIGNALS AND SYSTEMS
13.3 Homework assignment
Prof. T. Erseghe

Exercises 13.3

Prove that the following Fourier transform pairs are correct by either forward or backward relation, or by the symmetry rule:

1. $s(t) = \delta(t - t_1)$ and $S(j\omega) = e^{-j\omega t_1}$,
2. $s(t) = e^{j\omega_1 t}$ and $S(j\omega) = 2\pi\,\delta(\omega - \omega_1)$,
3. $s(t) = \text{triang}(t)$ and $S(j\omega) = \text{sinc}^2(\frac{\omega}{2\pi})$,
4. $s(t) = \text{sinc}^2(t)$ and $S(j\omega) = \text{triang}(\frac{\omega}{2\pi})$,

Then evaluate the Fourier transform of the following signals:

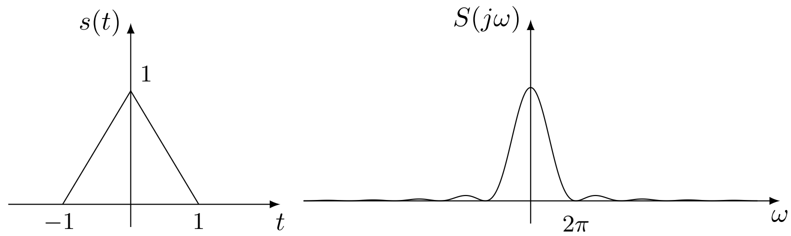
5. $s(t) = e^{-a|t|}$, $a > 0$,
6. $s(t) = t\ \text{rect}(t)$,
7. $s(t) = \cos(\omega_0 t)\ \text{rect}(\frac{\omega_0}{\pi} t)$,

Solutions.

1. In this case we apply the forward relation (Fourier transform) to the delta, to have
$$S(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_1)\, e^{-j\omega t}\, dt = e^{-j\omega t_1} = ,$$
where we exploited the sifting property of the delta.

2. In this case we apply the inverse relation (inverse Fourier transform) to the delta, to have
$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_1)\, e^{j\omega t}\, d\omega = \; e^{j\omega_1 t}$$
where we exploited the sifting property of the delta. Alternatively, we could have used the symmetry rule starting from the couple $x(t) = \delta(t-a)$ and $X(j\omega) = e^{-j\omega a}$ of the previous exercise, to have
$$s(t) = X(jt) = e^{-jat}\;, \qquad S(j\omega) = 2\pi\, x(-\omega) = 2\pi\, \delta(-\omega - a) = 2\pi\, \delta(\omega + a)\;.$$
which reveals the correct couple for $a = -\omega_1$.

3. In this case we apply the forward relation (Fourier transform) to the triangle, to have
$$\begin{aligned} S(j\omega) &= \int_{-\infty}^{\infty} \mathrm{triang}(t)\, e^{-j\omega t}\, dt \\ &= \int_{-1}^{0} (1+t)\, e^{-j\omega t}\, dt + \int_{0}^{1} (1-t)\, e^{-j\omega t}\, dt \end{aligned}$$
which, with a little effort, can be solved through an integration by parts, to give
$$\begin{aligned} S(j\omega) &= \begin{cases} 1 & ,\ \omega = 0 \\ \left.\dfrac{[j - \omega(1+t)]\, e^{-j\omega t}}{j\omega^2}\right|_{-1}^{0} + \left.\dfrac{[-j - \omega(1-t)]\, e^{-j\omega t}}{j\omega^2}\right|_{0}^{1} & ,\ \omega \neq 0 \end{cases} \\ &= \begin{cases} 1 & ,\ \omega = 0 \\ \dfrac{j - \omega - je^{j\omega} - je^{-j\omega} + j + \omega}{j\omega^2} & ,\ \omega \neq 0 \end{cases} \\ &= \begin{cases} 1 & ,\ \omega = 0 \\ \dfrac{2(1-\cos(\omega))}{\omega^2} = \dfrac{4\sin^2(\frac{1}{2}\omega)}{\omega^2} & ,\ \omega \neq 0 \end{cases} \\ &= \mathrm{sinc}^2(\tfrac{\omega}{2\pi}) \end{aligned}$$
The Fourier couple is illustrated in the figure below.



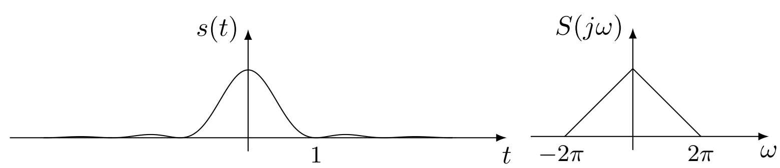
4. In this case we could apply the inverse relation (inverse Fourier transform) to the triangle, and follow an equivalent procedure to the exercise above. Alternatively, we use the symmetry rule starting from the couple $x(t) = \text{triang}(t)$ and $X(j\omega) = \text{sinc}^2(\frac{\omega}{2\pi})$ of the previous exercise, to have

$$y(t) = X(jt) = \text{sinc}^2(\tfrac{t}{2\pi}) \ , \qquad Y(j\omega) = 2\pi \ \text{triang}(-\omega) = 2\pi \ \text{triang}(\omega) \ .$$

and successively the scale property to obtain the correct couple

$$s(t) = y(2\pi\, t) = \text{sinc}^2(t) \ , \qquad S(j\omega) = \tfrac{1}{2\pi}\, Y(\tfrac{\omega}{2\pi}) = \text{triang}(\tfrac{\omega}{2\pi}) \ .$$

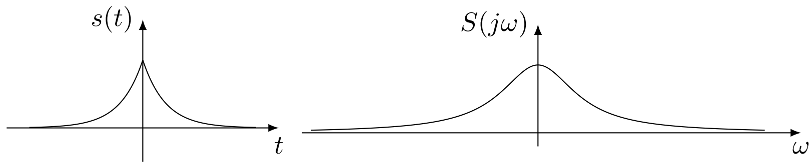
The two approaches are in practice equivalent, since the symmetry rule is exploiting the fact that the inverse Fourier transform (although not exactly in the same form) has already been calculated. The Fourier couple is illustrated in the figure below.



5. In this case we apply the forward relation (Fourier transform), to obtain

$$\begin{aligned}
S(j\omega) &= \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt \\
&= \left. \frac{e^{(a-j\omega)t}}{(a-j\omega)} \right|_{-\infty}^{0} + \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_{0}^{\infty} \\
&= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\
&= \frac{2a}{a^2+\omega^2}
\end{aligned}$$

providing the pair illustrated in the figure below.



6. In this case we apply the forward relation (Fourier transform), to have

$$\begin{aligned} S(j\omega) &= \int_{-\infty}^{\infty} t \text{ rect}(t)\, e^{-j\omega t}\, dt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} t\, e^{-j\omega t}\, dt \end{aligned}$$

which, with a little effort, can be solved through an integration by parts, to give

$$\begin{aligned} S(j\omega) &= \begin{cases} 0 & , \omega = 0 \\ \left.\dfrac{[j - \omega t]\, e^{-j\omega t}}{j\omega^2}\right|_{-\frac{1}{2}}^{\frac{1}{2}} & , \omega \neq 0 \end{cases} \\ &= \begin{cases} 1 & , \omega = 0 \\ \dfrac{[j - \frac{\omega}{2}]e^{-j\frac{\omega}{2}} - [j + \frac{\omega}{2}]e^{j\frac{\omega}{2}}}{j\omega^2} & , \omega \neq 0 \end{cases} \\ &= \begin{cases} 1 & , \omega = 0 \\ j\dfrac{\omega\cos(\frac{\omega}{2}) - 2\sin(\frac{\omega}{2})}{\omega^2} & , \omega \neq 0 \end{cases} \end{aligned}$$

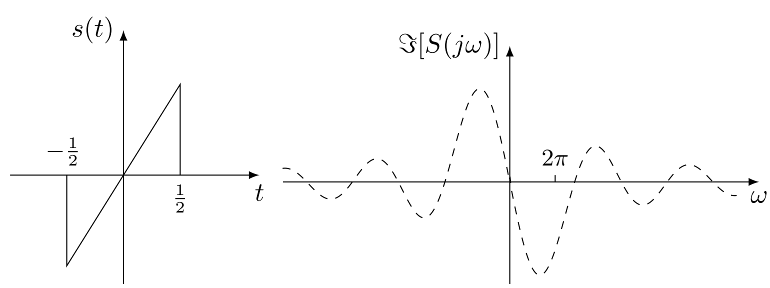
Incidentally, if we observe that

$$\begin{aligned} \text{sinc}'(x) &= \frac{d}{dx}\frac{\sin(\pi x)}{\pi x} \\ &= \frac{\cos(\pi x)}{x} - \frac{\sin(\pi x)}{\pi x^2} = \frac{\pi x \cos(\pi x) - \sin(\pi x)}{\pi x^2} \\ &= \frac{\cos(\pi x) - \text{sinc}(x)}{x} \end{aligned}$$

with $\text{sinc}'(0) = 0$ (by left and right limits), then we can compactly write

$$S(j\omega) = j\tfrac{1}{2\pi}\,\text{sinc}'(\tfrac{\omega}{2\pi})\ .$$

The Fourier couple is illustrated in the figure below.

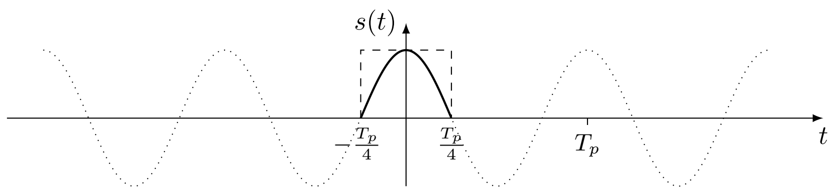


Note that the signal is real and odd, and therefore its transform is imaginary and odd, as we expected from symmetries.

7. The signals is an arc of a cosine, as we can appreciate by setting $\omega_0 = \frac{2\pi}{T_p}$ and by correspondingly writing the signal in the form

$$s(t) = \cos(2\pi \tfrac{t}{T_p})\ \mathrm{rect}(\tfrac{t}{\frac{1}{2}T_p}) \ ,$$

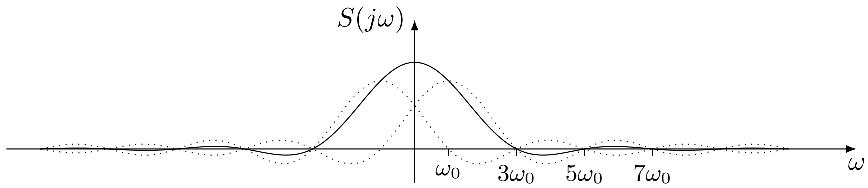
providing the result illustrated in the figure below.



Hence, by applying the forward relation (Fourier transform), we obtain

$$\begin{aligned}
S(j\omega) &= \int_{-\infty}^{\infty} \cos(2\pi \tfrac{t}{T_p})\ \mathrm{rect}(\tfrac{t}{\frac{1}{2}T_p})\, e^{-j\omega t}\, dt \\
&= \int_{-\frac{1}{4}T_p}^{\frac{1}{4}T_p} \cos(2\pi \tfrac{t}{T_p})\, e^{-j\omega t}\, dt \\
&= \int_{-\frac{1}{4}T_p}^{\frac{1}{4}T_p} \tfrac{1}{2}\big[e^{j2\pi\frac{t}{T_p}} + e^{-j2\pi\frac{t}{T_p}}\big]\, e^{-j\omega t}\, dt \\
&= \int_{-\frac{1}{4}T_p}^{\frac{1}{4}T_p} \tfrac{1}{2}\big[e^{j(\frac{2\pi}{T_p}-\omega)t} + e^{-j(\frac{2\pi}{T_p}+\omega)t}\big]\, dt \\
&= \frac{e^{j(\omega_0-\omega)t}}{2j(\omega_0-\omega)} + \frac{e^{-j(\omega_0+\omega)t}}{-2j(\omega_0+\omega)}\Bigg|_{-\frac{1}{4}T_p}^{\frac{1}{4}T_p} \\
&= \frac{\sin((\omega_0-\omega)\frac{1}{4}T_p)}{\omega_0-\omega} + \frac{\sin((\omega_0+\omega)\frac{1}{4}T_p)}{\omega_0+\omega} \\
&= \frac{\sin((\omega-\omega_0)\frac{1}{4}T_p)}{\omega-\omega_0} + \frac{\sin((\omega+\omega_0)\frac{1}{4}T_p)}{\omega+\omega_0} \\
&= \tfrac{1}{4}T_p\ \mathrm{sinc}(\tfrac{\omega-\omega_0}{4\pi/T_p}) + \tfrac{1}{4}T_p\ \mathrm{sinc}(\tfrac{\omega+\omega_0}{4\pi/T_p}) \\
&= \tfrac{\pi}{2\omega_0}\ \mathrm{sinc}(\tfrac{\omega-\omega_0}{2\omega_0}) + \tfrac{\pi}{2\omega_0}\ \mathrm{sinc}(\tfrac{\omega+\omega_0}{2\omega_0})
\end{aligned}$$

where we exploited $\omega_0 = \frac{2\pi}{T_p}$. The Fourier transform is illustrated in the figure below.



$S(j\omega)$
ω
ω_0
$3\omega_0$
$5\omega_0$
$7\omega_0$

FOUNDATIONS OF SIGNALS AND SYSTEMS
13.5 Solved exercises

Prof. T. Erseghe

Exercises 13.5

Solve the following by exploiting the properties of the Fourier transform:

1. prove that any real-valued continuous-time aperiodic signal can be expressed through the trigonometric (and real-valued) integral

$$s(t) = \int_0^\infty \tfrac{1}{\pi}|S(j\omega)|\,\cos(\omega t + \varphi(\omega))\,d\omega$$

   where $S(j\omega) = |S(j\omega)|e^{j\varphi(\omega)}$ is its Fourier transform;

2. evaluate the Fourier transform of $\mathrm{rect}(\frac{t-t_1}{T})$;
3. evaluate the Fourier transform of $x(t)\cos(\omega_0 t)$ as a function of $X(j\omega)$;
4. evaluate the Fourier transform of the signum signal $\mathrm{sgn}(t)$;
5. evaluate the Fourier transform of the hyperbola signal $\frac{j}{\pi t}$;
6. evaluate the Fourier transform of unit step function $1(t)$;
7. evaluate the Fourier transform of $\mathrm{triang}(t)$ knowing the transform pair $\mathrm{rect}(t) \to \mathrm{sinc}(\frac{\omega}{2\pi})$;
8. evaluate the Fourier transform of

$$s(t) = 1 - 3\sin(\tfrac{\pi}{5}t) + 5\cos(3t - \tfrac{\pi}{3}) - 4\cos^2(\tfrac{2\pi}{3}t)\ ;$$

9. evaluate the Fourier transform of $\mathrm{sinc}^2(t)$;
10. evaluate the convolution $\mathrm{sinc} * \mathrm{sinc}(t)$;
11. evaluate the area and the energy of $\mathrm{sinc}(\frac{t-t_1}{T})$.

Solutions.

1. Being the signal real-valued, we exploit the Hermitian symmetry $S(j\omega) = S^*(-j\omega)$ of the Fourier transform, that is $S(-j\omega) = S^*(j\omega) = |S(j\omega)|e^{-j\varphi(\omega)}$.

Hence, we can interpret the inverse Fourier transform as

$$\begin{aligned}
s(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\omega)\, e^{j\omega t} d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} S(j\omega)\, e^{j\omega t} d\omega + \frac{1}{2\pi}\int_{0}^{\infty} S(j\omega)\, e^{j\omega t} d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\infty} S(-ju)\, e^{-jut} du + \frac{1}{2\pi}\int_{0}^{\infty} S(j\omega)\, e^{j\omega t} d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\infty} S^*(ju)\, e^{-jut} du + \frac{1}{2\pi}\int_{0}^{\infty} S(j\omega)\, e^{j\omega t} d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\infty} |S(ju)|\, e^{-j(ut+\varphi(u))} du + \frac{1}{2\pi}\int_{0}^{\infty} |S(j\omega)|\, e^{j(\omega t+\varphi(\omega))} d\omega \\
&= \frac{1}{\pi}\int_{0}^{\infty} |S(j\omega)|\, \cos(\omega t + \varphi(\omega)) d\omega
\end{aligned}$$

Note, however, that this result is true, provided that no delta function is present at $\omega = 0$, and this must be treated separately.

2. We write the signal in the form

$$s(t) = y(t - t_1)\ , \qquad y(t) = x(t/T)\ , \qquad x(t) = \text{rect}(t)$$

by recalling that the expression consists of a scale operation (scale by T) followed by a time shift (by t_1). Hence, by exploiting the Fourier transform of a rectangle, $X(j\omega) = \text{sinc}(\frac{\omega}{2\pi})$, from the scale property we have

$$Y(j\omega) = T X(j\omega T) = T\,\text{sinc}(T\tfrac{\omega}{2\pi}) = T\,\text{sinc}(\tfrac{\omega}{2\pi/T})$$

and from the time-shift property

$$S(j\omega) = Y(j\omega)\, e^{-j\omega t_1} = T\,\text{sinc}(\tfrac{\omega}{2\pi/T})\, e^{-j\omega t_1}\ .$$

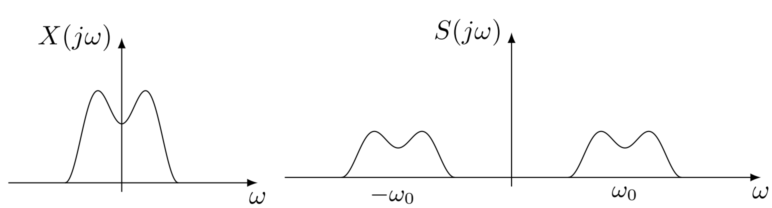
3. By Euler's identity, the signal can be written as

$$s(t) = x(t)\cos(\omega_0 t) = \tfrac{1}{2}\, x(t)\, e^{j\omega_0 t} + \tfrac{1}{2}\, x(t)\, e^{-j\omega_0 t}$$

so that by application of the modulation rule we get

$$S(j\omega) = \tfrac{1}{2}\, X(j(\omega - \omega_0)) + \tfrac{1}{2}\, X(j(\omega + \omega_0))$$

whose effect is depicted in the figure below.



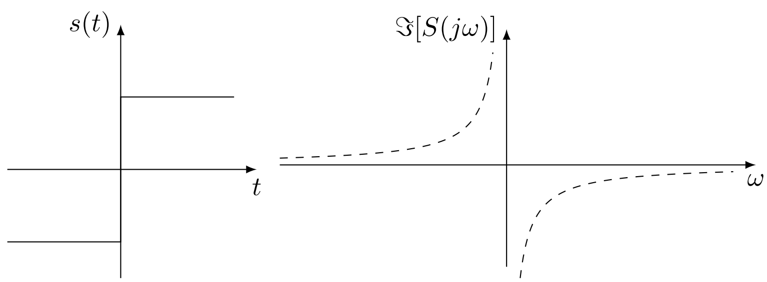
4. In this case, we can resort to the derivative property, since the derivative of $s(t) = \mathrm{sgn}(t)$ is

$$x(t) = s'(t) = 2\,\delta(t)\ , \qquad X(j\omega) = 2\ ,$$

where the Fourier transform is very easy to calculate. By exploiting the inversion rule for the Fourier transform of the derivative, we obtain

$$S(j\omega) = \frac{X(j\omega)}{j\omega} + m_s\,2\pi\,\delta(\omega) = \frac{2}{j\omega} = \frac{-2j}{\omega}\ ,$$

since the sign has $m_s = 0$ mean value. The Fourier couple is illustrated in the figure below.



5. We can infer the Fourier transform of the hyperbola by applying the symmetry rule to the couple of the previous exercise, that is $x(t) = \mathrm{sgn}(t)$ and $X(j\omega) = \frac{-2j}{\omega}$, to have

$$y(t) = X(jt) = \frac{-2j}{t}\ , \qquad Y(j\omega) = 2\pi\,x(-\omega) = -2\pi\,\mathrm{sgn}(\omega)\ .$$

By then defining $s(t) = -\frac{1}{2\pi}y(t)$ we get

$$s(t) = -\frac{1}{2\pi}y(t) = \frac{j}{\pi t}\ , \qquad S(j\omega) = -\frac{1}{2\pi}Y(j\omega) = \mathrm{sgn}(\omega)\ .$$

In oder to demonstrate that there are always many possible ways for deriving results, as an alternative take, we also use the product by t rule. In this case we can write

$$z(t) = t\,s(t) = \frac{j}{\pi}\ , \qquad Z(j\omega) = jS'(j\omega) = \frac{j}{\pi}\cdot 2\pi\,\delta(\omega) = 2j\,\delta(\omega)$$

revealing that

$$S'(j\omega) = 2\,\delta(\omega)\ .$$

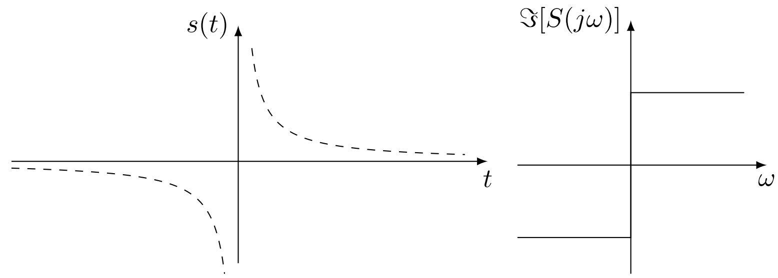
Hence, by integration, it must be

$$S(j\omega) = 2\cdot 1(\omega) + C\ ,$$

for some complex-valued constant C. Now, from the symmetries of the Fourier transform we know that $s(t)$ is imaginary and odd, hence its transform must be real and odd, so that C must be real-valued, and it also must be equal to $C = -1$ in order to guarantee the symmetry, that is

$$S(j\omega) = 2 \cdot 1(\omega) - 1 = \text{sgn}(\omega) \ .$$

The Fourier couple is illustrated in the figure below.



6. For the unit step signal we can exploit the relation

$$s(t) = 1(t) = \tfrac{1}{2}\text{sgn}(t) + \tfrac{1}{2}$$

so that, by resorting to the outcome of the previous exercise, we have

$$S(j\omega) = \tfrac{1}{2} \cdot \frac{-2j}{\omega} + \tfrac{1}{2} \cdot 2\pi\,\delta(\omega) = \frac{-j}{\omega} + \pi\,\delta(\omega) \ .$$

The same result can be obtained by the derivative property since $x(t) = s'(t) = \delta(t)$ and $X(j\omega) = 1$, so that

$$S(j\omega) = \frac{X(j\omega)}{j\omega} + m_s\,2\pi\,\delta(\omega) = \frac{1}{j\omega} + \tfrac{1}{2}\,2\pi\,\delta(\omega) = \frac{-j}{\omega} + \pi\,\delta(\omega) \ ,$$

since the unit step has $m_s = \frac{1}{2}$ mean value.

7. For the function $s(t) = \text{triang}(t)$ we suggest two possible solutions using the Fourier properties. As a first go, we exploit the convolution property by noting that $s(t) = x * x(t)$ where $x(t) = \text{rect}(t)$ whose known Fourier transform is $X(j\omega) = \text{sinc}(\frac{\omega}{2\pi})$, hence it is

$$S(j\omega) = X(j\omega)X(j\omega) = X^2(j\omega) = \text{sinc}^2(\tfrac{\omega}{2\pi}) \ ,$$

which is by far the fastest way. However, given that the triangle is a piecewise-linear function, then we could exploit the derivative property (and its inversion) to have

$$x(t) = s'(t) = \text{rect}(t + \tfrac{1}{2}) - \text{rect}(t - \tfrac{1}{2})$$

whose Fourier transform provides

$$X(j\omega) = \text{sinc}(\tfrac{\omega}{2\pi})\,[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}] = \text{sinc}(\tfrac{\omega}{2\pi})\,2j\sin(\tfrac{\omega}{2})\ .$$

By then inverting the derivative we have

$$S(j\omega) = \frac{X(j\omega)}{j\omega} + m_s\,2\pi\,\delta(\omega) = \frac{\text{sinc}(\frac{\omega}{2\pi})\,2j\sin(\frac{\omega}{2})}{j\omega} = \text{sinc}^2(\tfrac{\omega}{2\pi})\ ,$$

since the triangle has $m_s = 0$ mean value.

8.

9. By exploiting the product property of the Fourier transform, and the known Fourier pair $x(t) = \text{sinc}(t)$ and $X(j\omega) = \text{rect}(\frac{\omega}{2\pi})$, we have

$$s(t) = x(t)x(t)\ , \qquad \tfrac{1}{2\pi}X * X(j\omega) = \tfrac{1}{2\pi}\left[2\pi\,\text{triang}(\frac{\omega}{2\pi})\right] = \text{triang}(\frac{\omega}{2\pi})\ ,$$

where we used the fact that the triangle obtained by the convolution of the two rectangles $X(j\omega)$ has height 2π since this is the extension of each rectangle.

10. In this case it is easier to first evaluate the convolution in the Fourier domain, then obtain the convolution result by inverse Fourier transform. That is, by denoting $x(t) = \text{sinc}(t)$ and $X(j\omega) = \text{rect}(\frac{\omega}{2\pi})$ we have that $s(t) = x * x(t)$ has a Fourier domain counterpart of the form (convolution property)

$$S(j\omega) = X(j\omega)\,X(j\omega) = X^2(j\omega) = \text{rect}^2(\tfrac{\omega}{2\pi}) = \text{rect}(\tfrac{\omega}{2\pi})$$

so that $s(t) = \text{sinc}(t)$.

11. Area and energy do not change under a time-shift, so we need to identify area and energy of $x(t) = \text{sinc}(t/T)$. This can be easily done in the Fourier domain, where (by the scale property)

$$X(j\omega) = T\ \text{rect}(\tfrac{\omega}{2\pi/T})\ ,$$

and we have

$$A_s = A_x = X(j0) = T\ , \qquad E_s = E_x = \tfrac{1}{2\pi}E_X = \tfrac{1}{2\pi}\cdot T^2\,\frac{2\pi}{T} = T\ ,$$

since in the calculation of the energy we used $|X(j\omega)| = T^2\ \text{rect}(\frac{\omega}{2\pi/T})$, and the fact that this is a rectangle of basis $2\pi/T$.

FOUNDATIONS OF SIGNALS AND SYSTEMS
13.6 Homework assignment
Prof. T. Erseghe

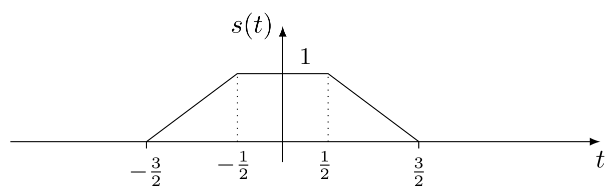
Exercises 13.6

Prove the following properties of the Fourier transform

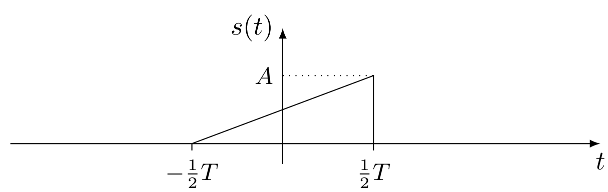
1. the time-shift property $x(t-t_1) \to X(j\omega)\, e^{-j\omega t_1}$;
2. the modulation property $x(t)\, e^{j\omega_0 t} \to X(j(\omega-\omega_0))$;
3. the product by t property $t\, x(t) \to jX'(j\omega)$.

Then solve the following by exploiting the properties of the Fourier transform:

4. evaluate the Fourier transform of t rect(t);
5. evaluate the Fourier transform of $\cos(\omega_0 t)$ rect$(\frac{\omega_0}{\pi}t)$;
6. evaluate the Fourier transform of the signal depicted in figure



7. evaluate the Fourier transform of the signal depicted in figure

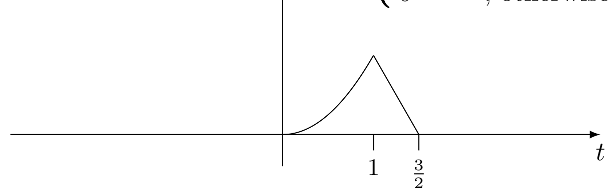


8. evaluate the area of $\text{sinc}^3(t)$;
9. evaluate the convolution between $\text{sinc}(\frac{1}{2}t)$ and $\text{sinc}(\frac{1}{3}t)$;
10. evaluate the convolution between $\text{sinc}(\frac{t}{T_1})$ and $\text{sinc}(\frac{t-t_0}{T_2})$ for $T_1 > T_2$;
11. given the signal $s(t) = t^5\, e^{-4t^2}$ which are valid symmetries for its Fourier transform among: a) real and even, b) imaginary and odd, c) real and odd, d) complex with no symmetries?
12. express the Fourier transform of $v(t) = s(-2t+t_0)$ as a function of $S(j\omega)$;

13. evaluate the Fourier transforms of $e^{-a|t|}$ and $e^{-a|t|}\,\mathrm{sgn}(t)$, $a > 0$, knowing the transform pair $e^{-at}1(t) \to \frac{1}{a+j\omega}$;

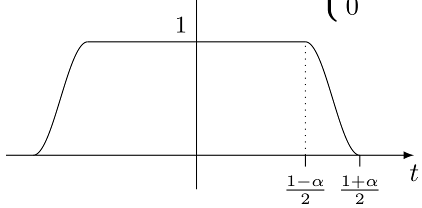
14. evaluate the Fourier transform of the signal depicted in figure

$$s(t) = \begin{cases} \frac{1}{2}t^2 & , 0 < t < 1 \\ \frac{3}{2} - t & , 1 < t < \frac{3}{2} \\ 0 & , \text{otherwise} \end{cases}$$



15. evaluate the Fourier transform of the raised-cosine pulse illustrated below

$$\mathrm{rcos}(t) = \begin{cases} 1 & , |t| < \frac{1-\alpha}{2} \\ \frac{1}{2} - \frac{1}{2}\sin\left(\frac{\pi}{\alpha}\left(|t| - \frac{1}{2}\right)\right) & , \frac{1-\alpha}{2} \le |t| \le \frac{1+\alpha}{2} \\ 0 & , \text{otherwise} \end{cases}$$



Solutions.

1. For the time-shift property, we investigate the Fourier transform of $y(t) = x(t - t_1)$, providing

$$\begin{aligned} Y(j\omega) &= \int_{-\infty}^{\infty} x(t - t_1)\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} x(u)\, e^{-j\omega(u+t_1)}\, du \\ &= e^{-j\omega t_1} \cdot \int_{-\infty}^{\infty} x(u)\, e^{-j\omega u}\, du \\ &= X(j\omega)\, e^{-j\omega t_1} \end{aligned}$$

where we denoted $u = t - t_1$.

2. For the modulation-in-time property, we investigate the Fourier transform of $y(t) = x(t)\, e^{j\omega_1 t}$, providing

$$\begin{aligned} Y(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{j\omega_1 t}\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_1)t}\, dt \\ &= X(j(\omega - \omega_1)) \; . \end{aligned}$$

3. For the product-by-t property, we investigate (j times) the derivative of the Fourier transform of $x(t)$, providing

$$\begin{aligned} jX'(j\omega) &= j\frac{d}{d\omega}\left(\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right) \\ &= \int_{-\infty}^{\infty} x(t)\, \frac{d}{d\omega}\left(j\, e^{-j\omega t}\right)\, dt \\ &= \int_{-\infty}^{\infty} x(t)\, j \cdot (-jt)\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} t \cdot x(t)\, e^{-j\omega t}\, dt \end{aligned}$$

where in the second equivalence we swapped derivative and integration, and where the final result evidences a Fourier transform of $t \cdot x(t)$ hence proving the result.

4. We can exploit the product-by-t property on the couple $x(t) = \text{rect}(t)$ and $X(j\omega) = \text{sinc}(\frac{\omega}{2\pi})$, to have

$$s(t) = t \cdot x(t) \; , \qquad S(j\omega) = jX'(j\omega) = \frac{j}{2\pi}\, \text{sinc}'(\tfrac{\omega}{2\pi}) \; ,$$

which corresponds to the outcome of Exercise 13.3.6. However, we appreciate the simplicity in obtaining the result by applying the properties of the Fourier transform.

5. In this case we can expand the cosine through Euler's identity, to have

$$s(t) = \cos(\omega_0 t)\ \mathrm{rect}(\tfrac{\omega_0}{\pi} t) = \tfrac{1}{2} e^{j\omega_0 t}\ \mathrm{rect}(\tfrac{\omega_0}{\pi} t) + \tfrac{1}{2} e^{-j\omega_0 t}\ \mathrm{rect}(\tfrac{\omega_0}{\pi} t)$$

then exploit the modulation property on the couple

$$x(t) = \mathrm{rect}(\tfrac{\omega_0}{\pi} t)\ , \qquad X(j\omega) = \tfrac{\pi}{\omega_0}\,\mathrm{sinc}(\tfrac{\pi}{\omega_0}\tfrac{\omega}{2\pi}) = \tfrac{\pi}{\omega_0}\,\mathrm{sinc}(\tfrac{\omega}{2\omega_0})$$

which we derived from the rect/sinc pair by use of the scale property. We have

$$\begin{aligned} S(j\omega) &= \tfrac{1}{2}\,X(j(\omega-\omega_0)) + \tfrac{1}{2}\,X(j(\omega+\omega_0)) \\ &= \tfrac{\pi}{2\omega_0}\,\mathrm{sinc}(\tfrac{\omega-\omega_0}{2\omega_0}) + \tfrac{\pi}{2\omega_0}\,\mathrm{sinc}(\tfrac{\omega+\omega_0}{2\omega_0}) \end{aligned}$$

which corresponds to the outcome of Exercise 13.3.7. However, we appreciate the simplicity in obtaining the result by applying the properties of the Fourier transform.

6. There are many different ways to solve this exercise. As a first solution, we observe that the signal can be written as the composition of triangular pulses in (at least) two ways, that is

$$s(t) = \mathrm{triang}(t-\tfrac{1}{2}) + \mathrm{triang}(t+\tfrac{1}{2}) = \tfrac{3}{2}\,\mathrm{triang}(\tfrac{t}{\frac{3}{2}}) - \tfrac{1}{2}\,\mathrm{triang}(\tfrac{t}{\frac{1}{2}})$$

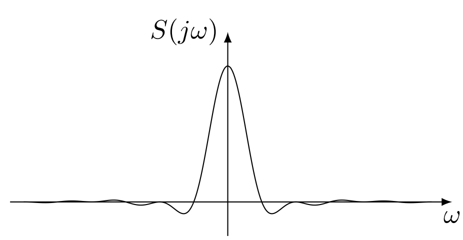
from which the Fourier transform readily follows from the triang/squared-sinc couple and either the time-shift or scale properties. For the first composition we have

$$\begin{aligned} S(j\omega) &= \mathrm{sinc}^2(\tfrac{\omega}{2\pi}) \cdot [e^{-j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}] \\ &= 2\,\mathrm{sinc}^2(\tfrac{\omega}{2\pi})\ \cos(\tfrac{\omega}{2}) \end{aligned}$$

while for the second it is

$$S(j\omega) = \tfrac{9}{4}\mathrm{sinc}^2(\tfrac{3\omega}{4\pi}) - \tfrac{1}{4}\mathrm{sinc}^2(\tfrac{\omega}{4\pi})\ ,$$

where the equivalence between the two expression might be proved by standard trigonometric identities. The resulting transform is illustrated in the figure below.



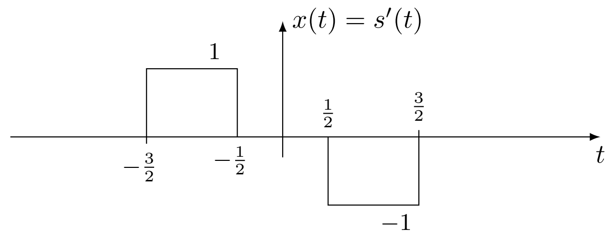
An alternative solution is to recognise that the signal is a trapezoid, hence it can be expressed as the convolution

$$s(t) = x * y(t) \ , \qquad x(t) = \mathrm{rect}(t) \ , \ y(t) = \mathrm{rect}(\tfrac{1}{2}t) \ .$$

Hence, by exploiting the convolution property we have

$$S(j\omega) = X(j\omega)Y(j\omega) = 2\,\mathrm{sinc}(\tfrac{\omega}{2\pi})\,\mathrm{sinc}(\tfrac{\omega}{\pi}) \ ,$$

where the equivalence with the previous results might be again proved by standard trigonometric identities. As a final solution, given that the signal is piecewise linear, we can easily resort to the derivative property (and its inversion rule). The derivative is illustrated in the figure below



and can be written as

$$x(t) = s'(t) = -\,\mathrm{rect}(t-1) + \mathrm{rect}(t+1) \ ,$$

whose Fourier transform is

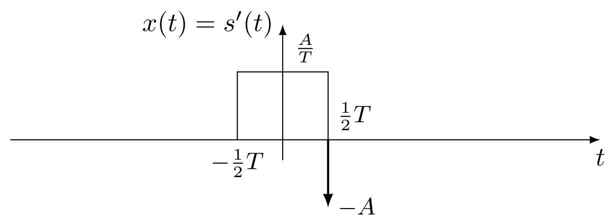
$$\begin{aligned} X(j\omega) &= \mathrm{sinc}(\tfrac{\omega}{2\pi})\,[e^{j\omega} - e^{-j\omega}] \\ &= 2j\,\mathrm{sinc}(\tfrac{\omega}{2\pi})\ \sin(\omega) \end{aligned}$$

where we used the rect/sinc couple and the time-shift property. By inverting the derivative rule (here we exploit the fact that the mean value is $m_s = 0$) we have

$$S(j\omega) = \frac{X(j\omega)}{j\omega} = \mathrm{sinc}(\tfrac{\omega}{2\pi})\,\frac{2\sin(\omega)}{\omega} = 2\,\mathrm{sinc}(\tfrac{\omega}{2\pi})\,\mathrm{sinc}(\tfrac{\omega}{\pi})$$

which perfectly corresponds to the result obtained by the convolution property.

7. In this case we are not able to write the signal as a (simple) function of known waveforms. Although one possibility is to express it as the product between a rectangle and a triangle, the convolution in the Fourier domain between a sinc and a squared-sinc is hardly solvable. However, the signal is piecewise linear with $m_s = 0$, hence the derivative (and its inversion) is a reasonable and simple way to proceed. We follow this path and identify in the figure below the signal derivative



where we note the presence of a delta in $t = \frac{1}{2}T$ due to the discontinuity. Therefore, we have

$$x(t) = \tfrac{A}{T}\,\mathrm{rect}(\tfrac{t}{T}) - A\delta(t - \tfrac{1}{2}T)\;, \qquad X(j\omega) = A\;\mathrm{sinc}(\tfrac{\omega}{2\pi/T}) - A\,e^{-j\omega\frac{T}{2}}$$

and by inversion of the derivative rule it is

$$S(j\omega) = \frac{X(j\omega)}{j\omega} = A\,\frac{\mathrm{sinc}(\frac{\omega}{2\pi/T}) - e^{-j\omega\frac{T}{2}}}{j\omega}\;.$$

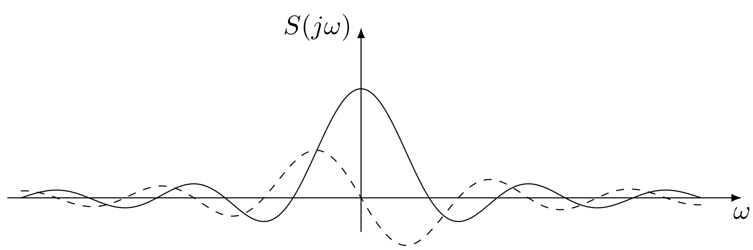
Although this result is a complete solution, we incidentally observe that, with a little effort, it can be re-written in the form

$$\begin{aligned} S(j\omega) &= A\,\frac{\sin(\frac{\omega}{2/T})}{\omega} + jA\,\frac{\cos(\frac{\omega}{2/T}) - \mathrm{sinc}(\frac{\omega}{2\pi/T})}{\omega} \\ &= \tfrac{AT}{2}\;\mathrm{sinc}(\tfrac{\omega}{2\pi/T}) + j\,\tfrac{AT}{2\pi}\;\mathrm{sinc}'(\tfrac{\omega}{2\pi/T}) \end{aligned}$$

where we exploited

$$\mathrm{sinc}'(x) = \frac{\cos(\pi x) - \mathrm{sinc}(x)}{x}\;.$$

The resulting transform is illustrated in the figure below.



As an alternative solution, we observe that we can write the signal in the form

$$\begin{aligned} s(t) &= (\tfrac{A}{2} + \tfrac{A}{T}t)\;\mathrm{rect}(\tfrac{t}{T}) \\ &= \tfrac{A}{2}\;\mathrm{rect}(\tfrac{t}{T}) + \tfrac{A}{T}\cdot t\;\mathrm{rect}(\tfrac{t}{T})\;, \end{aligned}$$

which suggests use of the product-by-t property. Hence by recalling the couple $x(t) = \text{rect}(\frac{t}{T})$ and $X(j\omega) = T\,\text{sinc}(\frac{T\omega}{2\pi})$, by use of the product-by-t property we have

$$\begin{aligned} S(j\omega) &= \tfrac{A}{2}\,X(j\omega) + j\tfrac{A}{T}X'(j\omega) \\ &= \tfrac{AT}{2}\,\text{sinc}(\tfrac{\omega}{2\pi/T}) + j\tfrac{AT}{2\pi}\,\text{sinc}'(\tfrac{\omega}{2\pi/T}) \end{aligned}$$

which perfectly corresponds to the previous solution method.

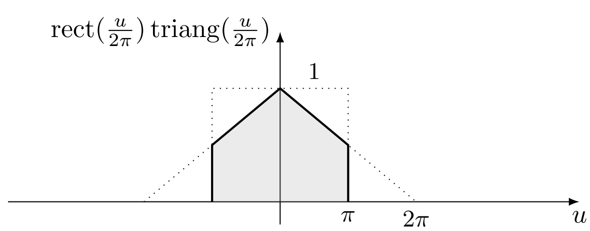
8. The area of $s(t) = \text{sinc}^3(t)$ is hardly derivable in the time domain, since no closed-form primitive is known for the integral. We can therefore exploit the Fourier relation $A_s = S(j0)$, provided that we are able to evaluate the Fourier transform at $\omega = 0$. Now, since both the transforms of $x(t) = \text{sinc}(t)$ and $y(t) = \text{sinc}^2(t)$ are known, respectively, $X(j\omega) = \text{rect}(\frac{\omega}{2\pi})$ and $Y(j\omega) = \text{triang}(\frac{\omega}{2\pi})$, then we can resort to the product property

$$s(t) = x(t)y(t)\ , \qquad S(j\omega) = \frac{1}{2\pi}X * Y(j\omega)\ .$$

Specifically, we do not need to calculate the entire form of the convolution, since we are only interested in the value at $\omega = 0$, that is

$$\begin{aligned} A_s &= S(j0) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(ju)Y(j(0-u))\,du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(ju)Y(-ju)\,du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{rect}(\tfrac{u}{2\pi})\,\text{triang}(\tfrac{u}{2\pi})\,du \end{aligned}$$

where we exploited the fact that the triangle is an even function. Hence, the area of interest is the one illustrated in the figure below



so that, by simple geometric considerations we have

$$A_s = \frac{1}{2\pi}\left[2\pi \cdot \frac{3}{4}\right] = \frac{3}{4}\ .$$

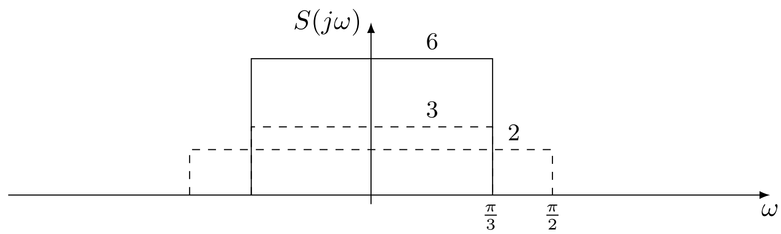
9. The convolution is better understood in the Fourier domain by exploiting the Fourier pairs

$$x(t) = \mathrm{sinc}(\tfrac{1}{2}t)\ , \qquad X(j\omega) = 2\,\mathrm{rect}(\tfrac{\omega}{\pi})$$
$$y(t) = \mathrm{sinc}(\tfrac{1}{3}t)\ , \qquad Y(j\omega) = 3\,\mathrm{rect}(\tfrac{\omega}{\frac{2}{3}\pi})$$

so that, by the convolution property from $s(t) = x * y(t)$ we get

$$S(j\omega) = X(j\omega)Y(j\omega) = 6\ \mathrm{rect}(\tfrac{\omega}{\pi})\ \mathrm{rect}(\tfrac{\omega}{\frac{2}{3}\pi}) = 6\ \mathrm{rect}(\tfrac{\omega}{\frac{2}{3}\pi})\ ,$$

as illustrated in the figure below.



Since $S(j\omega) = 2Y(j\omega)$ then we easily get $s(t) = 2y(t) = 2\ \mathrm{sinc}(\tfrac{1}{3}t)$.

10. This exercise is similar to the previous one, with the addition of a time shift. We have

$$x(t) = \mathrm{sinc}(\tfrac{t}{T_1})\ , \qquad X(j\omega) = T_1\,\mathrm{rect}(\tfrac{\omega}{2\pi/T_1})$$
$$y(t) = \mathrm{sinc}(\tfrac{t-t_0}{T_2})\ , \qquad Y(j\omega) = T_2\,\mathrm{rect}(\tfrac{\omega}{2\pi/T_2})\,e^{-j\omega t_0}$$

hence $s(t) = x * y(t)$ has Fourier transform

$$\begin{aligned} S(j\omega) &= X(j\omega)Y(j\omega) \\ &= T_1T_2\ \mathrm{rect}(\tfrac{\omega}{2\pi/T_1})\ \mathrm{rect}(\tfrac{\omega}{2\pi/T_2})\,e^{-j\omega t_0} \\ &= T_1T_2\ \mathrm{rect}(\tfrac{\omega}{2\pi/T_1})\,e^{-j\omega t_0} \\ &= T_2\,X(j\omega)\,e^{-j\omega t_0} \end{aligned}$$

where we exploited, in the third equality, the fact that $T_1 > T_2$, hence the extension of the first rectangle (which is $2\pi/T_1$) is smaller than the extension of the second rectangle (which satisfies $2\pi/T_2 > 2\pi/T_1$). By inversion, we readily have

$$s(t) = T_2\,x(t - t_0) = T_2\ \mathrm{sinc}(\tfrac{t-t_0}{T_1})\ .$$

11. The signal $s(t) = t^5\,e^{-4t^2}$ is real valued and it also is odd, and in fact

$$s(-t) = (-t)^5\,e^{-4(-t)^2} = -t^5\,e^{-4t^2} = -s(t)\ .$$

Being real valued and odd, its Fourier transform is Hermitian and odd, that is imaginary valued and odd. As a consequence, only b) is valid.

12. We write the signal in the form

$$v(t) = s_-(2t - t_0) = s_-\left(\frac{t - \frac{1}{2}t_0}{\frac{1}{2}}\right)$$

and denote $x(t) = s_-(t) = s(-t)$ with Fourier transform $X(j\omega) = S(-j\omega)$. Hence, from the scale and time-shift properties we have

$$V(j\omega) = \tfrac{1}{2}\, X(j\tfrac{\omega}{2})\, e^{-j\omega \frac{t_0}{2}} = \tfrac{1}{2}\, S(-j\tfrac{\omega}{2})\, e^{-j\frac{\omega}{2} t_0} \ .$$

13. In this case let $s(t) = e^{-at}\, 1(t)$, so that we can write

$$x(t) = e^{-a|t|} = s(t) + s(-t) = 2s_e(t)$$
$$y(t) = e^{-a|t|}\,\mathrm{sgn}(t) = s(t) - s(-t) = 2s_o(t)$$

that is the two signals are related to the even and odd components of the signal $s(t)$. Therefore, by applying the Fourier transform, and the rule on time-reversal, we have

$$X(j\omega) = S(j\omega) + S(-j\omega) = 2S_e(j\omega) = 2\Re[S(j\omega)]$$
$$Y(j\omega) = S(j\omega) - S(-j\omega) = 2S_o(j\omega) = 2j\Im[S(j\omega)]$$

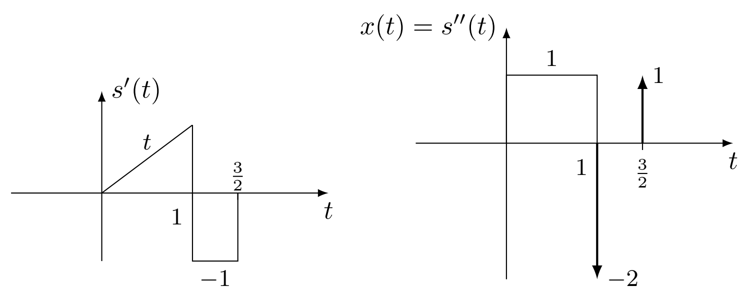
where in the last equivalences we exploited the fact that $s(t)$ is real-valued, hence $S(j\omega)$ has the Hermitian symmetry corresponding to even real part and odd imaginary part. Since

$$S(j\omega) = \frac{1}{a + j\omega} = \frac{a - j\omega}{a^2 + \omega^2}$$

then it is

$$X(j\omega) = \frac{2a}{a^2 + \omega^2} \ , \qquad Y(j\omega) = \frac{-2j\omega}{a^2 + \omega^2} \ .$$

14. In this case the signal is piecewise quadratic, therefore we need to apply the derivative rule twice, to have the derivatives illustrated in figure



that is we have

$$x(t) = s''(t) = \mathrm{rect}(t - \tfrac{1}{2}) - 2\delta(t - 1) + \delta(t - \tfrac{3}{2}) \ ,$$

with Fourier transform

$$X(j\omega) = \mathrm{sinc}(\tfrac{\omega}{2\pi})\, e^{-j\frac{\omega}{2}} - 2e^{-j\omega} + e^{-j\frac{3}{2}\omega} \ .$$

By then inverting (twice) the derivative rule (we observe that the mean value is $m_s = 0$), we finally obtain

$$S(j\omega) = \frac{X(j\omega)}{(j\omega)^2} = -\frac{X(j\omega)}{\omega^2} = \frac{2e^{-j\omega} - e^{-j\frac{3}{2}\omega} - \mathrm{sinc}(\frac{\omega}{2\pi})\, e^{-j\frac{\omega}{2}}}{\omega^2} \ .$$

15. For the raised cosine, we apply the derivation rule (and its inverse). The derivative of the signal has the form

$$x(t) = s'(t) = \mathrm{rcos}(t) = \begin{cases} -\frac{\pi}{2\alpha}\cos(\frac{\pi}{\alpha}(t-\frac{1}{2})) & , -\frac{\alpha}{2} \le t - \frac{1}{2} \le \frac{\alpha}{2} \\ +\frac{\pi}{2\alpha}\cos(\frac{\pi}{\alpha}(t+\frac{1}{2})) & , -\frac{\alpha}{2} \le t + \frac{1}{2} \le \frac{\alpha}{2} \\ 0 & , \text{otherwise} \end{cases}$$

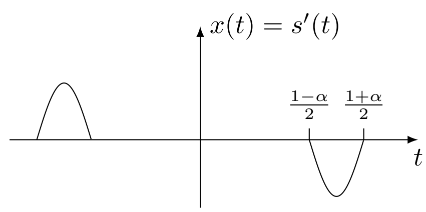
that is

$$\begin{aligned} x(t) &= -\tfrac{\pi}{2\alpha}\cos(\tfrac{\pi}{\alpha}(t-\tfrac{1}{2}))\ \mathrm{rect}(\tfrac{t-\frac{1}{2}}{\alpha}) + \tfrac{\pi}{2\alpha}\cos(\tfrac{\pi}{\alpha}(t+\tfrac{1}{2}))\ \mathrm{rect}(\tfrac{t+\frac{1}{2}}{\alpha}) \\ &= y(t+\tfrac{1}{2}) - y(t-\tfrac{1}{2}) \end{aligned}$$

with

$$y(t) = \tfrac{\pi}{2\alpha}\cos(\tfrac{\pi t}{\alpha})\ \mathrm{rect}(\tfrac{t}{\alpha})$$

as illustrated in the figure below



Therefore for the Fourier transform, by the time-shift property we have

$$X(j\omega) = Y(j\omega)\,[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}] = 2j\,Y(j\omega)\,\sin(\tfrac{\omega}{2}) \ ,$$

and by inversion of the derivative (since the mean value is $m_s = 0$) we have

$$S(j\omega) = \frac{X(j\omega)}{j\omega} = Y(j\omega)\,\frac{\sin(\frac{\omega}{2})}{\frac{\omega}{2}} = Y(j\omega)\ \mathrm{sinc}(\tfrac{\omega}{2\pi}) \ .$$

The Fourier transform of $y(t)$ can be obtained by writing the signal in the form

$$y(t) = z(t)\,\cos(\omega_0 t) \ , \qquad z(t) = \tfrac{\pi}{2\alpha}\ \mathrm{rect}(\tfrac{t}{\alpha}) \ , \ \omega_0 = \tfrac{\pi}{\alpha} \ ,$$

so that by modulation and scale properties we have

$$Y(j\omega) = \tfrac{1}{2}Z(j(\omega-\omega_0)) + \tfrac{1}{2}Z(j(\omega+\omega_0)) \ , \qquad Z(j\omega) = \tfrac{\pi}{2}\ \mathrm{sinc}(\tfrac{\omega}{2\pi/\alpha}) \ .$$

Hence, by combining the results we obtain

$$Y(j\omega) = \tfrac{\pi}{4}\,\mathrm{sinc}(\tfrac{\omega-\frac{\pi}{\alpha}}{2\pi/\alpha}) + \tfrac{\pi}{4}\,\mathrm{sinc}(\tfrac{\omega+\frac{\pi}{\alpha}}{2\pi/\alpha})$$

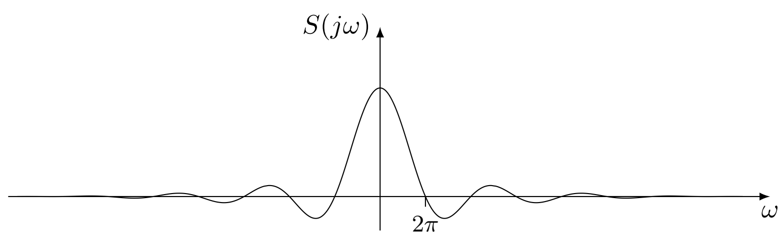
and the transform can be written in the form

$$S(j\omega) = \mathrm{ircos}(\tfrac{\omega}{2\pi})\ ,$$

with

$$\begin{aligned}\mathrm{ircos}(x) &= \tfrac{\pi}{4}\ \mathrm{sinc}(x)\left[\mathrm{sinc}(\alpha x - \tfrac{1}{2}) + \mathrm{sinc}(\alpha x + \tfrac{1}{2})\right] \\ &= \frac{\mathrm{sinc}(x)\ \cos(\pi\alpha x)}{1-(2\alpha x)^2}\ .\end{aligned}$$

The resulting transform is illustrated in the figure below.



FOUNDATIONS OF SIGNALS AND SYSTEMS
14.2 Solved exercises
Prof. T. Erseghe

Exercises 14.2

Solve the following by exploiting the sampling/periodic repetition link between Fourier transforms, then compare the result to those obtained in previous exercises:

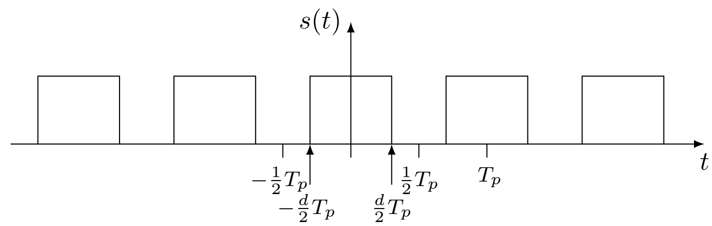
1. evaluate the Fourier coefficients of the square wave $s(t) = \mathrm{rep}_{T_p} \mathrm{rect}(\frac{t}{dT_p})$, $0 < d < 1$ (see Exercise 10.2.3);
2. evaluate the DFT coefficients of the square wave
$$s(n) = \begin{cases} 1 & , n \in [0, M) \pmod N \\ 0 & , \text{otherwise} \end{cases}$$
for $N > M$ (see Exercise 12.5.5).

The above provide the basic application of the concept, which is what essential in this course. Nevertheless, an **advanced** use of the sampling/periodic repetition link can provide (with some non negligible effort) very interesting outcomes, which are made available for the interested readers. As an example, in the following we show how to derive some fundamental equivalences:

3. by exploiting the outcomes of Exercise 12.5.5 and 14.2.2 prove that
$$\mathrm{sinc}_M(t) = \frac{\sin(\pi t)}{M \sin(\pi \frac{t}{M})} = \begin{cases} \mathrm{rep}_M \mathrm{sinc}(t) & , M \text{ odd} \\ \\ \mathrm{rep}_{2M} \mathrm{sinc}(t) - \mathrm{sinc}(t - M) & , M \text{ even} \end{cases}$$
revealing that what we called the periodic sinc function sinc_M is truly a periodic repetition (in some form) of a sinc;
4. by exploiting the outcomes of Exercise 10.2.4 prove that
$$\mathrm{rep}_N \mathrm{sinc}(t) = \mathrm{sinc}_N(t) \cdot \begin{cases} \cos(\frac{\pi}{N} t) & , N \text{ even} \\ 1 & , N \text{ odd} \end{cases}$$
providing a compact result for a periodic repetition of a sinc.

Solutions.

1. The square wave is illustrated in the figure below.



In this case, we have

$$s(t) = \text{rep}_{T_p} x(t) \;, \qquad x(t) = \text{rect}(\tfrac{t}{dT_p}) \;,$$

where from standard Fourier couples we know that

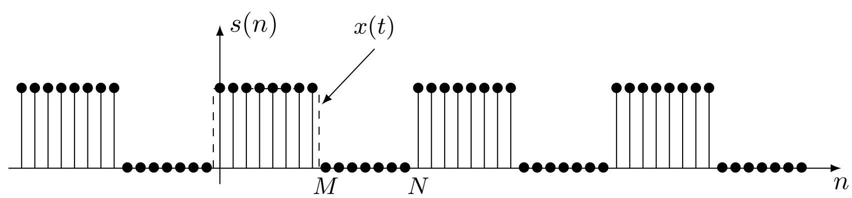
$$X(j\omega) = dT_p \;\text{sinc}(dT_p \tfrac{\omega}{2\pi}) = dT_p \;\text{sinc}(\tfrac{d\omega}{\omega_0}) \;,$$

where $\omega_0 = 2\pi/T_p$, while from the sampling relation in the Fourier domain it is

$$S_k = \frac{1}{T_p} X(jk\omega_0) = \frac{1}{T_p} \cdot dT_p \;\text{sinc}(kd) = d \;\text{sinc}(kd) \;.$$

The result perfectly corresponds to that of Exercise 10.2.3.

2. The square wave is illustrated in the figure below.



This allows writing the signal in the form

$$s(n) = \text{rep}_N x(t)\Big|_{t=n} \;, \qquad x(x) = \text{rect}(\tfrac{t - \frac{1}{2}(M-1)}{M}) \;,$$

where $T_p = N$, and where, by Fourier properties,

$$X(j\omega) = M \;\text{sinc}(M \tfrac{\omega}{2\pi}) \, e^{-j \frac{\omega}{2}(M-1)} \;.$$

From the sampling/periodic repetition relation in the Fourier domain it is (recall that $T_p = N$ and $\omega_0 = 2\pi/T_p = 2\pi/N$)

$$\begin{aligned} S_k &= \text{rep}_N \frac{1}{T_p} X(jk\omega_0) \\ &= \text{rep}_N \tfrac{M}{N} \;\text{sinc}(\tfrac{M}{N} k) \, e^{-j\pi \frac{M-1}{N} k} \;. \end{aligned}$$

The comparison with Exercise 12.5.5 is fully detailed in the next exercise.

3. If we wish to compare the result of the previous exercise, namely

$$S_k = \text{rep}_N \tfrac{M}{N} \;\text{sinc}(\tfrac{M}{N} k) \, e^{-j\pi \frac{M-1}{N} k} \;,$$

with the results of Exercise 12.5.5, namely

$$S_k = \tfrac{M}{N} \;\text{sinc}_M(\tfrac{M}{N} k) \, e^{-j \frac{M-1}{N} k\pi} \;,$$

then we preliminarily need to expand the periodic repetition. We have

$$
\begin{aligned}
S_k &= \sum_{n=-\infty}^{\infty} \tfrac{M}{N}\,\mathrm{sinc}(\tfrac{M}{N}(k-nN))\, e^{-j\pi\frac{M-1}{N}(k-nN)} \\
&= \sum_{n=-\infty}^{\infty} \tfrac{M}{N}\,\mathrm{sinc}(\tfrac{M}{N}k-nM)\, e^{j\pi(M-1)n}\, e^{-j\pi\frac{M-1}{N}k} \\
&= \tfrac{M}{N}\, e^{-j\pi\frac{M-1}{N}k}\cdot \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \mathrm{sinc}(\tfrac{M}{N}k-nM) & , M \text{ odd} \\ \displaystyle\sum_{n=-\infty}^{\infty} \mathrm{sinc}(\tfrac{M}{N}k-nM)\cdot(-1)^n & , M \text{ even} \end{cases} \\
&= \tfrac{M}{N}\, e^{-j\pi\frac{M-1}{N}k}\cdot \begin{cases} \mathrm{rep}_M\,\mathrm{sinc}(x) & , M \text{ odd} \\ \mathrm{rep}_{2M}\,\mathrm{sinc}(x)-\mathrm{sinc}(x-M) & , M \text{ even} \end{cases} \Bigg|_{x=\frac{M}{N}k}
\end{aligned}
$$

which reveals the identity stated in the exercise text.

4. The case for N odd has already been proven in the previous exercise. Therefore we concentrate on N even and look for the Fourier series coefficients of

$$
s(t) = \mathrm{rep}_{T_p} x(t)\ , \qquad x(t) = \mathrm{sinc}(t)\ , \qquad T_p = N\ ,
$$

where from standard Fourier couples we know that

$$
X(j\omega) = \mathrm{rect}(\tfrac{\omega}{2\pi})\ .
$$

From the sampling/ relation in the Fourier domain it is (recall that $T_p = N$ and $\omega_0 = 2\pi/T_p = 2\pi/N$)

$$
S_k = \frac{1}{T_p} X(jk\omega_0) = \tfrac{1}{N}\,\mathrm{rect}(\tfrac{k}{N}) = \begin{cases} \frac{1}{N} & , |k| < \frac{N}{2} \\ \frac{1}{2N} & , |k| = \frac{N}{2} \\ 0 & , \text{otherwise} \end{cases}
$$

since in this case the samples also include the border of the rectangular signal. In a compact way, the resulting coefficients can be written in the form

$$
S_k = \tfrac{1}{2N}\ \mathrm{rect}(\tfrac{k}{N+1}) + \tfrac{1}{2N}\ \mathrm{rect}(\tfrac{k}{N-1})\ .
$$

Now, from Exercise 10.2.4 we know the Fourier pair

$$
x(t) = M\,\mathrm{sinc}_M(\tfrac{Mt}{T_p}) = \frac{\sin(\frac{\pi M t}{T_p})}{\sin(\frac{\pi t}{T_p})}\ , \qquad X_k = \mathrm{rect}(\tfrac{k}{M})\ ,
$$

for M odd, which we use in the present context with $T_p = N$ and $M = N \pm 1$, to have

$$\begin{aligned} s(t) &= \tfrac{1}{2N} \left[\frac{\sin(\frac{\pi(N+1)t}{N})}{\sin(\frac{\pi t}{N})} + \frac{\sin(\frac{\pi(N-1)t}{N})}{\sin(\frac{\pi t}{N})} \right] \\ &= \frac{\sin(\pi t)}{N \sin(\frac{\pi t}{N})} \cos(\tfrac{\pi t}{N}) \end{aligned}$$

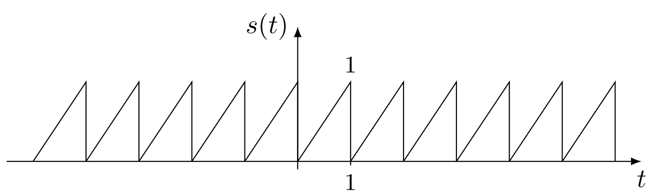
where we exploited the trigonometric equivalence $\sin(\alpha+\beta)+\sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$. This proves the result.

FOUNDATIONS OF SIGNALS AND SYSTEMS
14.3 Homework assignment
Prof. T. Erseghe

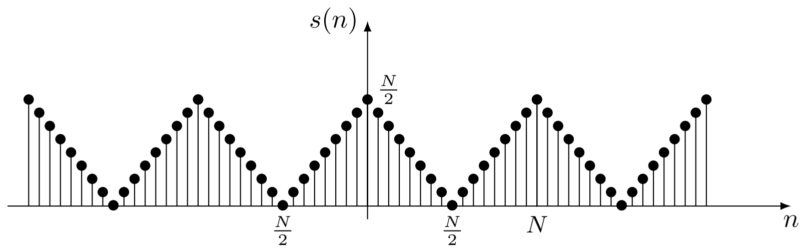
Exercises 14.3

Solve the following by exploiting the sampling/periodic repetition link between Fourier transforms, then compare the result to those obtained in previous exercises and appreciate how the approach can dramatically simplify calculations:

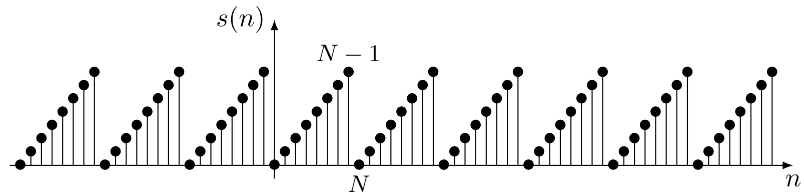
1. evaluate the Fourier coefficients of the rectified sinusoid $s(t) = |\cos(2\pi f_0 t)|$ (see Exercise 10.3.4);
2. evaluate the Fourier coefficients of the triangular wave $s(t) = \text{rep}_{T_p} \text{triang}(\frac{2t}{T_p})$ (see Exercise 10.3.5);
3. evaluate the Fourier coefficients of $s(t) = [\cos(2\pi f_0 t)]^+$, where $[x]^+ = x \cdot 1(x)$ is the positive part operator (see Exercise 10.3.7);
4. evaluate the Fourier coefficients of the saw-tooth waveform (see Exercise 10.6.3)



5. evaluate the Fourier coefficients of the periodic sinc function defined as $s(t) = \text{rep}_{T_p} M \,\text{sinc}(\frac{Mt}{T_p})$ for odd $M = 1 + 2N$ (see Exercise 10.2.4);
6. evaluate the DFT coefficients of $s(n) = |\cos(\frac{2}{M}\pi n)|$ (see Exercise 12.3.4);
7. evaluate the DFT coefficients of the triangular waveform (see Exercise 12.6.5)



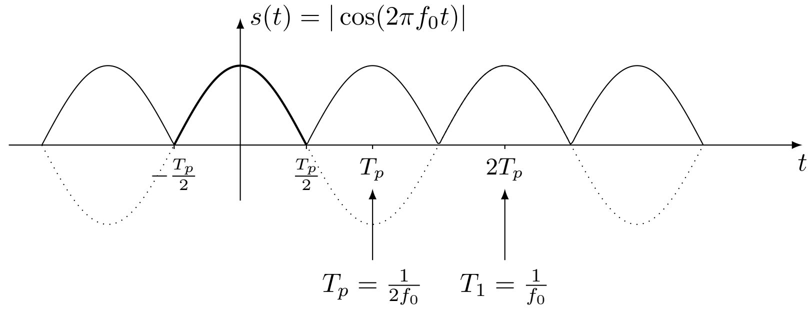
8. evaluate the DFT coefficients of the saw-tooth waveform (see Exercise 12.6.4)



s(n)
N − 1
N
n

Solutions.

1. We consider $f_0 > 0$, with no loss in generality. By mimicking Exercise 10.3.4, the rectified sinusoid is illustrated in the figure below



and allows writing the signal in the form

$$s(t) = \text{rep}_{T_p} x(t) \;, \qquad x(t) = \cos(2\pi f_0 t)\ \text{rect}(\tfrac{t}{T_p}) \;, \quad T_p = \tfrac{1}{2f_0} \;.$$

The Fourier transform of $x(t)$ is easily found to be, by cosine modulation properties,

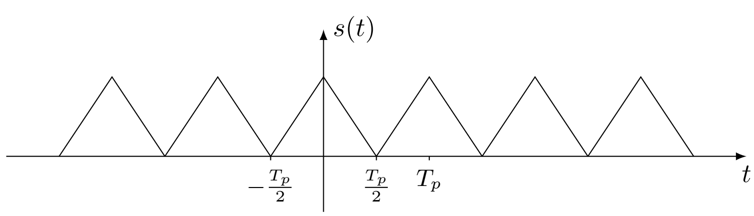
$$\begin{aligned} X(j\omega) &= \tfrac{T_p}{2}\,\text{sinc}(\tfrac{\omega-2\pi f_0}{2\pi/T_p}) + \tfrac{T_p}{2}\,\text{sinc}(\tfrac{\omega+2\pi f_0}{2\pi/T_p}) \\ &= \tfrac{T_p}{2}\,\text{sinc}(\tfrac{\omega}{\omega_0} - \tfrac{1}{2}) + \tfrac{T_p}{2}\,\text{sinc}(\tfrac{\omega}{\omega_0} + \tfrac{1}{2}) \end{aligned}$$

where $\omega_0 = 2\pi/T_p$, while from the sampling relation in the Fourier domain it is

$$S_k = \frac{1}{T_p} X(jk\omega_0) = \tfrac{1}{2}\,\text{sinc}(k - \tfrac{1}{2}) + \tfrac{1}{2}\,\text{sinc}(k + \tfrac{1}{2}) \;,$$

which perfectly corresponds to the result of Exercise 10.3.4, although the present derivation is by far the simplest procedure.

2. The triangular wave is illustrated in the figure below



In this case, we have

$$s(t) = \text{rep}_{T_p} x(t) \;, \qquad x(t) = \text{triang}(\tfrac{t}{\frac{1}{2}T_p}) \;,$$

where from standard Fourier couples we know that

$$X(j\omega) = \tfrac{1}{2}T_p \,\mathrm{sinc}^2(\tfrac{1}{2}T_p \,\tfrac{\omega}{2\pi}) = \tfrac{1}{2}T_p \,\mathrm{sinc}^2(\tfrac{\omega}{2\omega_0}) \ ,$$

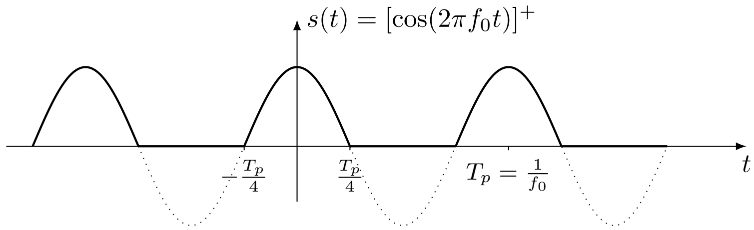
with $\omega_0 = 2\pi/T_p$, while from the sampling relation in the Fourier domain it is

$$S_k = \frac{1}{T_p} X(jk\omega_0) = \tfrac{1}{2}\mathrm{sinc}^2(\tfrac{k}{2}) \ .$$

The result perfectly corresponds to that of Exercise 10.3.5 since

$$\tfrac{1}{2}\mathrm{sinc}^2(\tfrac{k}{2}) = \begin{cases} \frac{1}{2} & , \ k = 0 \\ \frac{2\,\sin^2(\frac{\pi}{2}k)}{(\pi k)^2} = \frac{1-(-1)^k}{(\pi k)^2} & , \ k \neq 0 \end{cases}$$

3. By mimicking Exercise 10.3.7, the signal is illustrated in the figure below



and allows writing

$$s(t) = \mathrm{rep}_{T_p} x(t) \ , \qquad x(t) = \cos(2\pi f_0 t)\ \mathrm{rect}(\tfrac{t}{\frac{1}{2}T_p}) \ , \quad T_p = \tfrac{1}{f_0} \ .$$

The Fourier transform of $x(t)$ is easily found to be, by cosine modulation properties,

$$\begin{aligned} X(j\omega) &= \tfrac{T_p}{4}\,\mathrm{sinc}(\tfrac{\omega-2\pi f_0}{4\pi/T_p}) + \tfrac{T_p}{4}\,\mathrm{sinc}(\tfrac{\omega+2\pi f_0}{4\pi/T_p}) \\ &= \tfrac{T_p}{4}\,\mathrm{sinc}(\tfrac{\omega}{2\omega_0} - \tfrac{1}{2}) + \tfrac{T_p}{4}\,\mathrm{sinc}(\tfrac{\omega}{2\omega_0} + \tfrac{1}{2}) \end{aligned}$$

where $\omega_0 = 2\pi/T_p$, while from the sampling relation in the Fourier domain it is

$$S_k = \frac{1}{T_p} X(jk\omega_0) = \tfrac{1}{4}\,\mathrm{sinc}(\tfrac{k-1}{2}) + \tfrac{1}{4}\,\mathrm{sinc}(\tfrac{k+1}{2}) \ ,$$

which perfectly corresponds to the result of Exercise 10.3.7, although the present derivation is by far the simplest procedure.

4. We write the saw-tooth waveform as

$$s(t) = \mathrm{rep}_{T_p=1} x(t) \ , \qquad x(t) = t\ \mathrm{rect}(t - \tfrac{1}{2}) \ .$$

The Fourier transform of $x(t)$ can be found, by using the product-by-t property, to be

$$\begin{aligned} X(j\omega) &= j\frac{d}{d\omega}\left(\operatorname{sinc}(\tfrac{\omega}{2\pi})\, e^{-j\frac{\omega}{2}}\right) \\ &= \tfrac{1}{2}\ \operatorname{sinc}(\tfrac{\omega}{2\pi})\, e^{-j\frac{\omega}{2}} + \tfrac{j}{2\pi}\operatorname{sinc}'(\tfrac{\omega}{2\pi})\, e^{-j\frac{\omega}{2}} \end{aligned}$$

while from the sampling relation in the Fourier domain it is (recall that $\omega_0 = 2\pi$ in this case)

$$\begin{aligned} S_k &= \frac{1}{T_p} X(jk\omega_0) \\ &= \tfrac{1}{2}\ \operatorname{sinc}(k)\, e^{-j\pi k} + \tfrac{j}{2\pi}\operatorname{sinc}'(k)\, e^{-j\pi k} \\ &= [\tfrac{1}{2}\ \operatorname{sinc}(k) + \tfrac{j}{2\pi}\operatorname{sinc}'(k)]\,(-1)^k \\ &= [\tfrac{1}{2}\ \delta(k) + \tfrac{j}{2\pi}\operatorname{sinc}'(k)]\,(-1)^k \\ &= \tfrac{1}{2}\ \delta(k) + \tfrac{j}{2\pi}\operatorname{sinc}'(k)\,(-1)^k\ , \end{aligned}$$

which corresponds to the result of Exercise 10.6.3 since

$$\operatorname{sinc}'(k) = \frac{\cos(\pi k) - \operatorname{sinc}(k)}{k} = \begin{cases} 0 & , k = 0 \\ \frac{1}{k}\,(-1)^k & , k \neq 0 \end{cases}$$

However, in this specific case, the relation with the Fourier transform involves a much more lengthy derivation.

5. In this case, we have

$$s(t) = \operatorname{rep}_{T_p} x(t)\ , \qquad x(t) = M\ \operatorname{sinc}(\tfrac{t}{\frac{1}{M}T_p})\ ,$$

where

$$X(j\omega) = M\cdot\tfrac{1}{M}T_p\operatorname{rect}(\tfrac{1}{M}T_p\,\tfrac{\omega}{2\pi}) = T_p\operatorname{rect}(\tfrac{\omega}{M\omega_0})\ ,$$

with $\omega_0 = 2\pi/T_p$. From the sampling relation in the Fourier domain it is

$$S_k = \frac{1}{T_p}X(jk\omega_0) = \operatorname{rect}(\tfrac{k}{M}) = \begin{cases} 1 & , |k| \leq N \\ 0 & , \text{otherwise} \end{cases} .$$

We incidentally observe that,

$$\begin{aligned} s(t) &= \operatorname{rep}_{T_p} M\ \operatorname{sinc}(\tfrac{Mt}{T_p}) = \sum_{n=-\infty}^{\infty} M\ \operatorname{sinc}(\tfrac{M(t-nT_p)}{T_p}) \\ &= \sum_{n=-\infty}^{\infty} M\ \operatorname{sinc}(\tfrac{Mt}{T_p} - nM) = M\ \operatorname*{rep}_{M}\operatorname{sinc}(t)\Big|_{t=Mt/T_p} \\ &= M\operatorname{sinc}_M(\tfrac{Mt}{T_p})\ , \end{aligned}$$

where we used the equivalence stated in Exercise 14.2.4. Therefore, the result perfectly corresponds to the outcomes of Exercise 10.2.4

6. We observe that, for M even, the period is $N = \frac{M}{2}$. Therefore we set $T_p = N$ and $T = 1$ and write

$$s(n) = \text{rep}_N x(t)\Big|_{t=n} = \text{rep}_N x(n)\ , \qquad x(t) = \cos(\tfrac{2\pi}{M}t)\ \text{rect}(\tfrac{t}{N})$$

where, by the properties of the Fourier transform, it is

$$\begin{aligned} X(j\omega) &= \tfrac{N}{2}\ \text{sinc}(\tfrac{\omega - 2\pi/M}{2\pi/N}) + \tfrac{N}{2}\ \text{sinc}(\tfrac{\omega+2\pi/M}{2\pi/N}) \\ &= \tfrac{N}{2}\ \text{sinc}(\tfrac{\omega}{\omega_0} - \tfrac{1}{2}) + \tfrac{N}{2}\ \text{sinc}(\tfrac{\omega}{\omega_0} + \tfrac{1}{2}) \end{aligned}$$

with $\omega_0 = 2\pi/T_p = 2\pi/N$. From the sampling+periodic repetition relation in the Fourier domain it straightforwardly is

$$\begin{aligned} S_k &= \text{rep}_N \tfrac{1}{T_p} X(jk\omega_0) \\ &= \text{rep}_N \tfrac{1}{2}\ \text{sinc}(k - \tfrac{1}{2}) + \tfrac{1}{2}\ \text{sinc}(k + \tfrac{1}{2}) \end{aligned}$$

In the comparison with the outcomes of Exercise 12.3.4, stating that

$$S_k = \tfrac{1}{2}\, \text{sinc}_N(k - \tfrac{1}{2})\ \cos((k - \tfrac{1}{2})\tfrac{\pi}{N}) + \tfrac{1}{2}\, \text{sinc}_N(k + \tfrac{1}{2})\ \cos((k + \tfrac{1}{2})\tfrac{\pi}{N})$$

we observe that the two results perfectly coincide because of the equivalence on the periodic repetition of a sinc stated in Exercise 14.2.4.

7. We can write the signal by setting $T_p = N$ and $T = 1$ in the form

$$s(n) = \text{rep}_N x(t)\Big|_{t=n} = \text{rep}_N x(n)\ , \qquad x(t) = \tfrac{N}{2}\ \text{triang}(2t/N)\ ,$$

where, by the properties of the Fourier transform, it is

$$X(j\omega) = \tfrac{N^2}{4}\, \text{sinc}^2(\tfrac{\omega}{4\pi/N}) = \tfrac{N^2}{4}\, \text{sinc}^2(\tfrac{\omega}{2\omega_0})$$

with $\omega_0 = 2\pi/T_p = 2\pi/N$. From the sampling+periodic repetition relation in the Fourier domain it straightforwardly is

$$\begin{aligned} S_k &= \text{rep}_N \tfrac{1}{T_p} X(jk\omega_0) \\ &= \tfrac{N}{4}\ \text{rep}_N \text{sinc}^2(\tfrac{1}{2}k) \end{aligned}$$

In the comparison with the solution obtained in Exercise 12.6.5, stating that

$$S_k = \tfrac{N}{4}\ \text{sinc}^2_{\frac{N}{2}}(\tfrac{1}{2}k)\ ,$$

we revealed yet another link between the periodic repetition of a (squared) sinc and the sinc_M function.

8. We can write the signal by setting $T_p = N$ and $T = 1$ in the form

$$s(n) = \text{rep}_N x(t)\Big|_{t=n} = \text{rep}_N x(n)\ , \qquad x(t) = t\, \text{rect}(\tfrac{t - \frac{N}{2} + \frac{1}{2}}{N})\ ,$$

where, by the properties of the Fourier transform, it is

$$\begin{aligned} X(j\omega) &= j\frac{d}{d\omega}\left(N\ \mathrm{sinc}(\tfrac{\omega}{2\pi/N})\, e^{-j\omega\frac{N-1}{2}}\right) \\ &= \left[\tfrac{N(N-1)}{2}\ \mathrm{sinc}(\tfrac{\omega}{\omega_0}) + j\tfrac{N^2}{2\pi}\ \mathrm{sinc}'(\tfrac{\omega}{\omega_0})\right]\ e^{-j\omega\frac{N-1}{2}} \end{aligned}$$

with $\omega_0 = 2\pi/T_p = 2\pi/N$. From the sampling+periodic repetition relation in the Fourier domain it is

$$\begin{aligned} S_k &= \mathrm{rep}_N\, \tfrac{1}{T_p} X(jk\omega_0) \\ &= \mathrm{rep}_N \left[\tfrac{N-1}{2}\ \mathrm{sinc}(k) + j\tfrac{N}{2\pi}\ \mathrm{sinc}'(k)\right]\ e^{-j2\pi(1-\frac{1}{N})k} \\ &= \mathrm{rep}_N \left[\tfrac{N-1}{2}\delta(k) + j\tfrac{N}{2\pi}\ \mathrm{sinc}'(k)\right]\ e^{j\frac{2\pi}{N}k} \end{aligned}$$

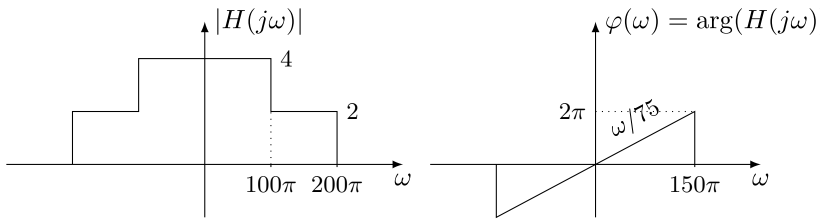
which in any case is a much more involved derivation and result than that obtained in Exercise 12.6.4, which in this specific case is the preferred way.

FOUNDATIONS OF SIGNALS AND SYSTEMS
14.5 Solved exercises
Prof. T. Erseghe

Exercises 14.5

Solve the following filtering exercises by exploiting the Fourier transform approach:

1. We say that a filter does not distort an input $x(t)$ if the output has the form $y(t) = Ax(t - t_0)$, for some real valued constants A (scale) and t_0 (time-shift), in which case the shape of the signal is kept. Is the filter with transfer function



distorting the signal $x(t) = \cos(50\pi t) + 5\cos(120\pi t)$, or not?

2. The signal $x(t) = A\cos^n(\omega_0 t)$, with $\omega_0 > 0$, is fed to an ideal low-pass filter with cut-off pulsation $\omega_c > 0$, that is $H(j\omega) = \text{rect}(\frac{\omega}{2\omega_c})$. Identify, in dependence of the value of n, the range of ω_0 that guarantees $y(t) = x(t)$.

3. A filter has transfer function $H(j\omega) = 1 + j\omega T$. Is this a real filter? Evaluate the filter output corresponding to the input $x(t) = \text{rect}(t/T)$.

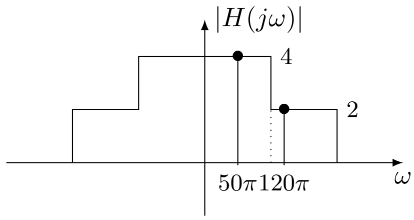
4. The input and output signals of a filter are $x(t) = \text{triang}(\frac{t}{3})$ and $y(t) = \text{triang}(\frac{t+2}{3}) + 2\,\text{triang}(\frac{t}{3}) + 4\,\text{triang}(\frac{t-1}{3})$. Evaluate the filter transfer function $H(j\omega)$, its impulse response $h(t)$, and the response to $x(t) = 1(t)$. Is the filter BIBO stable?

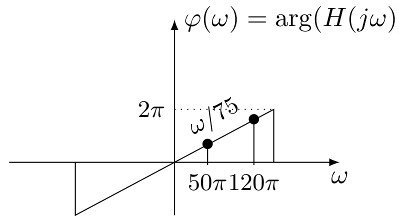
Solutions.

1. We observe that the filter is real, because the Hermitian symmetry in the Fourier domain ensures an even absolute value and an odd phase. Therefore, the output can be found by simple application of the property of a sinusoid through a real filter. Specifically, we have

$$y(t) = |H(j50\pi)|\,\cos(50\pi t + \varphi(50\pi)) + 5\,|H(j120\pi)|\,\cos(120\pi t + \varphi(120\pi))$$

where the values of interest can be inferred from the figure

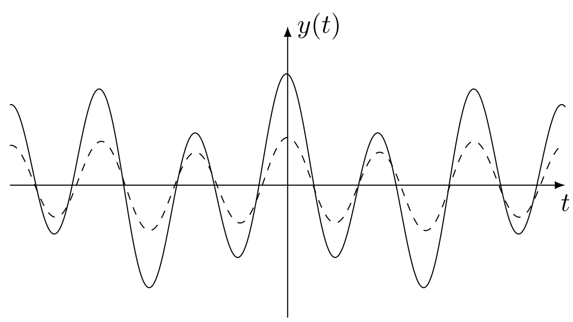




to have

$$\begin{aligned} y(t) &= 4\,\cos(50\pi t + \tfrac{50\pi}{75}) + 5 \cdot 2\,\cos(120\pi t + \tfrac{120\pi}{75}) \\ &= 4\,\cos(50\pi(t + \tfrac{1}{75})) + 10\,\cos(120\pi(t + \tfrac{1}{75})) \end{aligned}$$

and therefore the signal is distorted since the two sinusoids are multiplied by different factors, as one can appreciate from the figure below where the dashed plot is signal $x(t)$.



2. In order for $y(t) = x(t)$, given the low-pass nature of the filter with cut-off pulsation ω_c, it suffices that the extension of the signal in the Fourier domain satisfies $e(X) \in (-\omega_c, \omega_c)$. Since the signal is the power n of a reference signal $s(t) = \cos(\omega_0 t)$ with Fourier extension $e(S) = [-\omega_0, \omega_0]$, then since $X(j\omega)$ is the repeated convolution of $S(j\omega)$ (repeated n times), by the rules of the extension of the convolution we have $e(X) = [-n\omega_0, n\omega_0]$, and therefore we simply need $n\omega_0 < \omega_c$.

3. The filter $H(j\omega) = 1 + j\omega T$ has even real part and odd imaginary part, hence it is Hermitian in the Fourier domain, and real valued in its impulse response. Hence, the filter is real. We evaluate the filter output through the product relation in the Fourier domain, to have

$$Y(j\omega) = X(j\omega)H(j\omega) = T\,\mathrm{sinc}(\frac{\omega}{2\pi/T}) \cdot [1 + j\omega T]$$

which, however, is better written in the form

$$Y(j\omega) = X(j\omega) \cdot [1 + j\omega T] = X(j\omega) + T \cdot j\omega X(j\omega)$$

since, by exploiting the derivative property of the Fourier transform, is mapped in the time-domain signal

$$
\begin{aligned}
y(t) &= x(t) + T\,x'(t) \\
&= \mathrm{rect}(\tfrac{t}{T}) + \mathrm{rect}'(\tfrac{t}{T}) \\
&= \mathrm{rect}(\tfrac{t}{T}) + \delta(\tfrac{t}{T} + \tfrac{1}{2}) - \delta(\tfrac{t}{T} - \tfrac{1}{2}) \\
&= \mathrm{rect}(\tfrac{t}{T}) + T\delta(t + \tfrac{T}{2}) - T\delta(t - \tfrac{T}{2})
\end{aligned}
$$

where we exploited the equivalence $\delta(t/T) = T\delta(t)$.

4. In this case we invert the standard filter relation $Y(j\omega) = X(j\omega)H(j\omega)$ to identify $H(j\omega)$ in what is called a deconvolution. We have

$$
\begin{aligned}
H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\
&= \frac{3\,\mathrm{sinc}^2(\frac{3\omega}{2\pi})\,[e^{j2\omega} + 2 + 4e^{-j\omega}]}{3\,\mathrm{sinc}^2(\frac{3\omega}{2\pi})} \\
&= e^{j2\omega} + 2 + 4e^{-j\omega}
\end{aligned}
$$

whose inverse transform is

$$
h(t) = \delta(t+2) + 2\delta(t) + 4\delta(t-1)\ .
$$

The filter is evidently BIBO stable, since $|h(t)| = h(t)$ and $A_h = 1+2+4 = 7$ is finite. The response to the unit step readily provides

$$
\begin{aligned}
y(t) &= h * 1(t) \\
&= [\delta(t+2) + 2\delta(t) + 4\delta(t-1)] * 1(t) \\
&= 1(t+2) + 2 \cdot 1(t) + 4 \cdot 1(t-1)\ .
\end{aligned}
$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
14.6 Homework assignment
Prof. T. Erseghe

Exercises 14.6

Solve the following filtering exercises by exploiting the Fourier transform approach:

1. The base-band derivative filter $H(j\omega) = j\omega\ \text{rect}(\frac{\omega}{2\omega_c})$ is fed with a sinusoid $x(t) = \cos(\omega_0 t)$. Evaluate the impulse response of the filter, and the filter output as a function of ω_0.

2. The signal $x(t) = A\text{sinc}^n(t/T)$ is fed to n ideal low-pass filter with cut-off pulsation $\omega_c > 0$, that is $H(j\omega) = \text{rect}(\frac{\omega}{2\omega_c})$. Identify, in dependence of the value of n, the range of ω_0 that guarantees $y(t) = x(t)$.

3. A filter has a transfer function of the form $H(j\omega) = e^{-\omega/\omega_0}\,\text{rect}(\frac{\omega}{\omega_0} - \frac{1}{2})$. Is it a real filter? Evaluate its impulse response.

4. Evaluate the output of a filter $H(j\omega) = \text{sinc}(\frac{5\omega}{\pi})$ when the input is $x(t) = e^{-j\frac{\pi}{5}t} + e^{j\frac{\pi}{10}t}$.

5. Evaluate the output to the low-pass filter with impulse response $h(t) = \text{sinc}(t/T)$, $T > 0$, to the input signals $x_1(t) = \cos(\omega_0 t + \frac{\pi}{4})$ and $x_2(t) = \text{sinc}(\frac{\omega_0}{2\pi}(t-5))$ for both $\omega_0 = \frac{2\pi}{5T}$ and $\omega_0 = \frac{6\pi}{T}$.

6. Evaluate the output to the RC filter with impulse response $h(t) = a\,e^{-at}\,1(t)$, $a > 0$, to the input signal $x(t) = A\cos(\omega_0 t)$. What is the value of a that guarantees that $y(t) = B\cos(\omega_0 t - \frac{\pi}{4})$?

7. Evaluate the output to the series of two identical RC filters with impulse response $h_1(t) = h_2(t) = a\,e^{-at}\,1(t)$, $a > 0$, to the input signal $x(t) = A\cos^2(\omega_0 t)$, and specify the output for $a = 2\omega_0$;

Solutions.

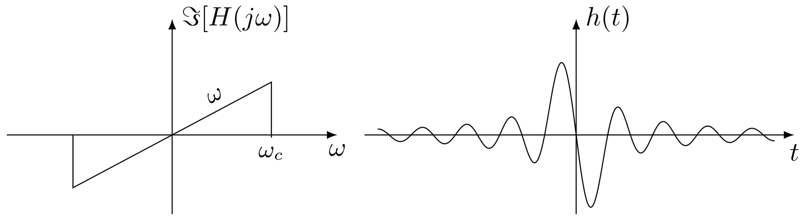
1. The filter can be written in the form

$$H(j\omega) = j\omega\, G(j\omega)\ , \qquad G(j\omega) = \mathrm{rect}(\tfrac{\omega}{4\pi f_c})\ ,$$

where $\omega_c = 2\pi f_c$, so that the inverse transform of $G(j\omega)$ is simply $g(t) = 2f_c\,\mathrm{sinc}(2f_c t)$ by standard application of the Fourier transform properties. By further recalling the derivative property, from the identity $H(j\omega) = j\omega\, G(j\omega)$ we also have

$$h(t) = g'(t) = (2f_c)^2 \mathrm{sinc}'(2f_c t)\ .$$

The filter is illustrated in the figure below.



For the response to the signal $x(t) = \cos(\omega_0 t)$ we can exploit the the property of a sinusoid through a real filter, by recalling that $|H(j\omega)| = |\omega|\,\mathrm{rect}(\frac{\omega}{2\omega_c})$ and $\varphi(\omega) = \frac{\pi}{2}\,\mathrm{sgn}(\omega)$ is almost constant. Therefore we have

$$\begin{aligned} y(t) &= |H(j\omega_0)|\, \cos(\omega_0 t + \varphi(\omega_0)) \\ &= \begin{cases} \omega_0\, \cos(\omega_0 t + \frac{\pi}{2}) & , \omega_0 \in (0, \omega_c) \\ -\omega_0\, \cos(\omega_0 t - \frac{\pi}{2}) & , \omega_0 \in (-\omega_c, 0) \\ 0 & , \text{otherwise} \end{cases} \\ &= \begin{cases} -\omega_0\, \sin(\omega_0 t) = x'(t) & , \omega_0 \in (-\omega_c, \omega_c) \\ 0 & , \text{otherwise} \end{cases} \end{aligned}$$

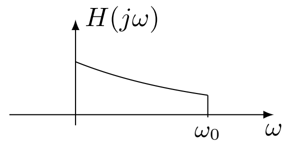
so that, in the active band, the filter acts as a derivative.

2. The signal has the form $x(t) = a\, s^n(t)$ with $s(t) = \mathrm{sinc}(t/T)$ and $S(j\omega) = T\,\mathrm{rect}(\frac{\omega}{2\pi/T})$, hence with Fourier domain extension $e(S) = (-\omega_T, \omega_T)$, $\omega_T = \pi/T$. Therefore, by considering that the n-product maps in the Fourier domain into an n-convolution, we have $e(X) = (-n\omega_T, n\omega_T)$. The output $y(t) = x(t)$ is verified in case the signal extension is included inside the filter cutoff frequency, that is if $n\omega_T \le \omega_c$. As a function of T, we have $T \ge \pi n/\omega_c$.

3. The Fourier domain respponse

$$H(j\omega) = e^{-\omega/\omega_0}\ \mathrm{rect}(\tfrac{\omega - \frac{1}{2}\omega_0}{\omega_0})$$

is real valued but evidently not real-symmetric, as depicted in the figure below.



The impulse response can be obtained by applying the inverse Fourier integral, to have

$$
\begin{aligned}
h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)\, e^{j\omega t}\, dt \\
&= \frac{1}{2\pi}\int_{0}^{\omega_0} e^{-\omega/\omega_0}\, e^{j\omega t}\, dt \\
&= \frac{1}{2\pi}\, \frac{e^{\omega(jt-1/\omega_0)}}{jt-1/\omega_0}\Big|_0^{\omega_0} \\
&= \frac{1}{2\pi}\frac{e^{j\omega_0 t-1}-1}{jt-1/\omega_0} = \frac{\omega_0}{2\pi}\,\frac{1-\frac{1}{e}e^{j\omega_0 t}}{1-j\omega_0 t}
\end{aligned}
$$

4. By applying the rules of filters to a composition of complex exponentials, we have

$$y(t) = H(-j\tfrac{\pi}{5})\, e^{-j\frac{\pi}{5}t} + H(j\tfrac{\pi}{10})\, e^{j\frac{\pi}{10}t}$$

where

$$
\begin{aligned}
H(-j\tfrac{\pi}{5}) &= \mathrm{sinc}(\tfrac{5\cdot -\frac{\pi}{5}}{\pi}) = \mathrm{sinc}(-1) = 0 \\
H(j\tfrac{\pi}{10}) &= \mathrm{sinc}(\tfrac{5\cdot\frac{\pi}{10}}{\pi}) = \mathrm{sinc}(\tfrac{1}{2}) = \frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}} = \tfrac{2}{\pi}
\end{aligned}
$$

so that $y(t) = \frac{2}{\pi}\, e^{j\frac{\pi}{10}t}$.

5. The filter is characterised by the couple

$$h(t) = \mathrm{sinc}(\tfrac{t}{T})\ , \qquad H(j\omega) = T\ \mathrm{rect}(\tfrac{\omega}{2\pi/T})\ .$$

For the input $x(1)$, by exploiting the property of filters with sinusoidal input, we have

$$
\begin{aligned}
y_1(t) &= |H(j\omega_0)|\, \cos(\omega_0 t + \tfrac{\pi}{4} + \varphi(\omega_0)) \\
&= |H(j\omega_0)|\, \cos(\omega_0 t + \tfrac{\pi}{4}) \\
&= |H(j\omega_0)|\, x_1(t) \\
&= \begin{cases} T\, x_1(t) & , |\omega_0| < \pi/T \\ 0 & , \text{otherwise} \end{cases}
\end{aligned}
$$

where we exploited the fact that the filter has zero phase, and the fact that the filter has two active levels, T and 0, above and below the cut pulsation π/T. Hence, for $\omega_0 = \frac{2\pi}{5T} < \frac{\pi}{T}$ the output is $y_1(t) = T\, x_1(t)$, while for $\omega_0 = \frac{6\pi}{T} > \frac{\pi}{T}$ the output is $y_1(t) = 0$.

For the second input signal we need to investigate its action in the Fourier domain, to have (here we denote $T_0 = 2\pi/\omega_0$)

$$\begin{aligned} Y_2(j\omega) &= X_2(j\omega)H(j\omega) \\ &= T_0\,\mathrm{rect}(\tfrac{\omega}{2\pi/T_0})\,e^{-j5\omega}\cdot T\ \mathrm{rect}(\tfrac{\omega}{2\pi/T}) \\ &= T_0\,T\,e^{-j5\omega}\cdot\begin{cases}\mathrm{rect}(\tfrac{\omega}{2\pi/T_0}) & ,\,T_0 > T \\ \mathrm{rect}(\tfrac{\omega}{2\pi/T}) & ,\,T > T_0\end{cases} \end{aligned}$$

since, in the product, the smallest rectangle identifies the output. By inverting to the time-domain, we obtain

$$y_2(t) = \begin{cases} T\,x_2(t) & ,\,\omega_0 < \frac{2\pi}{T} \\ T_0\,h(t-5) & ,\,\omega_0 > \frac{2\pi}{T}\end{cases}$$

where we replaced back ω_0. Therefore, for $\omega_0 = \frac{2\pi}{5T} < \frac{2\pi}{T}$ the output is $y_2(t) = T\,x_2(t)$, while for $\omega_0 = \frac{6\pi}{T} > \frac{2\pi}{T}$ the output is $y_2(t) = T_0\,h(t-5)$.

6. For the RC filter we have (see Exercise 13.2.6)

$$h(t) = a\,e^{-at}\,1(t)\ , \qquad H(j\omega) = \frac{1}{1+j\omega/a} = \frac{1-j\omega/a}{1+(\omega/a)^2}$$

with

$$|H(j\omega)| = \frac{1}{\sqrt{1+(\omega/a)^2}}\ , \qquad \varphi(\omega) = \tan^{-1}(\tfrac{-\omega}{a}) = -\tan^{-1}(\tfrac{\omega}{a})$$

where the expression of the phase was derived by solving

$$\begin{aligned} \Re[H(j\omega)] &= |H(j\omega)|\,\cos(\varphi(\omega)) = \frac{1}{1+(\omega/a)^2} \\ \Im[H(j\omega)] &= |H(j\omega)|\,\sin(\varphi(\omega)) = \frac{-\omega/a}{1+(\omega/a)^2} \end{aligned}$$

to have

$$\frac{\Im[H(j\omega)]}{\Re[H(j\omega)]} = \tan(\varphi(\omega)) = \frac{-\omega}{a}\ .$$

Therefore, by exploiting the filtering properties under a sinusoidal input, we have

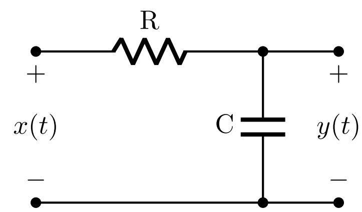
$$\begin{aligned} y(t) &= A\,|H(j\omega_0)|\,\cos(\omega_0 t + \varphi(\omega_0)) \\ &= \frac{A}{\sqrt{1+(\omega_0/a)^2}}\,\cos(\omega_0 t - \tan^{-1}(\tfrac{\omega_0}{a})) \end{aligned}$$

so that we have $y(t) = B\cos(\omega_0 t - \frac{\pi}{4})$ under the condition

$$\tfrac{\pi}{4} = \tan^{-1}(\tfrac{\omega_0}{a}) \quad\Longrightarrow\quad \tfrac{\omega_0}{a} = \tan(\tfrac{\pi}{4}) = 1$$

that is for $a = \omega_0$.

Remark: the name RC filter comes from the fact that the filter action is that of an RC circuit with input voltage $x(t)$ and output voltage $y(t)$, as illustrated in figure



If we denote with $i(t)$ the current flowing through the resistor R and the capacitor C, then the equations determining the link between input and output are of the form

$$x(t) = y(t) + Ri(t)$$
$$i(t) = C\,y'(t)$$

the first considering that the voltage $x(t)$ is the sum of voltages on the resistance R and at the output, while the second carries the link between the capacity voltage $y(t)$ and the current flowing through the capacitor. The relations are better understood in the Fourier domain, where they map into

$$X(j\omega) = Y(j\omega) + R \cdot I(j\omega)$$
$$I(j\omega) = j\omega\, C(j\omega)$$

where we used the derivative property. Hence, by substitution of the second in the first we obtain

$$X(j\omega) = Y(j\omega) \cdot (1 + j\omega RC) \quad \Longrightarrow \quad Y(j\omega) = \frac{X(j\omega)}{1 + j\omega RC}$$

which identifies a transfer function of the form

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

which is equivalent to that of the exercise by setting $a = 1/RC$.

7. We take the transfer function of the RC filter from the previuous exercise, and note that, in the series of two RC filters the overall transfer function takes the form

$$G(j\omega) = H(j\omega)H(j\omega) = H^2(j\omega) = |H(j\omega)|^2\, e^{2\varphi(\omega)}$$

Moreover, the signal has the form

$$x(t) = A\cos^2(\omega_0 t) = \tfrac{A}{2} + \tfrac{A}{2}\, \cos(2\omega_0 t)$$

so that the output is

$$\begin{aligned} y(t) &= \tfrac{A}{2}\, |H(j0)|^2 + \tfrac{A}{2}\, |H(j2\omega_0)|^2\, \cos(2\omega_0 t + 2\varphi(2\omega_0)) \\ &= \frac{A}{2} + \frac{A}{2(1 + (2\omega_0/a)^2)}\, \cos(2\omega_0 t - 2\tan^{-1}(\tfrac{2\omega_0}{a})) \end{aligned}$$

For $a = 2\omega_0$ we have

$$\begin{aligned} y(t) &= \tfrac{A}{2} + \tfrac{A}{4}\ \cos(2\omega_0 t - 2\tan^{-1}(1)) \\ &= \tfrac{A}{2} + \tfrac{A}{4}\ \cos(2\omega_0 t - \tfrac{\pi}{2}) \\ &= \tfrac{A}{2} + \tfrac{A}{4}\ \sin(2\omega_0 t) \end{aligned}$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
15.2 Solved exercises
Prof. T. Erseghe

Exercises 15.2

Solve the following by either using the forward/inverse transform or the properties of the discrete-time Fourier transform:

1. prove the correctness of Fourier pair $s(n) = \delta(n)$ and $S(e^{j\theta}) = 1$;
2. prove the correctness of Fourier pair $s(n) = 1$ and $S(e^{j\theta}) = 2\pi\,\mathrm{comb}_{2\pi}(\theta)$;
3. prove the correctness of Fourier pair $s(n) = d\,\mathrm{sinc}(nd)$, $0 < d < 1$, and $S(e^{j\theta}) = \mathrm{rep}_{2\pi}\,\mathrm{rect}(\frac{\theta}{2\pi d})$;
4. evaluate the discrete-time Fourier transform of $s(n) = e^{j\theta_0 n}$;
5. evaluate the discrete-time Fourier transform of $s(n) = \mathrm{sgn}(n)$;
6. evaluate the discrete-time Fourier transform of $s(n) = 1_0(n)$;
7. evaluate the discrete-time Fourier transform of $s(n) = n\,\alpha^n\,1_0(n)$, $|\alpha| < 1$. For which values of α the absolute value $|S(e^{j\theta})|$ is an even function?

Solutions.

1. We apply the forward transform, to have

$$S(e^{j\theta}) = \sum_{n=-\infty}^{\infty} \delta(n)\,e^{-j\theta n} = 1$$

where we used the sifting properties of the delta.

2. In this case it is convenient to prove the result by inverse transform, to have

$$\begin{aligned} s(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\theta})\,e^{j\theta n}\,d\theta \\ &= \int_{-\pi}^{\pi} \mathrm{rep}_{2\pi}\delta(\theta)\,e^{j\theta n}\,d\theta \\ &= \int_{-\pi}^{\pi} \delta(\theta)\,e^{j\theta n}\,d\theta = 1 \end{aligned}$$

where we used the range $(-\pi, \pi)$ for the integration, in such a way to reveal that the only contribution of $S(e^{j\theta}) = 2\pi\mathrm{rep}_{2\pi}\delta(\theta)$ which is used in the integral is the delta centred in 0.

3. Also in this case it is convenient to proceed by inverse transform, to have

$$
\begin{aligned}
s(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{rep}_{2\pi}\,\mathrm{rect}(\tfrac{\theta}{2\pi d})\, e^{j\theta n}\; d\theta \\
&= \frac{1}{2\pi}\int_{-\pi d}^{\pi d} e^{j\theta n}\; d\theta \\
&= \begin{cases} \frac{2\pi d}{2\pi} = d & , n = 0 \\ \dfrac{e^{j\theta n}}{j2\pi n}\Big|_{-\pi d}^{\pi d} = \dfrac{e^{j\pi dn} - e^{-j\pi dn}}{j2\pi n} = \dfrac{\sin(\pi dn)}{\pi n} & , n \neq 0 \end{cases} \\
&= d\ \mathrm{sinc}(nd)
\end{aligned}
$$

where, again, we used the range $(-\pi, \pi)$ for the integration, in such a way to reveal the only contribution which is used in the integral is the rectangle centred in 0, ranging from $-\pi d$ to πd.

4. By using the Fourier pair of Exercise 15.2.2, namely

$$
x(n) = 1\ , \qquad X(e^{j\theta}) = 2\pi\,\mathrm{comb}_{2\pi}(\theta)\ ,
$$

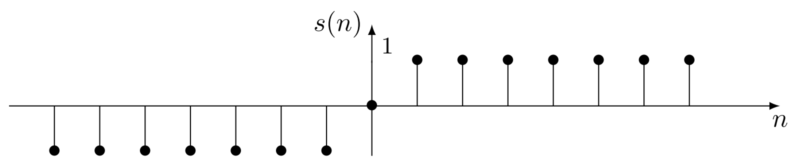
then the transform can be found by exploiting the modulation property since

$$
s(n) = e^{j\theta_0 n} = x(n)\, e^{j\theta_0 n}\ .
$$

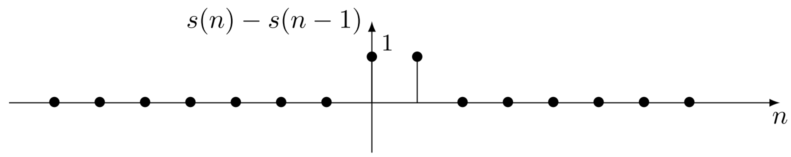
The resulting transform is therefore of the form

$$
S(e^{j\theta}) = X(e^{j(\theta-\theta_0)}) = 2\pi\,\mathrm{comb}_{2\pi}(\theta - \theta_0)\ .
$$

5. For the signum



we can use the increment property. We therefore identify the increment signal



which we can write as

$$
y(n) = s(n) - s(n-1) = \delta(n) + \delta(n-1)
$$

with discrete-time Fourier transform

$$Y(e^{j\theta}) = 1 + e^{-j\theta}$$

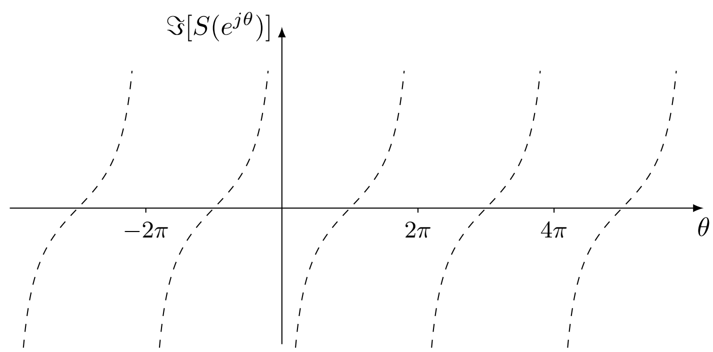
obtained from the properties of deltas and the time-shift property. By inverting the increment we obtain

$$S(e^{j\theta}) = \frac{Y(e^{j\theta})}{1 - e^{-j\theta}} + 2\pi m_s \mathrm{comb}_{2\pi}(\theta) = \frac{1 + e^{-j\theta}}{1 - e^{-j\theta}}$$

since the signum has $m_s = 0$. A more explicit result is obtained by working on the resulting expression to have

$$S(e^{j\theta}) = \frac{1 + e^{-j\theta}}{1 - e^{-j\theta}} \frac{e^{j\frac{\theta}{2}}}{e^{j\frac{\theta}{2}}} = \frac{e^{j\frac{\theta}{2}} + e^{-j\frac{\theta}{2}}}{e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}}} = \frac{2\cos(\frac{\theta}{2})}{2j\sin(\frac{\theta}{2})} = -j\cot(\tfrac{\theta}{2})$$

as illustrated in the figure below.



6. For the discrete unit step $s(n) = 1_0(n)$ we can follow two equivalent paths. If we proceed by the application of the increment we have

$$y(n) = s(n) - s(n-1) = \delta(n) \ , \qquad Y(e^{j\theta}) = 1 \ ,$$

hence by reversing the increment we obtain (recall that we have $m_s = \frac{1}{2}$)

$$S(e^{j\theta}) = \frac{Y(e^{j\theta})}{1 - e^{-j\theta}} + 2\pi m_s \mathrm{comb}_{2\pi}(\theta) = \frac{1}{1 - e^{-j\theta}} + \pi \mathrm{comb}_{2\pi}(\theta) \ .$$

Alternatively, we can recall the link with the signum function, namely

$$s(n) = 1_0(n) = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{sgn}(n) + \tfrac{1}{2}\delta(n)$$

whose Fourier transform reads as

$$S(e^{j\theta}) = \pi \mathrm{comb}_{2\pi}(\theta) - j\tfrac{1}{2}\cot(\tfrac{\theta}{2}) + \tfrac{1}{2} \ ,$$

which is equivalent to the previous result since

$$1 - j\cot(\tfrac{\theta}{2}) = 1 + \frac{1 + e^{-j\theta}}{1 - e^{-j\theta}} = \frac{2}{1 - e^{-j\theta}} \ .$$

7. In this case we can write the signal in the form $s(n) = n\,x(n)$ with $x(n) = \alpha^n\,1_0(n)$, and exploit the product-by-n property. For the transform of $x(n)$ we have

$$
\begin{aligned}
X(e^{j\theta}) &= \sum_{n=-\infty}^{\infty} \alpha^n\,1_0(n)\,e^{-j\theta n} \\
&= \sum_{n=0}^{\infty} (\alpha\,e^{-j\theta})^n = \frac{1}{1-\alpha\,e^{-j\theta}}\ ,
\end{aligned}
$$

so that by use of the product-by-n property we obtain

$$
\begin{aligned}
S(e^{j\theta}) &= j\frac{d}{d\theta}\left(\frac{1}{1-\alpha\,e^{-j\theta}}\right) \\
&= -j\frac{j\,\alpha\,e^{-j\theta}}{(1-\alpha\,e^{-j\theta})^2} = \frac{\alpha\,e^{-j\theta}}{(1-\alpha\,e^{-j\theta})^2}\ .
\end{aligned}
$$

For assessing the parity of the absolute value, we calculate

$$
\begin{aligned}
|S(e^{j\theta})| &= \frac{|\alpha|}{|1-\alpha\,e^{-j\theta}|^2} \\
&= \frac{|\alpha|}{(1-\alpha\,e^{-j\theta})(1-\alpha\,e^{-j\theta})^*} = \frac{|\alpha|}{(1-\alpha\,e^{-j\theta})(1-\alpha^*\,e^{j\theta})} \\
&= \frac{|\alpha|}{1-\alpha\,e^{-j\theta}-\alpha^*\,e^{j\theta}+|\alpha|^2} \\
&= \frac{|\alpha|}{1-|\alpha|\,e^{-j(\theta-\varphi_\alpha)}-|\alpha|\,e^{j(\theta-\varphi_\alpha)}+|\alpha|^2} \\
&= \frac{|\alpha|}{1-2|\alpha|\,\cos(\theta-\varphi_\alpha)+|\alpha|^2}
\end{aligned}
$$

where we used $\alpha = |\alpha|\,e^{j\varphi_\alpha}$. Therefore, the only possibility for having an even function is that $\varphi_\alpha = 0$ or π (to have $\pm\cos(\theta)$), that is a real-valued α (positive or negative).

FOUNDATIONS OF SIGNALS AND SYSTEMS
15.3 Homework assignment
Prof. T. Erseghe

Exercises 15.3

Prove the following properties of the discrete-time Fourier transform:

1. time-reversal property $x(-n) \to X(e^{-j\theta})$;
2. conjugation property $x^*(n) \to X^*(e^{-j\theta})$;
3. time-shift property $x(n-n_0) \to X^*(e^{j\theta})\, e^{-j\theta n_0}$;
4. modulation property $x(n)e^{j\theta_0 n} \to X(e^{j(\theta-\theta_0)})$;
5. convolution property $x * y(n) \to X(e^{j\theta})Y(e^{j\theta})$;
6. product property $x(n)y(n) \to \frac{1}{2\pi} X *_{\text{cir}} Y(e^{j\theta})$.
7. product by n property $n\, x(n) \to j\, X'(e^{j\theta})$.

Then, solve the following by either using the forward/inverse transform or the properties of the discrete-time Fourier transform:

8. evaluate the discrete-time Fourier transform of $\text{rect}(\frac{n}{N})$ for $N = 1 + 2M$;
9. evaluate the discrete-time Fourier transform of $\cos(n\theta_0 + \varphi_0)$;
10. evaluate the discrete-time Fourier transform of $d\, \text{sinc}^2(nd)$, $0 < d < 1$;
11. evaluate the signal whose discrete-time Fourier transform is $2j\, e^{j\theta}/(2 + e^{j\theta})$;
12. evaluate and draw the discrete-time Fourier transform of $s(n) = \frac{3}{4}\, \sin(\frac{\pi}{2}n) + \frac{1}{4}\, \sin(\frac{3\pi}{2}n)$.

Solutions.

1. For the time-reversal property we have $y(n) = x(-n)$ with direct transform
$$\begin{aligned} Y(e^{j\theta}) &= \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\theta n} \\ &= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j(-\theta)m} \\ &= X(e^{-j\theta}) \end{aligned}$$

2. For the conjugation property we have $y(n) = x^*(n)$ with direct transform
$$\begin{aligned} Y(e^{j\theta}) &= \sum_{n=-\infty}^{\infty} x^*(n)\, e^{-j\theta n} \\ &= \left(\sum_{n=-\infty}^{\infty} x(n)\, e^{-j(-\theta)n} \right)^* \\ &= X^*(e^{-j\theta}) \end{aligned}$$
where we used the equality $(e^{-j\theta n})^* = e^{j\theta n} = e^{-j(-\theta)n}$.

3. For the time-shift property we have $y(n) = x(n-n_0)$ with direct transform
$$\begin{aligned} Y(e^{j\theta}) &= \sum_{n=-\infty}^{\infty} x(n-n_0)\, e^{-j\theta n} \\ &= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\theta(m+n_0)} \\ &= e^{-j\theta n_0} \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\theta m} \\ &= X(e^{j\theta})\, e^{-j\theta n_0} \end{aligned}$$
where we used $m = n - n_0$.

4. For the modulation property we have $y(n) = x(n)e^{j\theta_0 n}$ with direct transform
$$\begin{aligned} Y(e^{j\theta}) &= \sum_{n=-\infty}^{\infty} x(n)\, e^{j\theta_0 n}\, e^{-j\theta n} \\ &= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\theta-\theta_0)n} \\ &= X(e^{j(\theta-\theta_0)}) \end{aligned}$$

5. For the convolution property we have $z(n) = x * y(n)$ with direct transform

$$\begin{aligned}
Y(e^{j\theta}) &= \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} x(m)y(n-m) \right) e^{-j\theta n} \\
&= \sum_{m=-\infty}^{\infty} x(m) \left(\sum_{n=-\infty}^{\infty} y(n-m)\, e^{-j\theta n} \right) \\
&= \sum_{m=-\infty}^{\infty} x(m) Y(e^{j\theta})\, e^{-j\theta m} \\
&= X(e^{j\theta}) Y(e^{j\theta})
\end{aligned}$$

where in the second equivalence we swapped the order of sums, and in the third we exploited the time-shift property.

6. For the product property we have $z(n) = x(n)y(n)$ with direct transform

$$\begin{aligned}
Y(e^{j\theta}) &= \sum_{n=-\infty}^{\infty} x(n)y(n)\, e^{-j\theta n} \\
&= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{1}{2\pi} \int_0^{2\pi} Y(e^{jv})\, e^{jvn}\, dv \right) e^{-j\theta n} \\
&= \frac{1}{2\pi} \int_0^{2\pi} Y(e^{jv}) \left(\sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\theta - v)n} \right) dv \\
&= \frac{1}{2\pi} \int_0^{2\pi} Y(e^{jv}) X(e^{j(\theta - v)}) \\
&= \frac{1}{2\pi} X *_{\text{cir}} Y(e^{j\theta})
\end{aligned}$$

where in the second equivalence we used the inverse transform to express $y(n)$, and in the third we swapped the order of sum and integral.

7. For the product-by-n property we derive the discrete-time Fourier transform expression, to have

$$\begin{aligned}
jX'(e^{j\theta}) &= j \frac{d}{d\theta} \left(\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\theta n} \right) \\
&= \sum_{n=-\infty}^{\infty} x(n) \frac{d}{d\theta} \left(j\, e^{-j\theta n} \right) \\
&= \sum_{n=-\infty}^{\infty} x(n) \cdot j \cdot -jn\, e^{-j\theta n} \\
&= \sum_{n=-\infty}^{\infty} n\, x(n)\, e^{-j\theta n}
\end{aligned}$$

revealing it as the discrete-time Fourier transform of $n\,x(n)$.

8. The signal, for $N = 1 + 2M$, has the form

$$s(n) = \mathrm{rect}(\tfrac{n}{N}) = \mathrm{rect}(\tfrac{n}{1+2M}) = \begin{cases} 1 & , |n| \le M \\ 0 & , \text{otherwise} \end{cases}$$

so that its Fourier transform is

$$\begin{aligned} S(e^{j\theta}) &= \sum_{n=-\infty}^{\infty} s(n)\, e^{-j\theta n} = \sum_{n=-M}^{M} e^{-j\theta n} \\ &= \sum_{m=0}^{2M} e^{-j\theta(m-M)} = e^{j\theta M} \sum_{m=0}^{2M} (e^{-j\theta})^m \\ &= e^{j\theta M}\, \frac{1 - e^{-j\theta(1+2M)}}{1 - e^{-j\theta}} \\ &= \frac{e^{j\theta M} - e^{-j\theta(1+M)}}{1 - e^{-j\theta}} \cdot \frac{e^{j\frac{\theta}{2}}}{e^{j\frac{\theta}{2}}} \\ &= \frac{e^{j\theta(M+\frac{1}{2})} - e^{-j\theta(M+\frac{1}{2})}}{e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}}} = \frac{\sin(\frac{\theta N}{2})}{\sin(\frac{\theta}{2})} \\ &= N\, \mathrm{sinc}_N(\tfrac{\theta N}{2\pi}) \end{aligned}$$

9. In this case it is needed to expand the sinusoid by Euler's identity, to have

$$\begin{aligned} s(n) &= \cos(n\theta_0 + \varphi_0) \\ &= \tfrac{1}{2} e^{j\varphi_0} \cdot e^{j\theta_0 n} + \tfrac{1}{2} e^{-j\varphi_0} \cdot e^{-j\theta_0 n} , \end{aligned}$$

Then, by considering the Fourier transform of a complex exponential, we obtain

$$S(e^{j\theta}) = \pi e^{j\varphi_0}\, \mathrm{comb}_{2\pi}(\theta - \theta_0) + \pi e^{-j\varphi_0}\, \mathrm{comb}_{2\pi}(\theta + \theta_0) .$$

10. In this case we can exploit the Fourier pair of Exercise 15.2.3, namely

$$x(n) = d\, \mathrm{sinc}^2(nd) , \qquad X(e^{j\theta}) = \mathrm{rep}_{2\pi}\, \mathrm{rect}(\tfrac{\theta}{2\pi d})$$

to write the signal in the form $s(n) = \frac{1}{d}\, x(n)\, x(n)$, so that by the product property we have

$$\begin{aligned} S(e^{j\theta}) &= \tfrac{1}{2\pi d}\, X_{\mathrm{circ}} X(e^{j\theta}) \\ &= \tfrac{1}{2\pi d} \left(2\pi d\, \mathrm{rep}_{2\pi}\, \mathrm{triang}(\tfrac{\theta}{2\pi d})\right) = \mathrm{rep}_{2\pi}\, \mathrm{triang}(\tfrac{\theta}{2\pi d}) \end{aligned}$$

where we exploited the properties of the circular convolution, and where we took into account that the self convolution of a rectangle of basis $2\pi d$ is a triangle of basis $4\pi d$ and height $2\pi d$. Observe that, when $d > \frac{1}{2}$, the periodic repetition is introducing aliasing.

11. In this case we cannot proceed by integration, since the primitive is not known in this specific case. However, if we write the transform in the form

$$S(e^{j\theta}) = \frac{j\, e^{j\theta}}{1 - (-\frac{1}{2}e^{j\theta})}$$

we can recognise that the denominator is the result of a geometric series, that is

$$\begin{aligned} S(e^{j\theta}) &= j\, e^{j\theta} \sum_{k=0}^{\infty} (-\tfrac{1}{2}e^{j\theta})^k \\ &= \sum_{k=0}^{\infty} j(-\tfrac{1}{2})^k e^{j\theta(k+1)} \\ &= \sum_{m=1}^{\infty} j(-\tfrac{1}{2})^{m-1} e^{j\theta m} \\ &= \sum_{n=-\infty}^{-1} j(-\tfrac{1}{2})^{-n-1} e^{-j\theta n} \end{aligned}$$

where $m = k+1 = -n$. Now the transform is in the form of a discrete-time Fourier transform, revealing that the signal in the time-domain is

$$\begin{aligned} s(n) &= \begin{cases} j(-\frac{1}{2})^{-n-1} = -2j\,(-2)^n & , n < 0 \\ 0 & , n \geq 0 \end{cases} \\ &= -2j\,(-2)^n\, 1_0(-1-n) \end{aligned}$$

12. For the signal

$$s(n) = \tfrac{3}{4}\,\sin(\tfrac{\pi}{2}n) + \tfrac{1}{4}\,\sin(\tfrac{3\pi}{2}n)$$

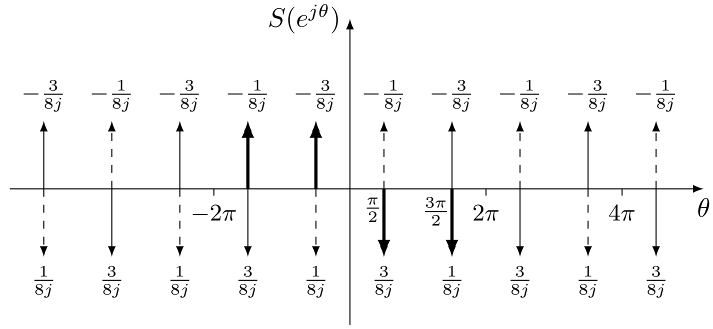
by applying the standard rule on the Fourier transform of a (sampled) sinusoid we have

$$\begin{aligned} S(e^{j\theta}) = \tfrac{3}{8j}\,\mathrm{rep}_{2\pi}\delta(\theta - \tfrac{\pi}{2}) - \tfrac{3}{8j}\,\mathrm{rep}_{2\pi}\delta(\theta + \tfrac{\pi}{2}) \\ + \tfrac{1}{8j}\,\mathrm{rep}_{2\pi}\delta(\theta - \tfrac{3\pi}{2}) - \tfrac{1}{8j}\,\mathrm{rep}_{2\pi}\delta(\theta + \tfrac{3\pi}{2}) \end{aligned}$$

where we observe that, thanks to the periodic repetition (or the sampling, which is equivalent), phase $\frac{3\pi}{2}$ is equivalent to phase $\frac{3\pi}{2} - \pi = -\frac{\pi}{2}$, so that

$$\begin{aligned} S(e^{j\theta}) = \tfrac{3}{8j}\,\mathrm{rep}_{2\pi}\delta(\theta - \tfrac{\pi}{2}) - \tfrac{3}{8j}\,\mathrm{rep}_{2\pi}\delta(\theta + \tfrac{\pi}{2}) \\ + \tfrac{1}{8j}\,\mathrm{rep}_{2\pi}\delta(\theta + \tfrac{\pi}{2}) - \tfrac{1}{8j}\,\mathrm{rep}_{2\pi}\delta(\theta - \tfrac{\pi}{2}) \end{aligned}$$

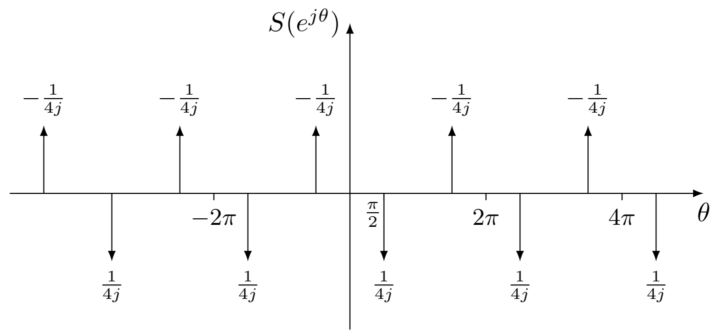
as illustrated in the figure below where deltas in bold represent the four delta functions prior to the periodic repetitions, the deltas in solid lines correspond to the contribution of phase $\frac{\pi}{2}$, and the deltas in dashed lines refer to the contributions of phase $\frac{3\pi}{2}$.



Therefore, we obtain

$$S(e^{j\theta}) = \frac{1}{4j}\,\mathrm{rep}_{2\pi}\delta(\theta - \tfrac{\pi}{2}) - \frac{1}{4j}\,\mathrm{rep}_{2\pi}\delta(\theta - \tfrac{\pi}{2})$$

that is $s(n) = \frac{1}{2}\sin(\frac{\pi}{2}n)$, a result that we could have easily determined by observing that $\sin(\frac{3\pi}{2}n) = \sin(-\frac{\pi}{2}n) = -\sin(\frac{\pi}{2}n)$. The transform is illustrated in the figure below.



FOUNDATIONS OF SIGNALS AND SYSTEMS
15.5 Solved exercises
Prof. T. Erseghe

Exercises 15.5

Solve the following by exploiting the relation between Fourier transform and DTFT:

1. evaluate the discrete-time Fourier transform of $d\,\mathrm{sinc}(nd)$, $0 < d < 1$;

Solutions.

1. In this case we can exploit the sampling relation with $T = 1$ applied to the signal couple

$$x(t) = d\ \mathrm{sinc}(dt)\ , \qquad X(j\omega) = \mathrm{rect}(\tfrac{\omega}{2\pi d})\ ,$$

whose Fourier transform is derived by simple application of the scale property. Since we have $s(n) = x(n)$, from the sampling relation with $T = 1$ we obtain

$$S(e^{j\theta}) = \mathrm{rep}_{2\pi} X(j\theta) = \mathrm{rep}_{2\pi}\,\mathrm{rect}(\tfrac{\theta}{2\pi d})$$

which is a square wave. The result is perfectly equivalent to that of Exercise 15.2.3, although the derivation is in this case effortless.

FOUNDATIONS OF SIGNALS AND SYSTEMS
15.6 Homework assignment
Prof. T. Erseghe

Exercises 15.6

Solve the following by exploiting the relation between Fourier transform and DTFT:

1. evaluate the discrete-time Fourier transform of $\mathrm{rect}(\frac{n}{N})$ for $N = 1 + 2M$;
2. evaluate the discrete-time Fourier transform of $d\,\mathrm{sinc}^2(nd)$, $0 < d < 1$;
3. evaluate the discrete-time Fourier transform of

$$s(n) = \begin{cases} \sin(n\frac{\pi}{N}) & , n \in [0, N] \\ 0 & , \text{otherwise} \end{cases}$$

Solutions.

1. We can exploit the sampling relation with $T = 1$ applied to the signal couple
$$x(t) = \mathrm{rect}(\tfrac{t}{N}) \;, \qquad X(j\omega) = N\,\mathrm{sinc}(\tfrac{\omega}{2\pi/N}) \;,$$
whose Fourier transform is derived by simple application of the scale property. Since we have $s(n) = x(n)$, from the sampling relation with $T = 1$ we obtain
$$\begin{aligned} S(e^{j\theta}) &= \mathrm{rep}_{2\pi} X(j\theta) \\ &= \mathrm{rep}_{2\pi} N\,\mathrm{sinc}(\tfrac{\theta}{2\pi/N}) \end{aligned}$$
which is a complete result. If we wish to relate this result to the periodic repetition of a sinc, providing a periodic sinc, then we need to further work on the outcome, to have
$$\begin{aligned} S(e^{j\theta}) &= \sum_{k=-\infty}^{\infty} N\,\mathrm{sinc}(\tfrac{\theta-k2\pi}{2\pi/N}) \\ &= \sum_{k=-\infty}^{\infty} N\,\mathrm{sinc}(\tfrac{N\theta}{2\pi} - kN) \\ &= N\,\mathrm{rep}_N\,\mathrm{sinc}(x)\Big|_{x=\frac{N\theta}{2\pi}} \\ &= N\,\mathrm{sinc}_N(\tfrac{N\theta}{2\pi}) \end{aligned}$$
which is now perfectly equivalent to the solution of Exercise 15.3.8.

2. We can exploit the sampling relation with $T = 1$ applied to the signal couple
$$x(t) = d\,\mathrm{sinc}^2(td) \;, \qquad X(j\omega) = \mathrm{triang}(\tfrac{\omega}{2\pi d}) \;,$$
whose Fourier transform is derived by simple application of the scale property. Since we have $s(n) = x(n)$, from the sampling relation with $T = 1$ we obtain
$$\begin{aligned} S(e^{j\theta}) &= \mathrm{rep}_{2\pi} X(j\theta) \\ &= \mathrm{rep}_{2\pi}\,\mathrm{triang}(\tfrac{\theta}{2\pi d}) \end{aligned}$$
which is perfectly equivalent to the solution of Exercise 15.3.10.

3. We can exploit the sampling relation with $T = 1$ applied to the signal
$$x(t) = \sin(\tfrac{\pi}{N}t)\;\mathrm{rect}(\tfrac{t-\frac{1}{2}N}{N}) \;,$$
Now the Fourier transform of $x(t)$ can be identified with some effort from the couple
$$y(t) = \mathrm{rect}(\tfrac{t-\frac{1}{2}N}{N}) \;, \qquad Y(j\omega) = N\,\mathrm{sinc}(\tfrac{\omega}{2\pi/N})\,e^{-j\omega\frac{N}{2}} \;,$$

by use of the modulation property, to have

$$
\begin{aligned}
X(j\omega) &= \tfrac{\pi}{j} Y(j(\omega - \tfrac{\pi}{N})) - \tfrac{\pi}{j} Y(j(\omega + \tfrac{\pi}{N})) \\
&= N\pi \left[\operatorname{sinc}(\tfrac{\omega}{2\pi/N} - \tfrac{1}{2}) - \operatorname{sinc}(\tfrac{\omega}{2\pi/N} + \tfrac{1}{2}) \right] e^{-j\omega \frac{N}{2}}
\end{aligned}
$$

Since we have $s(n) = x(n)$, from the sampling relation with $T = 1$ we obtain

$$
\begin{aligned}
S(e^{j\theta}) &= \operatorname{rep}_{2\pi} X(j\theta) \\
&= N\pi \operatorname{rep}_{2\pi} \left\{ \left[\operatorname{sinc}(\tfrac{N\theta}{2\pi} - \tfrac{1}{2}) - \operatorname{sinc}(\tfrac{N\theta}{2\pi} + \tfrac{1}{2}) \right] e^{-j\pi \frac{N\theta}{2\pi}} \right\}
\end{aligned}
$$

It is possible to relate the result to periodic sinc functions. However, we skip this path since the derivation is troublesome.

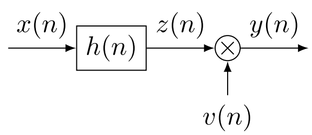
FOUNDATIONS OF SIGNALS AND SYSTEMS
16.2 Solved exercises
Prof. T. Erseghe

Exercises 16.2

Solve the following by exploiting the Fourier transforms approach to filters:

1. Identify the class of discrete-time filters such that $x(n) = e^{-j\frac{\pi}{6}n}$ and $y(n) = \frac{1}{8}e^{-j\frac{\pi}{6}n}$.

2. The signal $x(n) = \delta(n-1) - \delta(n+1)$ is first filtered by an ideal low-pass filter $h(n)$ with cut-phase $\theta_c = \frac{\pi}{2}$, then the output is multiplied by $v(n) = 1 - e^{j\pi n} = 1 - (-1)^n$ to get the output $y(n)$, as illustrated in figure



   Identify $y(n)$.

3. Identify the impulse response of a discrete-time high-pass filter with cut-phase $\theta_c \in (0, \pi)$.

4. The signal

$$x(n) = \sum_{k=-\infty}^{\infty} \text{triang}(\tfrac{1}{2}(n-8k)) - \text{triang}(\tfrac{1}{2}(n-4-8k)) \ ,$$

   is fed to an ideal high-pass filter with cut-phase $\theta_c = \frac{\pi}{2}$. Evaluate the output $y(n)$.

Solutions.

1. Obviously, the filter with $h(n) = \frac{1}{6}\delta(n)$ is a viable solution, but, in general, since in this case it is $y(n) = H(e^{j\frac{\pi}{6}})\, x(n)$, it suffices to have $H(e^{j\frac{\pi}{6}}) = \frac{1}{6}$, which the only effective constraint required to the class.

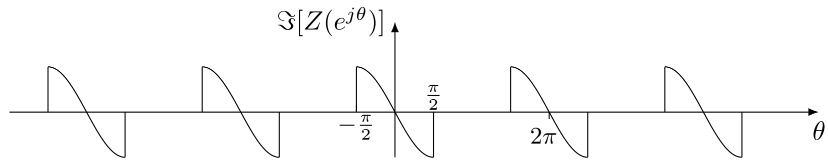
2. We proceed by analysing the system in the Fourier domain. For the input, by transforming the two deltas we have

$$X(e^{j\theta}) = e^{-j\theta} - e^{j\theta} = -2j\sin(\theta) \ .$$

   The filter is, by assumption, $H(e^{j\theta}) = \text{rep}_{2\pi}\,\text{rect}(\frac{\theta}{\pi})$ (i.e., a square wave with duty-cycle $d = \frac{1}{2}$), so that

$$\begin{aligned} Z(e^{j\theta}) &= X(e^{j\theta})\, H(e^{j\theta}) \\ &= -2j\sin(\theta) \cdot \text{rep}_{2\pi}\,\text{rect}(\tfrac{\theta}{\pi}) \\ &= \text{rep}_{2\pi} - 2j\sin(\theta)\ \text{rect}(\tfrac{\theta}{\pi}) \end{aligned}$$

as illustrated in the figure below



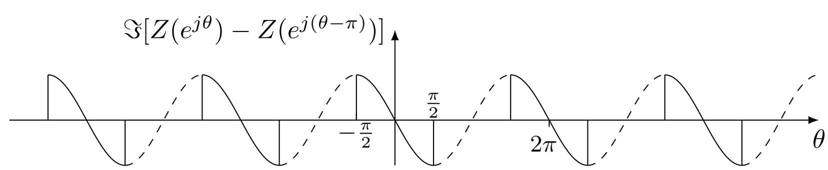
The Fourier transform of $y(n)$ can be inferred from the relation

$$y(n) = z(n)\, v(n) = z(n) - z(n)\, e^{j\pi n}$$

providing, by use of the modulation property

$$Y(e^{j\theta}) = Z(e^{j\theta}) - Z(e^{j(\theta-\pi)})$$

where the contribution added is illustrated in dashed lines in the figure below



Hence, we obtain

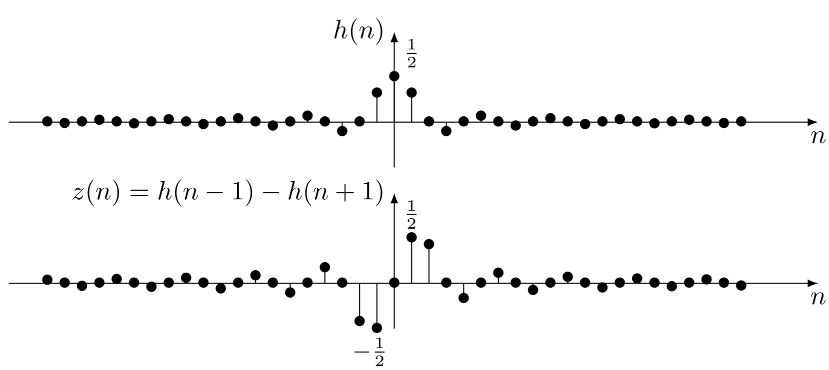
$$Y(e^{j\theta}) = -2j\sin(\theta)$$

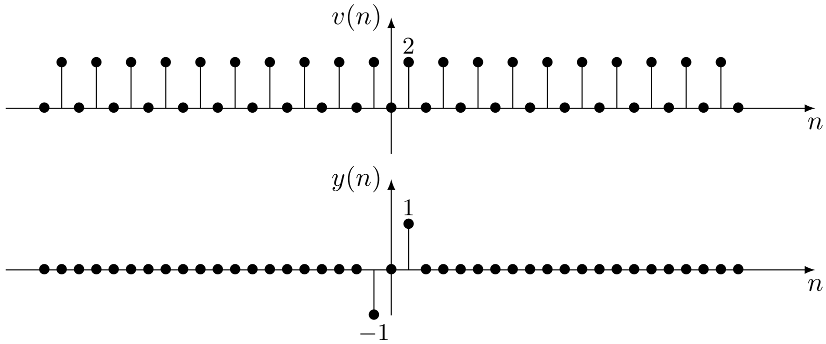
and therefore $y(n) = x(n)$.

This exercise can also be solved in the time domain, which can be done graphically, by observing that $h(n) = \frac{1}{2}\,\text{sinc}(\frac{n}{2})$ and therefore $z(n) = x * h(n) = h(n-1) - h(n+1)$, while it is

$$v(n) = 1 - (-1)^n = \begin{cases} 2 & , n \text{ odd} \\ 0 & , n \text{ even} \end{cases}$$

Therefore, the outcome is the one depicted in the figure below





From the figure we see that $h(n)$ is zero for n even, except at $n = 0$, which implies that $z(n)$ is zero for n odd, except at $n = \pm 1$. Since $v(n)$ is zero for n even, the only two values that are maintained in the product $y(n) = z(n)v(n)$ are the ones at $n = \pm 1$, which provides the result $y(n) = x(n)$.

3. The filter expression, in the Fourier domain is

$$H(e^{j\theta}) = 1 - \underset{2\pi}{\operatorname{rep}} \operatorname{rect}(\tfrac{\theta}{2\theta_c})$$

whose inverse Fourier transform can be obtained by separately inverting the two contributions. For the constant contribution, we exploit the Fourier couple $\delta(n) \to 1$, while for the periodic repetition of the rectangle we can interpret it as a sampling operation applied to the Fourier-transform couple

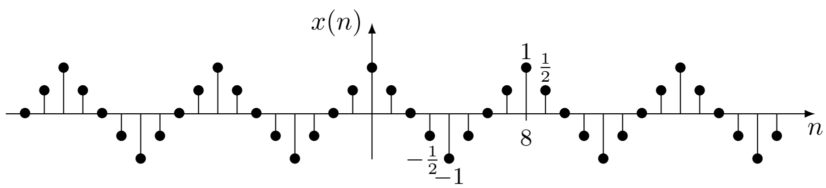
$$\tfrac{\theta_c}{\pi} \operatorname{sinc}(\tfrac{\theta_c}{\pi} t) \quad \to \quad \operatorname{rect}(\tfrac{\omega}{2\theta_c})$$

so that

$$h(n) = \delta(n) - \tfrac{\theta_c}{\pi} \operatorname{sinc}(\tfrac{\theta_c}{\pi} n) \ ,$$

where $d = \frac{\theta_c}{\pi}$ is the duty-cycle of the square wave.

4. The signal, depicted in the figure below



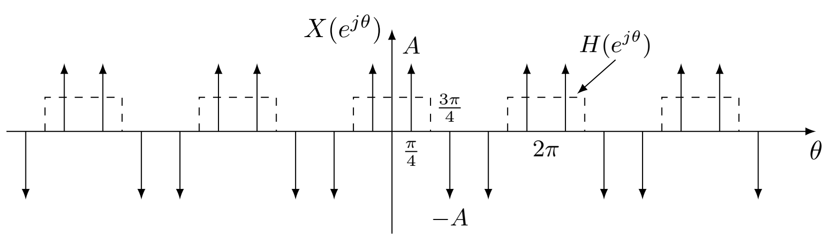
is periodic of period $N = 8$. Its DFT can be computed by hand, and we exploit the even symmetry by setting the range to $[-3, 4]$, to have

$$
\begin{aligned}
X_k &= \tfrac{1}{8} \sum_{n=-3}^{4} x(n)\, e^{-jk\frac{\pi}{4}n} \\
&= \tfrac{1}{8}\Big(1 + \tfrac{1}{2}e^{-jk\frac{\pi}{4}} + \tfrac{1}{2}e^{jk\frac{\pi}{4}} - \tfrac{1}{2}e^{-jk\frac{3\pi}{4}} - \tfrac{1}{2}e^{jk\frac{3\pi}{4}} - e^{-jk\pi}\Big) \\
&= \tfrac{1}{8}\Big(1 + \cos(\tfrac{\pi}{4}k) - \cos(\tfrac{3\pi}{4}k) - (-1)^k\Big) \\
&= \begin{cases} 0 & , k \text{ even} \\ A & , k = \pm 1 \pmod 8 \\ -A & , k = \pm 3 \pmod 8 \end{cases} \qquad A = \frac{2+\sqrt{2}}{8} \ ,
\end{aligned}
$$

which reveals a discrete-time Fourier transform with even symmetry of the form

$$
X(e^{j\theta}) = \sum_{k \in \{\pm 3, \pm 1\}} 2\pi X_k \,\mathrm{comb}_{2\pi}(\theta - \tfrac{\pi}{4}k) \ .
$$

as illustrated in the figure below.



Therefore, after filtering it is

$$
Y(e^{j\theta}) = X(e^{j\theta}) \cdot H(e^{j\theta}) = \sum_{k \in \{\pm 1\}} 2\pi A \,\mathrm{comb}_{2\pi}(\theta - \tfrac{\pi}{4}k) \ .
$$

or, equivalently, at the DFT level

$$
Y_k = X_k\, H(e^{j\frac{\pi}{4}k}) = \begin{cases} A & , k = \pm 1 \pmod 8 \\ 0 & , \text{otherwise} \end{cases}
$$

so that

$$
y(n) = \sum_{k=\pm 1} A\, e^{jk\frac{\pi}{4}n} = A\, e^{j\frac{\pi}{4}n} + A\, e^{-j\frac{\pi}{4}n} = 2A\, \cos(\tfrac{\pi}{4}n) \ .
$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
16.3 Homework assignment
Prof. T. Erseghe

Exercises 16.3

Solve the following by exploiting the Fourier transforms approach to filters:

1. Identify the class of discrete-time filters such that $x(n) = e^{-j\frac{\pi}{6}n}$ and $y(n) = \frac{1}{6}e^{j\frac{\pi}{6}n}$.

2. A discrete-time filter has transfer function $H(e^{j\theta}) = 1 + \frac{1}{2}\sin(\theta)$. Is this filter real? Evaluate the output when $x(n) = 1_0(n)$.

3. A discrete-time filter has impulse response $h(n) = 1_0(n) - 1_0(n-3)$. Evaluate the transfer function $H(e^{j\theta})$, as well as the output when the input signal is $x(n) = \mathrm{rep}_{10}\delta(n)$.

4. A discrete-time filter has impulse response $h(n) = n\,e^{-\frac{5}{3}n}\,1_0(n)$. Evaluate the transfer function $H(e^{j\theta})$, as well as the output when the input signal is $x(n) = A\,\cos(\frac{\pi}{2}n)$.

5. Identify the impulse response of a discrete-time band-pass filter that selects the phases in the range $[\theta_0 - \frac{1}{2}\theta_b; \theta_0 + \frac{1}{2}\theta_b] \subset (-\pi, \pi)$.

6. The discrete-time signal $x(n) = \cos(n) + \sin(3n) + e^{-j6n}$ is filtered by an ideal low-pass filter with cut-phase $\theta_c = \frac{\pi}{2}$. Evaluate the filter output $y(n)$.

Solutions.

1. The general rule for complex exponentials is $y(n) = H(e^{-j\frac{\pi}{6}})\,x(n) = H(e^{j\frac{\pi}{6}})\,e^{-j\frac{\pi}{6}n}$ which cannot meet value $\frac{1}{6}\,e^{j\frac{\pi}{6}n}$ for any choice of the constant $H(e^{-j\frac{\pi}{6}})$. Hence, there exists no such filter and the class is empty.

2. For the filter to be real we need to have the Hermitian symmetry (even real part, odd imaginary part) in the Frequency domain. However, the Fourier transform is real but not even nor odd. The output can, in this case, better evidenced in the time-domain. We have

$$H(e^{j\theta}) = 1 + \tfrac{1}{2}\sin(\theta) = 1 + \tfrac{1}{4j}e^{j\theta} - \tfrac{1}{4j}e^{-j\theta}$$

which, by inverse transform, provides

$$h(n) = \delta(n) + \tfrac{1}{4j}\,\delta(n+1) - \tfrac{1}{4j}\,\delta(n-1)\ .$$

Therefore, the output has the form

$$\begin{aligned} y(n) &= x * h(n) \\ &= 1_0(n) + \tfrac{1}{4j}\,1_0(n+1) - \tfrac{1}{4j}\,1_0(n-1) \\ &= 1_0(n) + \tfrac{1}{4j}\,\delta(n+1) + \tfrac{1}{4j}\,\delta(n)\ . \end{aligned}$$

3. We preliminary observe that

$$h(n) = 1_0(n) - 1_0(n-3) = \delta(n) + \delta(n-1) + \delta(n-2)\ ,$$

so that

$$H(e^{j\theta}) = 1 + e^{-j\theta} + e^{-j2\theta} = (1 + 2\cos(\theta))\,e^{-j\theta}\ .$$

For $x(n) = \text{comb}_{10}(n)$ it is much easier to proceed in the time-domain, to have

$$\begin{aligned} y(n) &= x * h(n) = x(n) + x(n-1) + x(n-2) \\ &= \text{rep}_{10}\delta(n) + \delta(n-1) + \delta(n-2)\ . \end{aligned}$$

4. In this case, given that the input is a sinusoid, the easiest approach is the Fourier-domain approach, since it is

$$y(n) = A\,|H(e^{j\frac{\pi}{2}})|\,\cos(\tfrac{\pi}{2}n + \varphi(\tfrac{\pi}{2}))\ .$$

We therefore evaluate the Fourier transform starting from the transform of $z(n) = e^{-\frac{5}{3}n}\,1_0(n)$, that is

$$Z(e^{j\theta}) = \sum_{n=0}^{\infty} e^{-\frac{5}{3}n}\,e^{-j\theta n} = \frac{1}{1 - e^{-(\frac{5}{3}+j\theta)}}$$

to have, from the product-by-n property

$$H(e^{j\theta}) = jZ'(e^{j\theta}) = \frac{e^{-(\frac{5}{3}+j\theta)}}{(1 - e^{-(\frac{5}{3}+j\theta)})^2}\ .$$

At phase $\theta = \frac{\pi}{2}$ we further have

$$H(e^{j\frac{\pi}{2}}) = \frac{-je^{-\frac{5}{3}}}{(1+je^{-\frac{5}{3}})^2} \ ,$$

from which we obtain

$$|H(e^{j\frac{\pi}{2}})| = \frac{e^{-\frac{5}{3}}}{1+e^{-\frac{10}{3}}} = \frac{1}{e^{\frac{5}{3}}+e^{-\frac{5}{3}}} \ , \qquad \varphi(\tfrac{\pi}{2}) = -\tfrac{\pi}{2} - 2\arctan(e^{-\frac{5}{3}}) \ .$$

5. In the Fourier domain, the filter reads as

$$H(e^{j\theta}) = \mathrm{rep}_{2\pi}\,\mathrm{rect}(\tfrac{\theta-\theta_0}{\theta_b}) + \mathrm{rect}(\tfrac{\theta+\theta_0}{\theta_b})$$

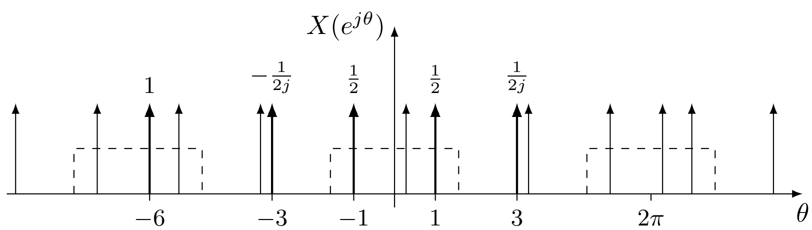
which we can inverse transform by applying the sampling plus modulation rules, starting from the Fourier transform couple

$$\tfrac{\theta_b}{2\pi}\ \mathrm{sinc}(\tfrac{\theta_b}{2\pi}t) \quad \longrightarrow \quad \mathrm{rect}(\tfrac{\omega}{\theta_b})$$

to have

$$h(n) = \tfrac{\theta_b}{\pi}\ \mathrm{sinc}(\tfrac{\theta_b}{2\pi}n)\ \cos(\theta_0 n) \ .$$

6. From the properties of the sinusoids, in the Fourier domain we have a collection of comb functions centred at ± 1, ± 3 and -6.



As illustrated in the figure above, ± 1 falls inside the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, hence the cosinus is kept. Instead, ± 3 falls outside the range of the filter, hence the sinus is cancelled. For the exponential, instead, which is centred at phase -6, we need to observe from the figure that it actually falls inside the active filter part, and is therefore kept. As a matter of fact, the filter response $H(e^{j\omega})$ is periodic 2π, hence it can be interpreted as a base-band rectangle only if the reference phase is mapped into the interval $(-\pi, \pi)$, in which case it is $-6 \pmod{2\pi} = 2\pi - 6 \simeq 0.2883$, which evidently falls within the active signal part. Therefore the output is

$$y(n) = \cos(n) + e^{-j6n} \ .$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
16.5 Homework assignment
Prof. T. Erseghe

Exercises 16.5

Solve the following exercises on signals, Fourier transforms, and filters:

1. The signal
$$x(t) = \sum_{k=-\infty}^{\infty} (-1)^k \operatorname{rect}(t-k) ,$$
is fed into a low-pass filter with cut pulsation $\omega_c = \frac{3}{2}\pi$. Evaluate the filter output $y(t)$;

2. Evaluate the Fourier transform of a real-valued signal which is even and periodic of period T_p, and which is defined as $s(t) = t^2$ in the interval $[0, \frac{1}{2}T_p]$;

3. Evaluate the Fourier transform of signal $s(t) = \operatorname{sinc}(t) \cdot \operatorname{sinc}(2(t-1))$;

4. Evaluate the area and the energy of the signals
$$s_1(t) = \operatorname{rect}^2(\tfrac{t}{10}) , \quad s_2(t) = \operatorname{sinc}(\tfrac{1}{8}(t-5))$$
$$s_3(t) = \sin(10\pi t) \operatorname{rect}(\tfrac{1}{10}(t-5)) , \quad s_4(t) = \operatorname{sinc}^2(t) ;$$

5. Evaluate the output of a continuous-time filter with impulse response $h(t) = e^{-|t|}$ when the input is $x(t) = 3\cos(2t)$;

6. The signal $s(t) = A \operatorname{sinc}^n(t/T) \cos(\omega_0 t)$ is fed to an ideal low-pass filter with cut pulsation ω_c. Identify the values of ω_0 (as a function of n, T and ω_c) that guarantee that the output is $y(t) = 0$;

7. Evaluate the convolution $s(t) = x * y(t)$ between the two signals $x(t) = \operatorname{sinc}(t)$ and $y(t) = \operatorname{sinc}(\frac{1}{2}t) \cos(\omega_0 t)$. Illustrate the result for $\omega_0 = \pi$. For what values of ω_0 is the area of $s(t)$ zero? For what values of ω_0 is $s(t)$ zero?

Solutions.

1. The signal has evidently period $T_p = 2$, and can be written in the more compact form
$$x(t) = 2\,\text{rep}_2\,\text{rect}(t) - 1\ ,$$
evidencing that it is a difference between a square-wave of duty cycle $d = \frac{1}{2}$ and a constant, hence its Fourier coefficients are
$$X_k = 2 \cdot \tfrac{1}{2}\,\text{sinc}(\tfrac{k}{2}) - 1 = \text{sinc}(\tfrac{k}{2}) - 1\ ,$$
associated to pulsations $k\omega_0 = k\pi$. Being the filter a low-pass filter with cut-pulsation $\omega_c = \frac{3}{2}\pi$, then only the coefficients for $k = -1, 0, 1$ are kept, that is the Fourier coefficients of the output are
$$Y_k = \begin{cases} X_k & , k = 0, \pm 1 \\ 0 & , \text{otherwise} \end{cases} = \begin{cases} \text{sinc}(\frac{1}{2}) = \frac{\sin(\frac{\pi}{2})}{\frac{\pi}{2}} = \frac{2}{\pi}, & , k = \pm 1 \\ 0 & , \text{otherwise} \end{cases}$$
since $X_0 = 0$. Therefore, by inverse Fourier transform (Fourier series, in this case) we have
$$y(t) = \sum_{k=-\infty}^{\infty} Y_k\, e^{jk\omega_0 t} = \tfrac{2}{\pi}\, e^{j\pi t} + \tfrac{2}{\pi}\, e^{-j\pi t} = \tfrac{4}{\pi}\, \cos(\pi t)\ .$$

2. For the even-symmetric signal, we can write
$$s(t) = \text{rep}_{T_p}\, t^2\, \text{rect}(\tfrac{t}{T_p})\ ,$$
so that its Fourier transform can be obtained from that of the pair
$$x(t) = \text{rect}(\tfrac{t}{T_p}) \quad \longrightarrow \quad X(j\omega) = T_p\, \text{sinc}(\tfrac{\omega}{2\pi/T_p})$$
by first applying twice the product-by-t property, to have
$$y(t) = t^2\, \text{rect}(\tfrac{t}{T_p}) \;\longrightarrow\; Y(j\omega) = j^2\, X''(j\omega) = -\tfrac{T_p^3}{4\pi^2}\, \text{sinc}''(\tfrac{\omega}{2\pi/T_p})$$
and by then sampling at $k\omega_0$, $\omega_0 = \frac{2\pi}{T_p}$, to have
$$S_k = \tfrac{1}{T_p}\, Y(jk\omega_0) = -\tfrac{T_p^2}{4\pi^2}\, \text{sinc}''(k) = \tfrac{T_p^2}{4\pi^2} \cdot \begin{cases} \frac{\pi^2}{3} & , k = 0 \\ \frac{2\,(-1)^k}{k^2} & , k \neq 0 \end{cases}$$
where the compact result was derived with some effort by expanding the second derivative of the sinc.

3. The transform can be approach by interpreting the signal as a product $s(t) = x(t)y(t)$ where
$$\begin{aligned} x(t) &= \text{sinc}(t) & &\longrightarrow & X(j\omega) &= \text{rect}(\tfrac{\omega}{2\pi}) \\ y(t) &= \text{sinc}(2(t-1)) & &\longrightarrow & Y(j\omega) &= \tfrac{1}{2}\, \text{rect}(\tfrac{\omega}{4\pi})\, e^{-j\omega} \end{aligned}$$

and by successively applying the product rule, that is

$$S(j\omega) = \frac{1}{2\pi} X * Y(j\omega) .$$

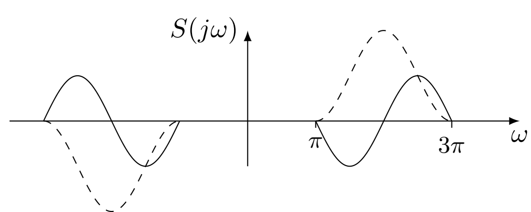
The convolution between the rectangle X (of width 2π) and the modulated rectangle Y (of width 4π) can be approached in the standard way where Y is kept fixed and X is shifted. The resulting expressions, considering the different regions and the overall extension $[-3\pi, 3\pi]$, provide

$$S(j\omega) = \frac{1}{4\pi} \cdot \begin{cases} \int_{-2\pi}^{\omega+\pi} e^{-ju}\, du & , \omega \in (-3\pi, -\pi) \\ \int_{\omega-\pi}^{\omega+\pi} e^{-ju}\, du & , \omega \in (-\pi, \pi) \\ \int_{\omega-\pi}^{2\pi} e^{-ju}\, du & , \omega \in (\pi, 3\pi) \\ 0 & , \text{otherwise} \end{cases}$$

After solving the integrals, we get

$$S(j\omega) = \frac{j}{4\pi} \cdot \begin{cases} -e^{-j\omega} - 1 & , \omega \in (-3\pi, -\pi) \\ 1 + e^{-j\omega} & , \omega \in (\pi, 3\pi) \\ 0 & , \text{otherwise} \end{cases}$$

as illustrated in the figure below, where we appreciate the Hermitian symmetry of $S(j\omega)$.



4. The first signal is $s_1(t) = \text{rect}(\frac{t}{10})$ and its area and energy can be easily evaluated in the time-domain, to have

$$A_1 = 10 \cdot 1 = 10 , \qquad E_1 = 10 \cdot 1^2 = 10 .$$

For the second signal, we need to map it to the Fourier domain, where the shape is rectangular. We have

$$S_2(j\omega) = 8 \text{ rect}(\tfrac{\omega}{\pi/4})\, e^{-j5\omega} ,$$

so that

$$A_2 = S_2(j0) = 8 , \qquad E_2 = \tfrac{1}{2\pi} \cdot \tfrac{\pi}{4} \cdot 8^2 = 8 .$$

For the third signal, again we need to map it into the Fourier domain, to have

$$S_3(j\omega) = \tfrac{10}{2j} \text{ rect}(\tfrac{(\omega-10\pi)}{\pi/5})\, e^{-j5(\omega-10\pi)} - \tfrac{10}{2j} \text{ rect}(\tfrac{(\omega+10\pi)}{\pi/5})\, e^{-j5(\omega+10\pi)}$$

with non-overlapping rectangles, so that

$$A_3 = S_3(j0) = 0 \ , \qquad E_3 = 2 \cdot \tfrac{1}{2\pi} \cdot \tfrac{\pi}{5} \cdot 5^2 = 5 \ .$$

For the last signal we have

$$S_4(j\omega) = \text{triang}(\tfrac{\omega}{2\pi})$$

so that

$$A_4 = S_4(j0) = 1 \ , \qquad E_4 = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \text{triang}^2(\tfrac{\omega}{2\pi})\, d\omega = \tfrac{2}{3} \ .$$

5. In this case we have

$$y(t) = 3\,|H(j2)|\,\cos(2t + \varphi(2)) \ ,$$

given that the Fourier transform $H(j\omega)$ is known. We evaluate it by applying the Fourier integral, to have

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{0} e^{t}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-t}\, e^{-j\omega t}\, dt \\ &= \frac{1}{1 - j\omega} + \frac{1}{1 + j\omega} = \frac{2}{1 + \omega^2} \end{aligned}$$

which is real-valued, even symmetric, and positive, hence $\varphi(\omega) = 0$. We have $H(j2) = \frac{2}{5}$, so that $y(t) = \frac{6}{5}\cos(2t)$.

6. From the Fourier pair $\text{sinc}(t/T) \to T\ \text{rect}(\frac{\omega}{2\pi/T})$ and the fact that a product of order n turns into a convolution of order n in the Fourier domain, we know that signal $\text{sinc}^n(t/T)$ has an extension in the Fourier domain of the form $[-\frac{\pi}{T}n, \frac{\pi}{T}n]$. Correspondingly, the cosinus multiplication defining $s(t)$ further ensures that the extension of $S(j\omega)$ is of the form

$$e(S) = [-\omega_0 - \tfrac{\pi}{T}n, -\omega_0 + \tfrac{\pi}{T}n] \cup [\omega_0 - \tfrac{\pi}{T}n, \omega_0 + \tfrac{\pi}{T}n] \ .$$

A low-pass filter with cut pulsation ω_c, instead, has an extension in the Fourier domain of

$$e(H) = [-\omega_c, \omega_c] \ ,$$

so that to have a zero-valued output it suffices to have disjoint extensions, that is,

$$\omega_c < \omega_0 - \tfrac{\pi}{T}n \quad \Longrightarrow \quad \omega_0 > \omega_c + \tfrac{\pi}{T}n \ .$$

7. The convolution is more easily approached as a product in the Fourier domain, where

$$\begin{aligned} X(j\omega) &= \text{rect}(\tfrac{\omega}{2\pi}) \\ Y(j\omega) &= \text{rect}(\tfrac{\omega - \omega_0}{\pi}) + \text{rect}(\tfrac{\omega + \omega_0}{\pi}) \\ S(j\omega) &= X(j\omega)\, Y(j\omega) \end{aligned}$$

where we observe that the extensions of the two transforms are

$$
\begin{aligned}
e(X) &= [-\pi, \pi] \\
e(Y) &= [-\omega_0 - \tfrac{\pi}{2}, -\omega_0 + \tfrac{\pi}{2}] \cup [\omega_0 - \tfrac{\pi}{2}, \omega_0 + \tfrac{\pi}{2}] \ .
\end{aligned}
$$

Now, for the area, from the properties of the convolution we have

$$
A_{x*y} = A_x \, A_y = X(j0) \, Y(j0) = 1 \cdot Y(j0) = \begin{cases} 0 & , 0 < \omega_0 - \frac{\pi}{2} \\ 2 & , \text{otherwise} \end{cases}
$$

since it is only a question of wether $0 \in e(Y)$. The requirement therefore is $\omega_0 > \frac{\pi}{2}$. For the signal to be zero, instead, the two extensions $e(X)$ and $e(Y)$ must be disjoint, that is we must have

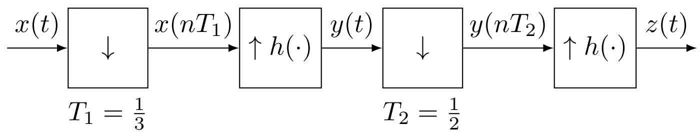
$$
\pi < \omega_0 - \tfrac{\pi}{2} \quad \Longrightarrow \quad \omega_0 > \tfrac{3}{2}\pi \ .
$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
17.3 Solved exercises
Prof. T. Erseghe

Exercises 17.3

Solve the following exercises on sampling, interpolation, and the sampling theorem:

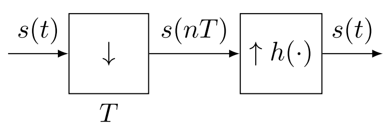
1. Identify a sampling/interpolation scheme for signal $s(t) = \text{sinc}^3(t)$;
2. Consider the following sampling/interpolation system



   where $h(t) = 2\,\text{sinc}(2t)$ and $x(t) = \text{sinc}^2(2t)$. Identify the Fourier transforms $Y(j\omega)$ and $Z(j\omega)$, as well as the output signal $z(t)$.

Solutions.

1. The signal is base-band, therefore we resort to the classical sampling and interpolation scheme



   with $h(t) = \text{sinc}(t/T)$ where we only need to identify the value of T in such a way that $e(S) \subset [-\pi/T, \pi/T]$. Now, from the couple $\text{sinc}(t) \to \text{rect}(\frac{\omega}{2\pi})$, and the fact that a product in time maps into a convolution in the Fourier domain, we know that $e(S) = [-3\pi, 3\pi]$, hence it suffices to choose

$$3\pi \le \frac{\pi}{T} \quad \Longrightarrow \quad T \le \tfrac{1}{3}\ .$$

   With the stricter choice $T = \frac{1}{3}$ we have

$$\text{sinc}^3(t) = \sum_{k=-\infty}^{\infty} \text{sinc}^3(\tfrac{1}{3}k)\ \text{sinc}(\tfrac{t-k/3}{1/3}) = \sum_{k=-\infty}^{\infty} \text{sinc}^3(\tfrac{1}{3}k)\ \text{sinc}(3t-k)$$

2. From the theory we know that

$$\begin{aligned} Y(j\omega) &= \tfrac{1}{T_1} H(j\omega)\, \text{rep}_{\frac{2\pi}{T_1}} X(j\omega) \\ &= 3H(j\omega)\, \text{rep}_{6\pi} X(j\omega) \end{aligned}$$

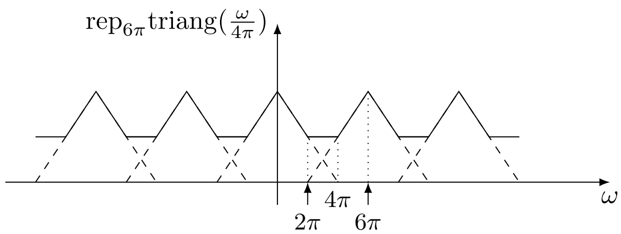
where we exploited the fact that $T_1 = \frac{1}{3}$. Since the Fourier transforms X and H are of the form

$$X(j\omega) = \tfrac{1}{2}\text{triang}(\tfrac{\omega}{4\pi}) \ , \qquad H(j\omega) = \text{rect}(\tfrac{\omega}{4\pi}) \ ,$$

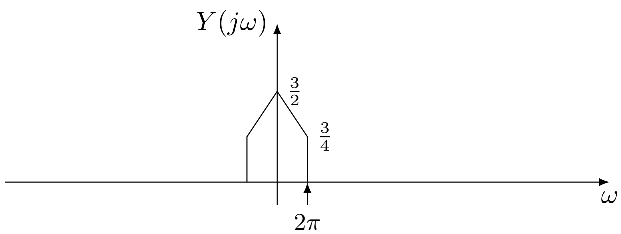
then it is

$$Y(j\omega) = \tfrac{3}{2} \ \text{rect}(\tfrac{\omega}{4\pi}) \, \text{rep}_{6\pi}\text{triang}(\tfrac{\omega}{4\pi}) \ ,$$

where the periodic repetition is illustrated in the figure below



so that the shape of Y is the one illustrated in figure



which we can write in the form

$$Y(j\omega) = 3H(j\omega)\, X(j\omega) = \tfrac{3}{4}\, \text{rect}(\tfrac{\omega}{4\pi}) + \tfrac{3}{4}\, \text{triang}(\tfrac{\omega}{2\pi}) \ .$$

Note that the extension in the Fourier domain is $e(Y) = [-2\pi, 2\pi]$.

The second sampling/interpolation system, leading to $z(t)$ instead provides

$$\begin{aligned} Z(j\omega) &= \tfrac{1}{T_2} H(j\omega)\, \text{rep}_{\frac{2\pi}{T_2}} Y(j\omega) \\ &= 2H(j\omega)\, \text{rep}_{4\pi} Y(j\omega) \end{aligned}$$

where we used $T_2 = \frac{1}{2}$, and where we note that, because of the extension $e(Y) = [-2\pi, 2\pi]$, there is no aliasing with a periodic repetition of 4π. As a matter of fact, this is a perfect sampling-reconstruction scheme (baseband Shannon's like) since T_2 is coherent with the bandwidth of Y. Therefore, in the absence of aliasing, we have

$$Z(j\omega) = 2\, H(j\omega)\, Y(j\omega) = 2\ \text{rect}(\tfrac{\omega}{4\pi})\, Y(j\omega) = 2\, Y(j\omega)$$

that is

$$Z(j\omega) = \tfrac{3}{2}\,\mathrm{rect}(\tfrac{\omega}{4\pi}) + \tfrac{3}{2}\,\mathrm{triang}(\tfrac{\omega}{2\pi})\ .$$

By inverse transform, we finally obtain

$$z(t) = y(t) = 3\,\mathrm{sinc}(2t) + \tfrac{3}{2}\,\mathrm{sinc}^2(t)\ .$$

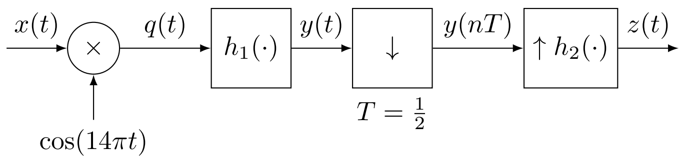
FOUNDATIONS OF SIGNALS AND SYSTEMS
17.4 Homework assignment
Prof. T. Erseghe

Exercises 17.4

Solve the following exercises on sampling, interpolation, and the sampling theorem:

1. Identify a sampling/interpolation scheme for signal $s(t) = \mathrm{sinc}^7(t)$;

2. Consider the following sampling/interpolation system



   where
$$X(j\omega) = \mathrm{rect}(\tfrac{\omega}{4\pi}) \left[1 - \mathrm{triang}(\tfrac{\omega}{2\pi})\right]$$
$$H_1(j\omega) = 2 - 2\ \mathrm{rect}(\tfrac{\omega}{28\pi})$$
$$H_2(j\omega) = \tfrac{1}{2}\ \mathrm{rect}(\tfrac{\omega}{4\pi})\ .$$

   Evaluate the output $z(t)$.

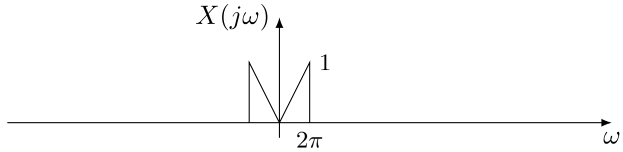
3. Identify an efficient sampling/interpolation scheme for the band-pass signal $s(t) = \mathrm{sinc}^2(t)\, e^{j\frac{19\pi}{2}t}$.

4. Identify an efficient sampling/interpolation scheme for the signal $s(t) = \mathrm{sinc}(\frac{t+1}{3})\cos(2t) - \frac{\pi}{12}\sin(4t)$. How does the result change if the signal is pre-filtered with a pass-band filter whose transfer function is $H(j\omega) = 1$ for $1 < |\omega| < 3$ and 0 elsewhere?

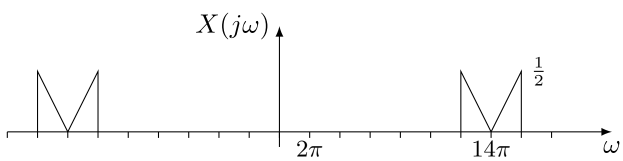
Solutions.

1. We can mimic Exercise 17.3.1 to observe that the signal extension in the Fourier domain is $e(S) = [-7\pi, 7\pi]$, and therefore we can apply the sampling theorem with
$$7\pi \le \frac{\pi}{T} \quad \Longrightarrow \quad T \le \tfrac{1}{7}\ .$$
With the stricter choice $T = \frac{1}{7}$ we have
$$\mathrm{sinc}^7(t) = \sum_{k=-\infty}^{\infty} \mathrm{sinc}^7(\tfrac{1}{7}k)\ \mathrm{sinc}(\tfrac{t-k/7}{1/7}) = \sum_{k=-\infty}^{\infty} \mathrm{sinc}^7(\tfrac{1}{7}k)\ \mathrm{sinc}(7t-k)\ .$$

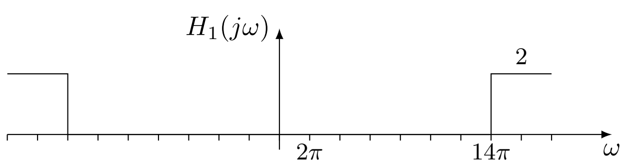
2. The solution can be better approached in the Fourier domain, by graphically interpreting all signals. The input signal is illustrated in the figure below

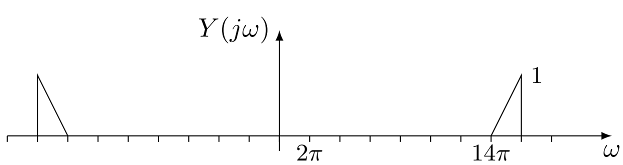


For signal $z(t)$, from the modulation property we have
$$Z(j\omega) = \tfrac{1}{2}\,X(j(\omega - 14\pi)) + \tfrac{1}{2}\,X(j(\omega + 14\pi))$$
so that its shape is the one in the next figure.



The action of the high-pass filter $h_1(t)$ is simply
$$Y(j\omega) = H_1(j\omega)\,Z(j\omega)\ ,$$
providing the result illustrated in the figure below.

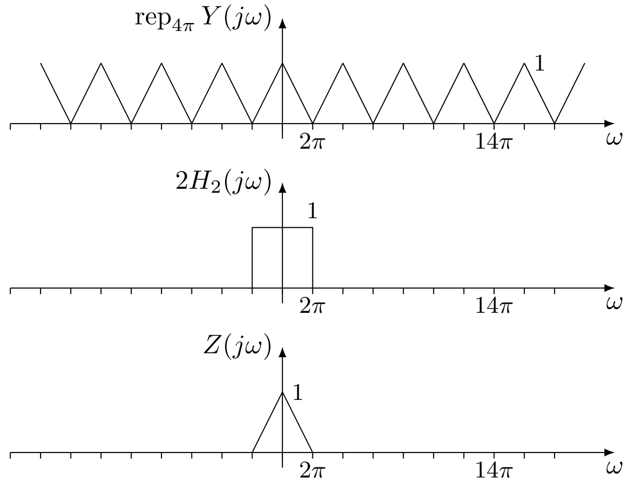




Finally, the action of the sampling/interpolation series has the form

$$\begin{aligned} Z(j\omega) &= \tfrac{1}{T} H_2(j\omega)\,\mathrm{rep}_{\frac{2\pi}{T}} Y(j\omega) \\ &= 2H_2(j\omega)\,\mathrm{rep}_{4\pi} Y(j\omega) \end{aligned}$$

providing



so that

$$Z(j\omega) = \mathrm{triang}(\tfrac{\omega}{2\pi})\ , \qquad z(t) = \mathrm{sinc}^2(t)\ .$$

3. In this case, the Fourier transform is

$$S(j\omega) = \mathrm{triang}(\tfrac{\omega - \frac{19\pi}{2}}{2\pi})$$

with extension $e(S) = [\frac{15\pi}{2}, \frac{23\pi}{2}]$. If we adopt a base-band approach to the sampling theorem we require that

$$\frac{23\pi}{2} \le \frac{\pi}{T} \quad \Longrightarrow \quad T \le \tfrac{2}{23} \simeq 0.087\ .$$

If, instead, we approach the solution from a band-pass perspective where we interpret the extension as $e(S) = \frac{19\pi}{2} + [-2\pi, 2\pi]$, we require that

$$2\pi \le \frac{\pi}{T} \quad \Longrightarrow \quad T \le \tfrac{1}{2}\ .,$$

which is a much more efficient choice (higher value of the sampling spacing T means less number of samples) requiring a filter of the form

$$\tfrac{1}{T}\, H(j\omega) = \mathrm{rect}(\tfrac{\omega - \frac{19\pi}{2}}{2\pi/T})$$

that is

$$h(t) = \mathrm{sinc}(\tfrac{t}{T})\, e^{j\frac{19\pi}{2}t}\ .$$

The reconstruction rule is, in this case

$$s(t) = \sum_{k=-\infty}^{\infty} s(kT)\, h(t-kT) = \sum_{k=-\infty}^{\infty} s(kT)\ \mathrm{sinc}(\tfrac{t}{T}-k)\, e^{j\frac{19\pi}{2}(t-kT)}$$

4. We preliminarily need to identify the Fourier transform, that is

$$\begin{aligned} S(j\omega) &= \tfrac{3}{2}\,\mathrm{rect}(\tfrac{\omega-2}{2\pi/3})\, e^{j(\omega-2)} + \tfrac{3}{2}\,\mathrm{rect}(\tfrac{\omega+2}{2\pi/3})\, e^{j(\omega+2)} \\ &\qquad - \tfrac{\pi^2}{12j}\,\delta(\omega-4) + \tfrac{\pi^2}{12j}\,\delta(\omega+4) \end{aligned}$$

revealing that the extension has the form

$$\begin{aligned} e(S) &= [-2-\tfrac{\pi}{3}, -2+\tfrac{\pi}{3}] \cup [2-\tfrac{\pi}{3}, 2+\tfrac{\pi}{3}] \cup \{4,-4\} \\ &\subset [-4,4]\,. \end{aligned}$$

so that it is required to have

$$4 < \frac{\pi}{T} \quad\Longrightarrow\quad T > \tfrac{\pi}{4}\ .$$

If the signal is prefiltered by

$$H(j\omega) = \mathrm{rect}(\tfrac{\omega-2}{2}) + \mathrm{rect}(\tfrac{\omega+2}{2})\ ,$$

then it is

$$\begin{aligned} \tilde{S}(j\omega) &= S(j\omega)\, H(j\omega) \\ &= \tfrac{3}{2}\,\mathrm{rect}(\tfrac{\omega-2}{2})\, e^{j(\omega-2)} + \tfrac{3}{2}\,\mathrm{rect}(\tfrac{\omega+2}{2})\, e^{j(\omega+2)} \end{aligned}$$

since $2 < \frac{2}{3}\pi$. The resulting extension is

$$e(S) = [-3,-1] \cup [1,3] \subset [-3,3]\ ,$$

so that in this case it is required to have

$$3 < \frac{\pi}{T} \quad\Longrightarrow\quad T > \tfrac{\pi}{3}\ .$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
18.2 Solved exercises
Prof. T. Erseghe

Exercises 18.2

Solve the following MatLab problems:

1. Evaluate numerically the Fourier transform of

$$x(t) = 2\,e^{-t}\,\cos(2\pi t)\,1(t)$$

and compare it with its analytical expression

$$X(j\omega) = \frac{1}{1 + j(\omega - 2\pi)} + \frac{1}{1 + j(\omega + 2\pi)}\;.$$

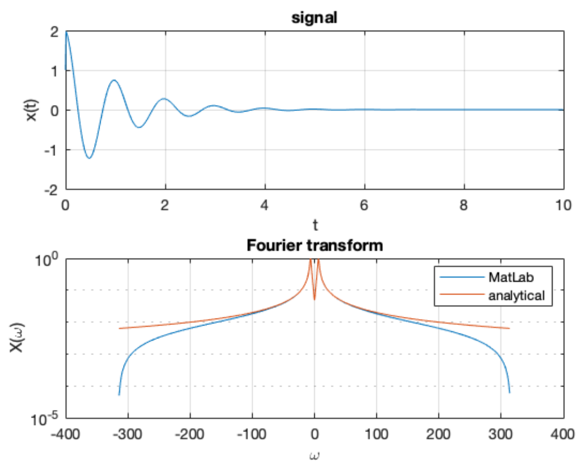
2. The file 'ex18_2_2.mat' contains in vector x pancreatic secretion values taken in the interval $[0, 300]$ min with a sampling spacing of $T = 0.1$ min. Plot the signal together with its Fourier transform (absolute values only).

Solutions.

1. In the code we define a small sampling step $T = 0.01$ and a number of samples $N = 1000$ such that the interval in which we sample the signal is $[0, 10]$, to ensure that the exponential outside the sampled range is small. The derivation of the Fourier transform is standard and it is compared with the analytical expression here called Xref. Note that the plot of the Fourier domain only shows the absolute values, and uses a logarithmic form (through function semilogy, which works as plot) since this is the standard way to correctly observe the Fourier transform behaviour, allowing for a correct interpretation of the result. Always use the logarithmic form! Note that there is full accordance between the MatLab outcome and the analytical expression up to $\omega = 100$, then some aliasing effect (due to the $1/\omega$ nature of the Fourier transform) is visible.

```
T = 0.01;
N = 1000;
t = (0:N-1)*T;
x = ((t>0)+.5*(t==0)).*(2*exp(-t).*cos(2*pi*t));
X = fftshift(T*fft(x));
om = (-round((N-1)/2):round(N/2)-1) *2*pi/(N*T);
Xref = 1./(1+1j*(om-2*pi))+1./(1+1j*(om+2*pi));

figure
subplot(2,1,1)
plot(t,x)
grid
```

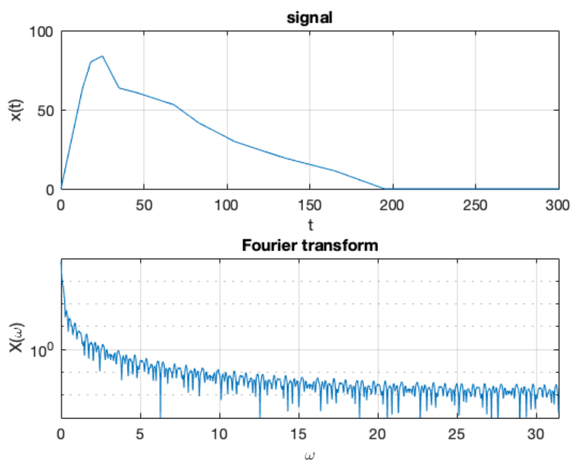
```
xlabel('t')
ylabel('x(t)')
title('signal')
subplot(2,1,2)
semilogy(om,abs(X),om,abs(Xref))
grid
xlabel('\omega')
ylabel('X(\omega)')
legend('MatLab','analytical')
title('Fourier transform')
```



2. In this case all the parameters are set so it it simply the case of applying the rules for correct calculation of the Fourier transform. Note that we restricted the Fourier plot to the positive axis, since this is symmetric by nature, and in fact a real-valued signal implies an Hermitian symmetry in the Fourier domain which, in turn, determines an even symmetric absolute value.

```
load('ex18_2_2.mat') % defines t, x, T
N = length(x);
X = fftshift(T*fft(x));
om = (-round((N-1)/2):round(N/2)-1) *2*pi/(N*T);

figure
subplot(2,1,1)
plot(t,x)
grid
xlabel('t')
ylabel('x(t)')
title('signal')
```

```
subplot(2,1,2)
semilogy(om,abs(X))
grid
xlabel('\omega')
ylabel('X(\omega)')
axis([0 max(om) 1e-3 1e4])
title('Fourier transform')
```



FOUNDATIONS OF SIGNALS AND SYSTEMS
18.3 Homework assignment
Prof. T. Erseghe

Exercises 18.3

Solve the following MatLab problems:

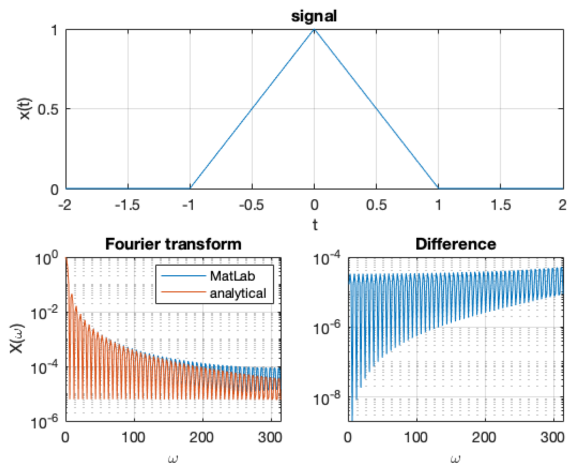
1. Evaluate numerically the Fourier transform of $x(t) = \text{triang}(t)$ and compare it with its analytical expression $X(j\omega) = \text{sinc}^2(\omega/(2\pi))$.

2. The file 'ex18_3_2.mat' contains in vector x pancreatic secretion values taken in the interval $[0, 300]$ min with a sampling spacing of $T = 0.1$ min as well as the impulse response g (on the same time samples) mapping to the plasma concentration $y = x*g$. Plot the signals together with their Fourier transform (absolute values only). Then evaluate the product $X \cdot G$ in the Fourier domain and inverse-transform it by use of the inverse MatLab functions ifftshift and ifft. Compare the result (which is a convolution evaluated in the Fourier domain) with the convolution taken in the time-domain: you should get a perfect correspondence!

Solutions.

1. The code can mimic that of Exercise 18.2.1, as follows, where we used the time span $[-2, 2]$ for the triangle and a sampling spacing $T = 0.01$. Note that, in this case, we correct the Fourier transform for the fact that the starting sample is not zero. Note also how we show the Fourier transform only in the positive axis where, again, perfect correspondence is available up to $\omega = 100$. In this case we also display the error, showing that it is of the order $2 \cdot 10^{-5}$ along the entire axis, and obviously much more visible where the absolute value of the Fourier transform gets smaller.

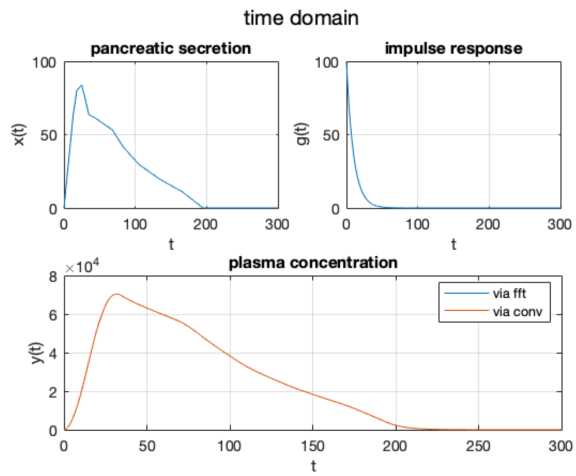
```
T = 0.01;
t = -2:T:2;
x = triang(t);
N = length(x);
om = (-round((N-1)/2):round(N/2)-1) *2*pi/(N*T);
X = fftshift(T*fft(x)).*exp(-1i*om*t(1));
Xref = sinc(om/(2*pi)).^2;

figure
subplot(2,1,1)
plot(t,x)
grid
xlabel('t')
ylabel('x(t)')
title('signal')
subplot(2,2,3)
semilogy(om,abs(X),om,abs(Xref))
axis([0 max(om) ylim])
grid
xlabel('\omega')
ylabel('X(\omega)')
legend('MatLab','analytical')
title('Fourier transform')
subplot(2,2,4)
semilogy(om,abs(X-Xref))
axis([0 max(om) ylim])
grid
xlabel('\omega')
title('Difference')

function s = triang(t)
s = (1-abs(t)).*(abs(t)<1);
end
```

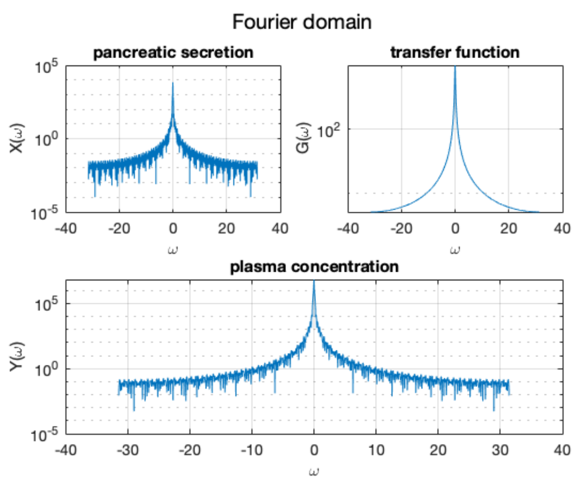


2. In this exercise we first evaluate the Fourier transforms of x and g separately (by using the standard approach), then make a product via the pointwise product operator. The inverse transform is calculated by applying the inverse function ifft and ifftshift in reverse order, to get the correct result. Note the perfect correspondence with the convolution calculated in the time-domain (which is truncated to the same range as x). Incidentally, one could observe that the compact expression "y=T*ifft(fft(x).*fft(g))" holds, where we neglected any use of ω or of the ifftshift operator. This is actually how MatLab calculates convolutions!!!

```
load('ex18_3_2.mat') % defines t, x, g, T
N = length(x);
X = fftshift(T*fft(x));
G = fftshift(T*fft(g));
om = (-round((N-1)/2):round(N/2)-1) *2*pi/(N*T);
Y = X.*G;
y = ifft(ifftshift(Y)/T);
y2 = T*conv(x,g);
y2 = y2(1:length(x));

figure(1)
subplot(2,2,1)
plot(t,x)
grid
xlabel('t')
ylabel('x(t)')
title('pancreatic secretion')
subplot(2,2,2)
plot(t,g)
grid
```

```
xlabel('t')
ylabel('g(t)')
title('impulse response')
subplot(2,1,2)
plot(t,y,t,y2)
grid
xlabel('t')
ylabel('y(t)')
legend('via fft','via conv')
title('plasma concentration')
sgtitle('time domain')

figure(2)
subplot(2,2,1)
semilogy(om,abs(X))
grid
xlabel('\omega')
ylabel('X(\omega)')
title('pancreatic secretion')
subplot(2,2,2)
semilogy(om,abs(G))
grid
xlabel('\omega')
ylabel('G(\omega)')
title('transfer function')
subplot(2,1,2)
semilogy(om,abs(Y))
grid
xlabel('\omega')
ylabel('Y(\omega)')
title('plasma concentration')
sgtitle('Fourier domain')
```


time domain
pancreatic secretion
x(t)
t
impulse response
g(t)
t
plasma concentration
×10^4
y(t)
t
via fft
via conv


Fourier domain
pancreatic secretion
X(ω)
ω
transfer function
G(ω)
ω
plasma concentration
Y(ω)
ω

FOUNDATIONS OF SIGNALS AND SYSTEMS
18.5 Homework assignment
Prof. T. Erseghe

Exercises 18.5

Solve the following MatLab problems:

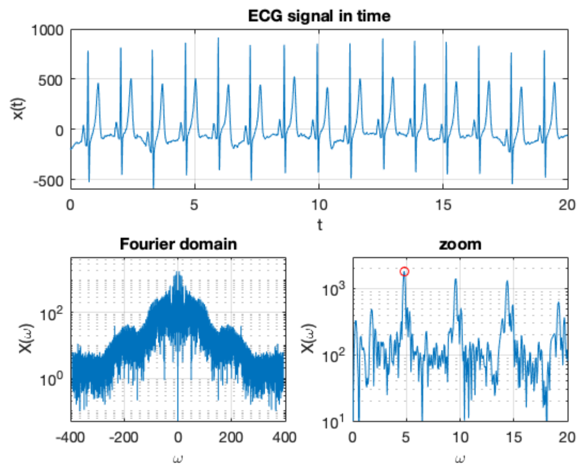
1. The file 'ex18_5_1.mat' contains in vector x some ECG samples taken with spacing $T = 1/125\,\mathrm{s}$. After removing the signal average value (use the mean MatLab function), plot the signal as well as its Fourier transform in absolute value, and determine the position $\omega_0 > 0$ of the first peak. By resorting to the expression $\omega_0 = 2\pi/T_p$, identify the ECG period $T_p = 2\pi/\omega_0$. You can use the MatLab function "[maxval,pos] = max(abs(X))" for this.

2. The file 'ex18_5_2.mat' contains in vector x some ECG samples taken with spacing $T = 1/125\,\mathrm{s}$ and corrupted by a sinusoidal noise. After removing the signal average value (use the mean MatLab function), plot the signal as well as its Fourier transform in absolute value. Then, filter the signal with an high-pass filter that rejects all pulsations in the range $|\omega| < \pi$, by applying a selection in the Fourier domain and then by applying an inverse transform. The sinusoidal noise should be absent in the filtered signal.

Solutions.

1. In the code we first subtract the average value, then use a trick to increase the definition in the Fourier domain (i.e., to increase the value of N), namely that of adding zero-valued contributions at the end of the signal. The search for the maximum is restricted in the range $[4, 8]$ since this is the range we can identify by looking at the plots. We also display the estimated period which turns out to be $T_p = 1.3151$.

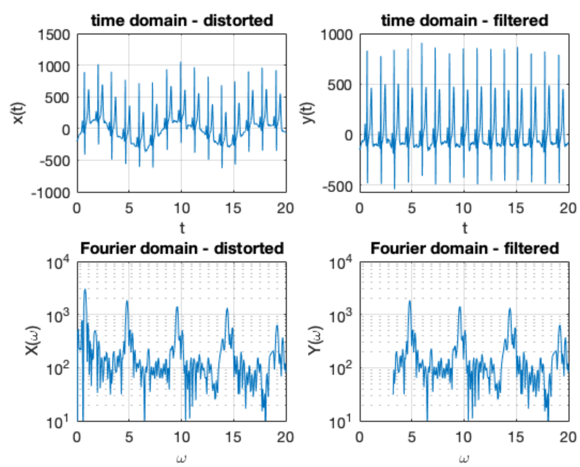
```
load('ex18_5_1.mat') % defines t, x, T
x = x - mean(x);
x = [x, zeros(1,2*length(x))]; % trick to tighthen
    Fourier sampling
N = length(x);
t = (0:N-1)*T;
X = fftshift(T*fft(x));
om = (-round((N-1)/2):round(N/2)-1) *2*pi/(N*T);

% find max: range [4,8] set by looking at the plot
[maxval,pos] = max(abs(X).*(om>4).*(om<8));
om0 = om(pos); % estimated omega0
disp(['estimated Tp = 2 pi/omega0 = ' num2str(2*pi
   /om0)])

figure(1)
subplot(2,1,1)
plot(t,x)
grid
xlabel('t')
ylabel('x(t)')
title('ECG signal in time')
axis([0 20 ylim])
subplot(2,2,3)
semilogy(om,abs(X))
grid
xlabel('\omega')
ylabel('X(\omega)')
title('Fourier domain')
subplot(2,2,4)
semilogy(om,abs(X))
hold on
semilogy(om0,maxval,'ro')
grid
xlabel('\omega')
ylabel('X(\omega)')
axis([0 20 1e1 3e3])
title('zoom')
```



2. We apply the same tricks as in the previous exercise. The high-pass filter is implemented by multyplying entrywise the Fourier coefficients X by a selection of the pulsation values "abs(om)>pi" which is active (i.e., equal to one) only for $|\omega| > \pi$. The filtered signal is then obtained by inverse transform.

```
load('ex18_5_2.mat') % defines t, x, T
x = x - mean(x);
x = [x, zeros(1,2*length(x))]; % trick to tighthen
    Fourier sampling
N = length(x);
t = (0:N-1)*T;
X = fftshift(T*fft(x));
om = (-round((N-1)/2):round(N/2)-1) *2*pi/(N*T);
Y = X.*(abs(om)>pi); % filter signal
y = ifft(ifftshift(Y)/T); % filtered signal in
   time

figure(1)
subplot(2,2,1)
plot(t,x)
grid
xlabel('t')
ylabel('x(t)')
title('time domain - distorted')
axis([0 20 ylim])
subplot(2,2,2)
plot(t,y)
grid
xlabel('t')
```

```
ylabel('y(t)')
title('time domain - filtered')
axis([0 20 ylim])
subplot(2,2,3)
semilogy(om,abs(X))
axis([xlim 5e-2 5e3])
grid
xlabel('\omega')
ylabel('X(\omega)')
axis([0 20 1e1 1e4])
title('Fourier domain - distorted')
subplot(2,2,4)
semilogy(om,abs(Y))
grid
xlabel('\omega')
ylabel('Y(\omega)')
axis([0 20 1e1 1e4])
title('Fourier domain - filtered')
```



FOUNDATIONS OF SIGNALS AND SYSTEMS
19.2 Solved exercises
Prof. T. Erseghe

Exercises 19.2

Evaluate the following Laplace transforms, by either applying the forward relation or the Laplace properties to known Laplace pairs, and identify their regions of convergence:

1. Evaluate the Laplace transform of the unit step $x(t) = 1(t)$;
2. Evaluate the Laplace transform of the Dirac delta $x(t) = \delta(t)$;
3. Evaluate the Laplace transform of the one-sided exponential $x(t) = e^{p_1 t}\,(t)$;
4. Evaluate the Laplace transform of the one-sided exponential $x(t) = -e^{p_1 t}\,1(-t)$;
5. Evaluate the Laplace transform of the ramp $x(t) = t\,1(t)$;
6. Evaluate the Laplace transform of $x(t) = \cos(\omega_0 t)\,1(t)$;

Solutions.

1. For the unit-step, we apply the forward rule, to have

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} 1(t)\, e^{-st}\, dt \\ &= \int_{0}^{\infty} e^{-st}\, dt = \frac{e^{-st}}{-s}\Big|_0^\infty \\ &= \frac{0-1}{-s} = \frac{1}{s} \end{aligned}$$

   where the integral converges only if $|e^{-s}| = e^{-\Re[s]} < 1$, that is $\Re[s] > 0$, which sets the region of convergence. Correctly, the signal is causal, and the ROC is the region on the right of the rightmost pole (the pole here is $s = 0$).

2. For the delta, we apply the forward rule, to have

$$X(s) = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt = e^{-s\cdot 0} = 1$$

   where we used the sifting property of the delta. Note that the integral converges for any s, hence the region of convergence is the entire complex plane.

3. For the one-sided complex exponential, we apply the forward rule, to have

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} 1(t)\, e^{p_1 t}\, e^{-st}\, dt \\ &= \int_{0}^{\infty} e^{-(s-p_1)t}\, dt = \frac{e^{-(s-p_1)t}}{-(s-p_1)}\Big|_0^{\infty} \\ &= \frac{0-1}{-(s-p_1)} = \frac{1}{s-p_1} \end{aligned}$$

where the integral converges only if $|e^{-(s-p_1)}| = e^{-\Re[s-p_1]} < 1$, that is $\Re[s-p_1] > 0$, which sets the region of convergence to $\Re[s] > \Re[p_1]$. Correctly, the signal is causal, and the ROC is the region on the right of the rightmost pole (the pole here is $s = p_1$).

The result could also have been derived from the couple $1(t) \to \frac{1}{s}$ of Exercise 19.2.1, by applying the modulation rule, providing a shift in the Laplace domain. The definition of the ROC, in this case, can be obtained by identifying the poles in the analytical expression $\frac{1}{s-p_1}$, and by recalling that the considered signal is causal.

4. For this one-sided complex exponential, we apply the forward rule, to have

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} -1(-t)\, e^{p_1 t}\, e^{-st}\, dt \\ &= \int_{-\infty}^{0} -e^{-(s-p_1)t}\, dt = \frac{e^{-(s-p_1)t}}{s-p_1}\Big|_{-\infty}^{0} \\ &= \frac{1-0}{s-p_1} = \frac{1}{s-p_1} \end{aligned}$$

where in this case the integral converges only if $|e^{s-p_1}| = e^{\Re[s-p_1]} < 1$, that is $\Re[s-p_1] < 0$, which sets the region of convergence to $\Re[s] < \Re[p_1]$. Correctly, the signal is anti-causal, and the ROC is the region on the left of the leftmost pole (the only pole here is $s = p_1$). Nicely, note that the analytical expression we obtain is identical to the one of Exercise 19.2.3, but with a different (complementary) region of convergence, thus correctly establishing that the signal information in the Laplace transform is kept by the couple "transform plus ROC."

5. For the ramp we can follow at least two different paths, exploiting the Laplace properties. As a first go we exploit the product-by-t property starting from the pair of Exercise 19.2.1, namely, $y(t) = 1(t)$ and $Y(s) = 1/s$, to have

$$x(t) = t\, y(t) \quad \longrightarrow \quad X(s) = -Y'(s) = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}\,.$$

Alternatively, we can observe that

$$x(t) = 1 * 1(t) = \int_{-\infty}^{\infty} 1(u)1(t-u)\, du = \begin{cases} 0 & , t < 0 \\ \int_0^t 1\, du = t & , t > 0 \end{cases}$$

so that by the convolution rule we have

$$X(s) = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2} \; .$$

The only pole in the analytical expression is $s = 0$, therefore, given that the considered signal is causal, the region of convergence must be $\Re[s] > 0$.

6. For the cosinus, we can exploit Euler's identity to write

$$x(t) = \tfrac{1}{2}\, e^{j\omega_0 t} + \tfrac{1}{2}\, e^{-j\omega_0 t}$$

and then apply the Laplace couple of Exercise 19.2.3, namely $y(t) = e^{p_1 t}\, 1(t)$ and $Y(s) = \frac{1}{s - p_1}$, with $p_1 = \pm j\omega_0$. By linearity we obtain

$$X(s) = \tfrac{1}{2} \frac{1}{s - j\omega_0} + \tfrac{1}{2} \frac{1}{s + j\omega_0} = \frac{s}{(s - j\omega_0)(s + j\omega_0)} = \frac{s}{s^2 - \omega_0^2}$$

which is an analytic expression containing the poles in $s = \pm j\omega_0$, hence the region of convergence is $\Re[s] > 0$.

FOUNDATIONS OF SIGNALS AND SYSTEMS
19.3 Homework assignment
Prof. T. Erseghe

Exercises 19.3

Evaluate the following Laplace transforms, by either applying the forward relation or the Laplace properties to known Laplace pairs, and identify their regions of convergence:

1. Evaluate the Laplace transform of the shifted Dirac delta $x(t) = \delta(t - t_0)$;
2. Evaluate the Laplace transform of the ramp $x_k(t) = \frac{t^k}{k!}\,1(t)$ (can be done through the convolution property, by induction);
3. Evaluate the Laplace transform of the exponential ramp $x_k(t) = \frac{t^k}{k!}\,e^{p_1 t}\,1(t)$;
4. Evaluate the Laplace transform of $x(t) = \sin(\omega_0 t)\,1(t)$;
5. Evaluate the Laplace transform of $x(t) = e^{p_1 t}\,(t) + e^{p_2 t}\,1(-t)$;
6. Evaluate the Laplace transform of the rectangle $x(t) = \mathrm{rect}(t)$;
7. Evaluate the Laplace transform of the triangle $x(t) = \mathrm{triang}(t)$;
8. Evaluate the Laplace transform of the kth derivative of the delta $x(t) = \delta^{(k)}(t)$.

Solutions.

1. For the shifted delta, we apply the forward rule, to have

$$X(s) = \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-st}\, dt = e^{-st_0}$$

where we used the sifting property of the delta. Note that the integral converges for any s, hence the region of convergence is the entire complex plane. Incidentally, the same result can be found by applying the time-shift rule to the Laplace pair of Exercise 19.2.2.

2. We prove the result by induction, by observing that

$$\begin{aligned} x_{k+1}(t) = x_k * 1(t) &= \int_{-\infty}^{\infty} \frac{u^k}{k!}\, 1(u) 1(t-u)\, du \\ &= \begin{cases} 0 & , t < 0 \\ \int_0^t \frac{u^k}{k!}\, du = \frac{t^{k+1}}{(k+1)!} & , t > 0 \end{cases} \end{aligned}$$

Hence by the convolution property we have

$$X_{k+1}(s) = \frac{X_k(s)}{s}$$

where from Exercise 19.2.1 we know that $X_0 = \frac{1}{s}$. Therefore, it must be

$$X_k(s) = \frac{1}{s^{k+1}}$$

with a $(k+1)$th-order pole at $s = 0$, hence the corresponding region of convergence is $\Re[s] > 0$.

3. In this case, the easiest way is to apply the complex exponential rule to the pair of Exercise 19.3.2, namely

$$y_k(t) = \frac{t^k}{k!}\, 1(t) \qquad Y_k(s) = \frac{1}{s^{k+1}} \,,$$

to have

$$x_k(t) = y_k(t)\, e^{p_1 t} \qquad X_k(s) = Y_k(s - p_1) = \frac{1}{(s - p_1)^{k+1}} \,.$$

The result has a $(k+1)$th-order pole at $s = p_1$, hence the corresponding region of convergence is $\Re[s] > \Re[p_1]$.

4. For the sinus, we can exploit Euler's identity to write

$$x(t) = \tfrac{1}{2j}\, e^{j\omega_0 t} - \tfrac{1}{2j}\, e^{-j\omega_0 t}$$

and then apply the Laplace couple of Exercise 19.2.3, namely $y(t) = e^{p_1 t}\,1(t)$ and $Y(s) = \frac{1}{s-p_1}$, with $p_1 = \pm j\omega_0$. By linearity we obtain

$$X(s) = \tfrac{1}{2j}\frac{1}{s - j\omega_0} - \tfrac{1}{2j}\frac{1}{s + j\omega_0} = \frac{\omega_0}{(s - j\omega_0)(s + j\omega_0)} = \frac{\omega_0}{s^2 - \omega_0^2}$$

which is an analytic expression containing the poles in $s = \pm j\omega_0$, hence the region of convergence is $\Re[s] > 0$.

5. In this case we can directly exploit the outcomes of Exercises 19.2.3 and 4, to have

$$X(s) = \frac{1}{s - p_1} - \frac{1}{s - p_2}$$

where the ROC is such that both integrals converge, that is $\Re[p_1] < \Re[s] < \Re[p_2]$. We observe that under the condition $\Re[p1] < \Re[p_2]$ we have an active ROC, while for $\Re[p_1] \geq \Re[p_2]$ the ROC is the empty set, since in this case no complex exponential can guarantee the damping effect that allows both integrals to converge.

6. For the rectangle, we apply the forward rule, to have

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} \text{rect}(t)\, e^{-st}\, dt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-st}\, dt = \frac{e^{-st}}{-s}\bigg|_{-\frac{1}{2}}^{\frac{1}{2}} \\ &= \frac{e^{-\frac{1}{2}s} - e^{\frac{1}{2}s}}{-s} = \frac{e^{\frac{1}{2}s} - e^{-\frac{1}{2}s}}{s} = \frac{\sinh(\frac{s}{2})}{\frac{s}{2}} \end{aligned}$$

where apparently there is a pole at $s = 0$, but actually the hyperbolic sine has a Taylor series starting at s, and it's a continuous function providing $X(0) = 1$. Hence the region of convergence is the entire complex plane.

7. For the triangle we can exploit the convolution rule, to have

$$x(t) = \text{rect} * \text{rect}(t)\ , \qquad X(s) = \left(\frac{\sinh(\frac{s}{2})}{\frac{s}{2}}\right)^2\ ,$$

where we exploited the outcomes of Exercise 19.3.6, and where the region of convergence is the entire complex plane.

8. For the derivatives of the ideal impulse, we can resort to the derivative property applied k-times to the couple of Exercise 19.2.2, namely

$$y(t) = \delta(t)\ , \qquad Y(s) = 1\ ,$$

to have

$$x(t) = y^{(k)}(t) = \delta^{(k)}(t)\ , \qquad X(s) = s^k\, Y(s) = s^k\ ,$$

where the region of convergence is the entire complex plane since there are no poles.

FOUNDATIONS OF SIGNALS AND SYSTEMS
19.5 Solved exercises
Prof. T. Erseghe

Exercises 19.5

Evaluate the inverse unilateral Laplace transform for the following rational functions:

1. $H(s) = \frac{s-3}{s+2}$;
2. $H(s) = \frac{1}{s^3+s^2-6s}$;
3. $H(s) = \frac{4s-1}{2s^2(s-1)}$.

Solutions.

1. The fraction is not proper, therefore, we first need to map it into a proper function by polynomial division. We have

$$\begin{array}{r|l} x-3 & x+2 \\ \cline{2-2} -\,x-2 & 1 \\ \cline{1-1} -\,5 & \end{array}$$

so that

$$H(s) = \frac{s-3}{s+2} = 1 - \frac{5}{s+2}$$

whose inverse transform readily provides

$$h(t) = \delta(t) - 5\,e^{-2t}\,1(t)\ .$$

2. The fraction is proper, therefore we do not need to divide it. We have

$$H(s) = \frac{1}{s(s^2+s-6)}$$

with poles (solutions) in the quadratic equation of the form

$$p_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2} \quad \longrightarrow \quad p_1 = 2\ ,\ p_2 = -3\ .$$

All the poles are distinct. Hence, we can write

$$H(s) = \frac{1}{s(s-2)(s+3)} = \frac{R_0}{s} + \frac{R_1}{s-2} + \frac{R_2}{s+3}$$

with residues given by

$$
\begin{aligned}
R_0 &= H(s)\cdot s\Big|_{s=0} = \frac{1}{(s-2)(s+3)}\Big|_{s=0} = -\tfrac{1}{6} \\
R_1 &= H(s)\cdot (s-2)\Big|_{s=2} = \frac{1}{s(s+3)}\Big|_{s=2} = \tfrac{1}{10} \\
R_2 &= H(s)\cdot (s+3)\Big|_{s=-3} = \frac{1}{s(s-2)}\Big|_{s=-3} = \tfrac{1}{15}
\end{aligned}
$$

By putting the result together, we find

$$
H(s) = -\tfrac{1}{6}\,\frac{1}{s} + \tfrac{1}{10}\,\frac{1}{s-2} + \tfrac{1}{15}\,\frac{1}{s+3}
$$

with inverse transform

$$
h(t) = -\tfrac{1}{6}\,1(t) + \tfrac{1}{10}\,e^{2t}\,1(t) + \tfrac{1}{15}\,e^{-3t}\,1(t)\ .
$$

3. The fraction is proper, therefore we do not need to divide it. However, in this case there is a pole of multiplicity 2, hence we can write

$$
H(s) = \frac{4s-1}{2s^2(s-1)} = \frac{R_0}{s^2} + \frac{R_1}{s} + \frac{R_2}{s-1}
$$

with residues given by

$$
\begin{aligned}
R_0 &= H(s)\cdot s^2\Big|_{s=0} = \frac{4s-1}{2(s-1)}\Big|_{s=0} = \tfrac{1}{2} \\
R_1 &= \frac{d}{ds}\Big(H(s)\cdot (s-2)\Big)\Big|_{s=0} = \frac{d}{ds}\left(\frac{4s-1}{2(s-1)}\right)\Big|_{s=0} \\
&= \frac{4}{2(s-1)} - \frac{4s-1}{2(s-1)^2}\Big|_{s=0} = -2 + \tfrac{1}{2} = -\tfrac{3}{2} \\
R_2 &= H(s)\cdot (s-3)\Big|_{s=1} = \frac{4s-1}{2s^2}\Big|_{s=1} = \tfrac{3}{2}
\end{aligned}
$$

where we observe that for the pole $s = 0$ with multiplicity 2, the residue R_1 needs a derivative. By putting together the results, we obtain

$$
H(s) = \tfrac{1}{2}\,\frac{1}{s^2} - \tfrac{3}{2}\,\frac{1}{s} + \tfrac{3}{2}\,\frac{1}{s-1}\ .
$$

By inverse transform we finally have

$$
h(t) = \tfrac{1}{2}\,t\,1(t) - \tfrac{3}{2}\,1(t) + \tfrac{3}{2}\,e^{t}\,1(t)\ .
$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
19.6 Homework assignment
Prof. T. Erseghe

Exercises 19.6

Evaluate the inverse unilateral Laplace transform for the following rational functions:

1. $H(s) = \frac{s+3}{(s-j)(s^2+1)}$;
2. $H(s) = \frac{1-s}{s^2+5s+6}$;
3. $H(s) = \frac{s^2-s}{s^2+1}$;
4. $H(s) = \frac{2s-1}{(s^2+1)(s^2+4)}$;
5. $H(s) = \frac{s^2(s+4)}{(s+1)(s+2)}$.

Solutions.

1. The fraction is proper, therefore we do not need to divide it. Furthermore, we have

$$H(s) = \frac{s+3}{(s-j)^2(s+j)} = \frac{A}{(s-j)^2} + \frac{B}{s-j} + \frac{C}{s+j}$$

where

$$A = \frac{s+3}{s+j}\Big|_{s=j} = \frac{1-3j}{2}$$
$$B = \frac{d}{ds}\Big(\frac{s+3}{s+j}\Big)\Big|_{s=j} = \frac{3-j}{4}$$
$$C = \frac{s+3}{(s-j)^2}\Big|_{s=-j} = \frac{j-3}{4} = -B$$

providing

$$\begin{aligned} h(t) &= \Big[A\,t\,e^{jt} + B\,e^{jt} - B\,e^{-jt}\Big]1(t) \\ &= \Big[(\tfrac{1}{2} - j\tfrac{3}{2})\,t\,e^{jt} + (\tfrac{1}{2} + j\tfrac{3}{2})\,\sin(t)\Big]1(t)\ . \end{aligned}$$

2. The fraction is proper, therefore we do not need to divide it. Furthermore, we have

$$H(s) = \frac{1-s}{s^2+5s+6} = \frac{1-s}{(s+3)(s+2)} = \frac{A}{s+2} + \frac{B}{s+3}$$

where

$$A = \frac{1-s}{s+3}\Big|_{s=-2} = 3$$
$$B = \frac{1-s}{s+2}\Big|_{s=-3} = -4$$

and therefore

$$h(t) = 3\,e^{-2t}\,1(t) - 4\,e^{-3t}\,1(t)\ .$$

3. The fraction is not proper, therefore we need to divide it, obtaining

$$H(s) = \frac{s^2-s}{s^2+1} = 1 - \frac{s+1}{s^2+1}$$

which is already in an invertible form involving sinusoids, that is

$$h(t) = \delta(t) - \cos(t)1(t) - \sin(t)1(t)\ .$$

4. The fraction is proper, therefore we do not need to divide it. Furthermore, we note that the denominator is a function of s^2, with poles of the form $p_{1,2} = \pm j$ and $p_{3,4} = \pm 2j$, which reveals the presence of sinusoids. Although we can proceed with the four poles, and identify four residues

(complex valued, but the final result will be real valued because of real-valued coefficients), we can exploit a residue mapping in $x = s^2$ of the form

$$\frac{1}{(x+1)(x+4)} = \frac{R_1}{x+1} + \frac{R_2}{x+4}$$

where

$$R_0 = \left.\frac{1}{x+4}\right|_{x=-1} = \tfrac{1}{3}$$
$$R_1 = \left.\frac{1}{x+1}\right|_{x=-4} = -\tfrac{1}{3}$$

to write

$$H(s) = \frac{2s-1}{(s^2+1)(s^2+4)} = \frac{\frac{2}{3}s - \frac{1}{3}}{s^2+1} + \frac{-\frac{2}{3}s + \frac{1}{3}}{s^2+4}$$

which is already in an invertible form, since it is the linear combination of sinusoids, providing

$$h(t) = \left[\tfrac{2}{3}\cos(t) - \tfrac{1}{3}\sin(t) - \tfrac{2}{3}\cos(2t) + \tfrac{1}{6}\sin(2t)\right] 1(t)\ .$$

5. The fraction is not proper, therefore we need to divide it. By expanding

$$H(s) = \frac{s^2(s+4)}{(s+1)(s+2)} = \frac{s^3+4s^2}{s^2+3s+2}$$

we have a polynomial division of the form

$$\begin{array}{rl|l} & x^3 + 4x^2 & x^2+3x+2 \\ \cline{3-3} - & x^3 - 3x^2 - 2x & x+1 \\ \cline{1-2} & x^2 - 2x & \\ - & x^2 - 3x - 2 & \\ \cline{1-2} & -5x - 2 & \end{array}$$

so that

$$H(s) = s + 1 - \frac{5s+2}{(s+1)(s+2)} = s + 1 - \frac{A}{s+1} - \frac{B}{s+2}$$

where

$$A = \left.\frac{5s+2}{s+2}\right|_{s=-1} = -3$$
$$B = \left.\frac{5s+2}{s+1}\right|_{s=-2} = 8$$

By inversion we finally obtain

$$h(t) = \delta'(t) + \delta(t) + 3\,e^{-t}\,1(t) - 8\,e^{-2t}\,1(t)\ .$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
20.2 Solved exercises
Prof. T. Erseghe

Exercises 20.2

Solve the following problems on differential equations:

1. Consider an RC filter driven by the differential equation

$$y(t) + RCy'(t) = x(t) \ ,$$

for input voltage $x(t) = A$ and initial condition on the output voltage $y(0_-) = V_0$. Evaluate the output response $y(t)$ for $t > 0$, and the steady state condition on y.

2. Consider the differential equation

$$y''(t) - y'(t) - 6y(t) = x'(t) - 3x(t) \ ,$$

for which it is required to evaluate the transfer function $H(s)$ and its BIBO stability properties, as well as the forced response for $x(t) = 1(t)$; then consider zero initial conditions and an input $x(t) = A\cos(\omega_0 t + \varphi_0)$ and identify for which values of ω_0 the output at steady state assumes the form $y(t) = \frac{1}{5}x(t - t_0)$.

3. Consider the spring-mass system system described by the differential equation

$$x(t) = ky(t) + my''(t) \ ,$$

where the input force is set to $x(t) = F_0\cos(\omega_0 t)$ and the initial conditions are $y(0_-) = y_0$ and $y'(0_-) = v_0$. Identify the output $y(t)$ for $t > 0$.

Solutions.

1. If we map the RC filter equation in the (unilateral) Laplace domain, we have

$$Y(s) + RC\,(sY(s) - V_0) = X(s)$$

so that

$$Y(s) = \frac{1}{1 + RC\,s}X(s) + \frac{RC\,V_0}{1 + RC\,s} = \frac{\beta}{s + \beta}\,X(s) + \frac{V_0}{s + \beta} \ ,$$

where $\beta = \frac{1}{RC}$ is the key constant of the RC filter. Incidentally, the impulse response provides

$$H(s) = \frac{\beta}{s + \beta} \quad \Longrightarrow \quad h(t) = \beta e^{-\beta t}\,1(t) \ ,$$

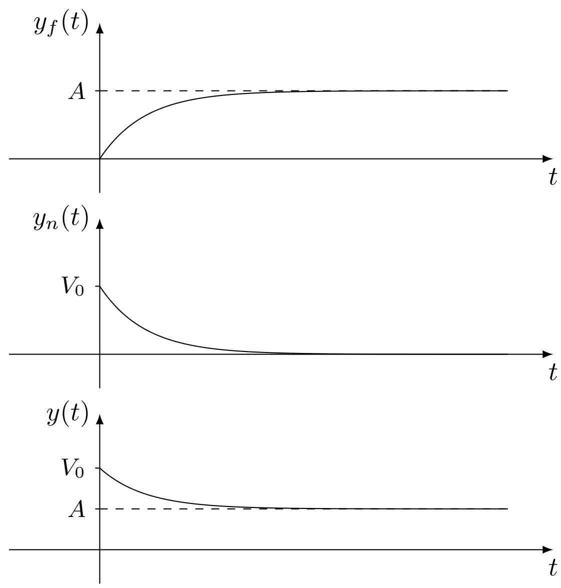
while from the unilateral transform of $x(t)$, namely, $X(s) = A/s$, we have

$$Y(s) = \frac{A\beta}{s(s+\beta)} + \frac{V_0}{s+\beta} = \frac{A}{s} - \frac{A}{s+\beta} + \frac{V_0}{s+\beta}$$

and therefore

$$y(t) = \underbrace{A\,(1 - e^{-\beta t})\,1(t)}_{y_f} + \underbrace{V_0\,e^{-\beta t}\,1(t)}_{y_n} \; .$$

At steady state, $t \gg 0$, the exponential goes to zero and we have $y(t) = A$, as depicted in figure.



2. From the differential equation we can straightforwardly identify the transfer function

$$H(s) = \frac{s-3}{s^2 - s - 6} = \frac{s-3}{(s-3)(s+2)} = \frac{1}{s+2} \quad \Longrightarrow \quad h(t) = e^{-2t}\,1(t)$$

which is evidently BIBO stable since the pole $p_1 = -2$ has negative real part. The forced response to the unit step, for which $X(s) = 1/s$, is simply

$$Y_f(s) = H(s)X(s) = \frac{1}{s(s+2)} = \tfrac{1}{2}\frac{1}{s} - \tfrac{1}{2}\frac{1}{s+2}$$

so that

$$y_f(t) = \tfrac{1}{2}\,(1 - e^{-2t})\;1(t) \; .$$

At steady state, instead, the system action on $x(t) = A\cos(\omega_0 t + \varphi_0)$ is simply that of filter $h(t)$, and from the properties of the Fourier transform we have

$$y(t) = |H(j\omega_0)|\, A\cos(\omega_0 t + \varphi_0 + \arg(H(j\omega_0)))$$

where

$$H(j\omega_0) = \left.\frac{1}{s+2}\right|_{s=j\omega_0} = \frac{1}{2+j\omega_0} .$$

Hence, it is

$$y(t) = |H(j\omega_0)|\, x(t-t_0) , \qquad t_0 = -\frac{\arg(H(j\omega_0))}{\omega_0}$$

the request thus being equivalent to

$$|H(j\omega_0)| = \tfrac{1}{5} \quad \Longrightarrow \quad 25 = \omega_0^2 + 4$$

so that it must be $\omega_0 = \pm\sqrt{21}$.

3. By mapping the differential equation in the (unilateral) Laplace domain we have

$$X(s) = kY(s) + m(s^2 Y(s) - sy_0 - v_0) ,$$

so that

$$\begin{aligned} Y(s) &= \frac{X(s) + my_0 s + mv_0}{ms^2 + k} \\ &= \underbrace{\tfrac{1}{m}\overbrace{\frac{1}{s^2+\omega_1^2}}^{H(s)} X(s)}_{Y_f(s)} + \underbrace{y_0 \frac{s}{s^2+\omega_1^2} + \tfrac{v_0}{\omega_1}\frac{\omega_1}{s^2+\omega_1^2}}_{Y_n(s)} , \qquad \omega_1 = \sqrt{\frac{k}{m}} \end{aligned}$$

where

$$X(s) = \frac{F_0 s}{s^2 + \omega_0^2}$$

Now, in case $\omega_0 \neq \omega_1$, we have

$$Y_f(s) = \tfrac{F_0 s}{m} \frac{1}{(s^2+\omega_0^2)(s^2+\omega_1^2)} = \tfrac{F_0 s}{m}\left[\frac{A}{s^2+\omega_0^2} + \frac{B}{s^2+\omega_1^2}\right]$$

where

$$B = -A , \quad A = \frac{1}{\omega_1^2 - \omega_0^2}$$

so that

$$Y(s) = \tfrac{F_0}{m(\omega_1^2-\omega_0^2)}\left[\frac{s}{s^2+\omega_0^2} - \frac{s}{s^2+\omega_1^2}\right] + y_0 \frac{s}{s^2+\omega_1^2} + \tfrac{v_0}{\omega_1}\frac{\omega_1}{s^2+\omega_1^2}$$

and therefore

$$y(t) = \underbrace{\tfrac{F_0}{m(\omega_1^2-\omega_0^2)}\left[\cos(\omega_0 t) - \cos(\omega_1 t)\right]}_{\text{forced}} + \underbrace{y_0 \cos(\omega_1 t) + \tfrac{v_0}{\omega_1}\sin(\omega_1 t)}_{\text{natural}}$$

for $t > 0$. Note that the natural response is sinusoidal, i,e, it does not vanish for large t, and in fact the transfer function here is not BIBO stable.

In case $\omega_0 = \omega_1$, instead, it is

$$Y_f(s) = \tfrac{F_0}{m}\frac{s}{(s^2+\omega_0^2)^2} = \tfrac{F_0}{2\omega_0 m}\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$$

where we note that, by the multiplication by t rule, we have

$$t\sin(\omega_0 t) \quad \Longrightarrow \quad -\frac{d}{ds}\left(\frac{\omega_0}{s^2+\omega_0^2}\right) = \frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$$

and therefore

$$y(t) = \underbrace{\tfrac{F_0}{2\omega_0 m}\, t\sin(\omega_0 t)}_{\text{forced}} + \underbrace{y_0\cos(\omega_0 t) + \tfrac{v_0}{\omega_0}\sin(\omega_0 t)}_{\text{natural}}$$

which diverges for large t since the system is not BIBO stable.

FOUNDATIONS OF SIGNALS AND SYSTEMS
20.3 Homework assignment
Prof. T. Erseghe

Exercises 20.3

Solve the following problems on differential equations:

1. The input to a system described by differential equation is $x(t) = \delta(t) + 2\,e^{-t}\,1(t)$, and the forced response has, in the unilateral Laplace domain, the form
$$Y_f(s) = \frac{1}{s(s+a)} \ ,$$
with steady-state condition $y_f(t) = 1$ for $t \gg 0$. We wish to: a) identify the value of the real parameter a; b) identify the transfer function $H(s)$ as well as the differential equation to which it relates; c) identify the BIBO stability properties of the system; d) identify the natural response under the initial conditions $y(0_-) = y'(0_-) = 0$ and $y''(0_-) = 1$.

2. Consider the system described by the differential equation
$$y'''(t) + 2y''(t) - 19y'(t) - 20y(t) = x''(t) + x'(t) \ ,$$
where we know that $p_1 = 1$ is a pole. We want to: a) determine the transfer function; b) determine wether the system is BIBO stable; c) determine the system output with input $x(t) = 1(t)$ and zero initial conditions; d) identify the transfer function $H_2(s)$ of a system that, in cascade with the given system, makes the cascade BIBO stable.

Solutions.

1. To identify the value of a, we transform the forced response to the time domain, to have

$$Y_f(s) = \frac{1}{s(s+a)} = \frac{1/a}{s} - \frac{1/a}{s+a} \ ,$$

so that

$$y_f(t) = \frac{1}{a}\left(1 - e^{-at}\right) 1(t) \ .$$

In order to have $y_f(t) = 1$ for $t \gg 0$, it needs to be $a > 0$, so that $y_f(t) = 1/a$ for $t \gg 0$, and therefore we have $a = 1$. The transfer function is then obtained by exploiting the relation

$$Y_f(s) = H(s)X(s) = \frac{1}{s(s+1)} X(s) \ , \quad X(s) = 1 + \frac{2}{s+1} = \frac{s+3}{s+1} \ ,$$

so that

$$H(s) = \frac{Y_f(s)}{X(s)} = \frac{s+1}{s(s+1)(s+3)} = \frac{1}{s(s+3)} = \frac{1}{s^2+3s} \ .$$

with poles $p_1 = 0$ and $p_2 = -3$, which identify a system which is not BIBO stable. The differential equation follows from $H(s)$ to have

$$x(t) = y''(t) + 3y'(t) \ .$$

The natural response for $y(0_-) = y'(0_-) = 0$ is identified by zero valued initial conditions (the value $y''(0_-) = 1$ is not part of the initial conditions) and therefore $y_n(t) = 0$.

2. The transfer function follows directly from the differential equation, to have

$$H(s) = \frac{s^2+s}{s^3+2s^2-19s-20} = \frac{s(s+1)}{(s+1)(s+5)(s-4)} = \frac{s}{(s+5)(s-4)} \ ,$$

with active poles $p_2 = -5$ and $p_3 = 4$. The system is not BIBO stable since $p_3 = 4$ has positive real part. The output, which in this case corresponds to the forced response, is given by

$$\begin{aligned} Y_f(s) = H(s)X(s) &= \frac{s}{(s+5)(s-4)} \cdot \frac{1}{s} = \frac{1}{(s+5)(s-4)} \\ &= -\tfrac{1}{9}\frac{1}{(s+5)} + \tfrac{1}{9}\frac{1}{(s-4)} \end{aligned}$$

to have

$$y_f(t) = -\tfrac{1}{9}e^{-5t}\,1(t) + \tfrac{1}{9}e^{4t}\,1(t) \ ,$$

which diverges, thus confirming that the system is not BIBO stable. To overcome BIBO stability we need to cancel the pole $p_3 = 4$, for example using the BIBO stable system

$$H_2(s) = \frac{s-4}{s+a}$$

with $\Re[a] > 0$, to have

$$H(s)H_2(s) = \frac{s}{(s+5)(s-4)} \cdot \frac{s-4}{s+a} = \frac{s}{(s+5)(s+a)} .$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
20.5 Solved exercises
Prof. T. Erseghe

Exercises 20.5

Evaluate the unilateral Z transform, or its inverse, for the following signals:

1. the discrete-time unit-step $x(n) = 1_0(n)$;
2. the discrete-ramp signals
$$x_k(n) = \frac{1}{k!}\,(n+k)\dots(n+2)(n+1)\,1_0(n)\ ,$$
with $x_0(n) = 1_0(n)$, to obtain $X_k(z) = (1-z^{-1})^{-(k+1)}$;
3. the shifted delta $x(n) = \delta(n-n_0)$ for $n_0 \geq 0$;
4. the one-sided exponential $x(n) = p_0^{n+1}\,1_0(n)$;
5. the modulated ramp
$$x_k(n) = \frac{1}{k!}\,(n+k)\dots(n+2)(n+1)\,p_0^{n+k+1}\,1_0(n)\ ;$$
6. the sinusoid $x(n) = \cos(\theta_0 n)\,1_0(n)$.

Then solve the following difference equations system:

7. Consider the discrete-time system described by equation
$$x(n) = y(n-2) + y(n-1) - 6y(n)\ ,$$
where $x(n) = A$, $y(-1) = k_1$, and $y(-2) = k_2$. Identify the system impulse response $h(n)$ and the output $y(n)$.

Solutions.

1. For the unit step, we have
$$X(z) = \sum_{n=-\infty}^{\infty} 1_0(n)\,z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}$$
valid in the region where $|z^{-1}| < 1$, that is for $|z| > 1$.
2. We observe that, starting from $x_0(n) = 1_0(n)$, with transform $X_0(z) = 1/(1-z^{-1})$, as we have already seen, signals are defined through the recursion
$$\begin{aligned} x_k(n) &= \tfrac{1}{k}\,(n+1)\,x_{k-1}(n+1) \\ &= \tfrac{1}{k!}\,(n+k)\dots(n+2)(n+1)\,1_0(n+1) \\ &= \tfrac{1}{k!}\,(n+k)\dots(n+2)(n+1)\,1_0(n) \end{aligned}$$

where we replaced $1_0(n+1)$ with $1_0(n)$ since in $n=-1$ the factor $n+1$ guarantees that the signal value is zero. The validity of this recursion suggests proving the result by induction. We therefore assume that the transform is correct at index $k-1$, and want to prove its correctness at index k. From the recursion, to obtain $x_k(n)$ from $x_{k-1}(n)$ we need to: 1) multiply by n and by the constant factor $\frac{1}{k}$, and 2) time-shift by -1, to have

$$x_k(n) = u(n+1) \;, \qquad u(n) = \tfrac{1}{k}\, n\, x_{k-1}(n) \;.$$

Now, from the multiplication-by-n rule, we easily identify the Z transform of $u(n)$ as

$$\begin{aligned} U(z) &= -\tfrac{1}{k}\, z\, \frac{dX_{k-1}(z)}{dz} \\ &= -\tfrac{1}{k}\, z\, \frac{d}{dz}\left(\frac{1}{(1-z^{-1})^k}\right) \\ &= -\tfrac{1}{k}\, z \cdot -k\, \frac{1}{(1-z^{-1})^{k+1}} \cdot z^{-2} = \frac{z^{-1}}{(1-z^{-1})^{k+1}} \end{aligned}$$

while it is easy to see that

$$\begin{aligned} X_k(z) &= \sum_{n=0}^{\infty} u(n+1)\, z^{-n} \\ &= \sum_{m=1}^{\infty} u(m)\, z^{-(m-1)} = \sum_{m=0}^{\infty} u(m)\, z^{-(m-1)} \\ &= z\, U(z) = \frac{1}{(1-z^{-1})^{k+1}} \;, \end{aligned}$$

where $m=n+1$, and where in the second row we exploited the fact that $u(0)=0$ by construction. This proves the result.

3. In this case we simply apply the forward transform

$$X(z) = \sum_{n=0}^{\infty} \delta(n-n_0)\, z^{-n} = z^{-n_0}$$

which is valid for $|z|>0$

4. In this case we simply apply the forward transform

$$X(z) = \sum_{n=0}^{\infty} p_0^{n+1}\, z^{-n} = p_0 \sum_{n=0}^{\infty} (p_0 z^{-1})^n = \frac{p_0}{1-p_0 z^{-1}} = \frac{1}{p_0^{-1} - z^{-1}}$$

with associated ROC of the form $|p_0 z^{-1}| < 1$, that is $|z| > |p_0|$.

5. For the modulated ramp we have

$$x_k(n) = p_0^{n+k+1} y_k(n)\ , \quad y_k(n) = \frac{1}{k!}\,(n+k)\dots(n+2)(n+1)\ 1_0(n)\ ;$$

so that, by the multiplication rule we have

$$X_n(z) = p_0^{k+1} Y_n(z/p_0)\ , \quad Y_n(z) = \frac{1}{(1-z^{-1})^{k+1}}$$

where we used the result from Exercise 20.5.2. Hence, it is

$$X_n(z) = \frac{1}{(p_0^{-1} - z^{-1})^{k+1}}$$

with associated ROC of the form $|z| > |p_0|$.

6. For the sinusoid we simply apply the forward transform

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} \tfrac{1}{2} e^{j\theta_0 n}\, z^{-n} + \sum_{n=0}^{\infty} \tfrac{1}{2} e^{-j\theta_0 n}\, z^{-n} \\ &= \tfrac{1}{2}\frac{1}{1-e^{j\theta_0} z^{-1}} + \tfrac{1}{2}\frac{1}{1-e^{-j\theta_0} z^{-1}} \\ &= \tfrac{1}{2}\frac{2 - e^{-j\theta_0} z^{-1} - e^{j\theta_0} z^{-1}}{(1-e^{j\theta_0} z^{-1})(1-e^{-j\theta_0} z^{-1})} \\ &= \frac{1-\cos(\theta_0) z^{-1}}{1 - 2\cos(\theta_0) z^{-1} + z^{-2}} \end{aligned}$$

7. The impulse response can be identified by simply looking at the equation to get the transfer function

$$H(z) = \frac{1}{z^{-2} + z^{-1} - 6} = \frac{1}{(z^{-1}+3)(z^{-1}-2)} = \frac{\frac{1}{5}}{z^{-1}-2} - \frac{\frac{1}{5}}{z^{-1}+3}$$

where all the polynomials are in z^{-1} (you can replace $x = z^{-1}$ if this is less confusing). Note that the equation is in z^{-1}, so that the poles are not 2 and -3, but are $p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{-3} = -\frac{1}{3}$, which make the system BIBO stable. From standard Z couples, we then have

$$h(n) = \frac{1}{5}\left(p_2^{n+1} - p_1^{n+1}\right) 1_0(n) = \frac{1}{5}\left((-\tfrac{1}{3})^{n+1} - (\tfrac{1}{2})^{n+1}\right) 1_0(n)$$

In order to identify the full output $y(n)$, we preliminarily need to map the equation in the unilateral Z domain, to have

$$X(z) = z^{-2} Y(z) + z^{-1} y(-1) + y(-2) + z^{-1} Y(z) + y(-1) - 6 Y(z)$$

so that

$$Y(z) = \frac{X(z)}{z^{-2} + z^{-1} - 6} - \frac{k_1 z^{-1} + k_1 + k_2}{z^{-2} + z^{-1} - 6}\ , \qquad X(z) = \frac{A}{1 - z^{-1}}\ ,$$

where $X(z)$ is the unilateral transform. Hence, for the forced response we have

$$Y_f(z) = \frac{-A}{(z^{-1}+3)(z^{-1}-2)(z^{-1}-1)} = \frac{-\frac{A}{20}}{z^{-1}+3} + \frac{-\frac{A}{5}}{z^{-1}-2} + \frac{\frac{A}{4}}{z^{-1}-1} \ ,$$

where we used the standard residues method in z^{-1} to get the result, which guarantees that

$$y_f(n) = A\Big(\tfrac{1}{20}(-\tfrac{1}{3})^{n+1} + \tfrac{1}{5}(\tfrac{1}{2})^{n+1} - \tfrac{1}{4}\Big)\, 1_0(n) \ .$$

For the natural response, instead, we have

$$Y_n(z) = -\frac{k_1 z^{-1} + k_1 + k_2}{(z^{-1}+3)(z^{-1}-2)} = -\frac{2k_1 - k_2}{5}\frac{1}{z^{-1}+3} - \frac{3k_1 + k_2}{5}\frac{1}{z^{-1}-2}$$

so that

$$y_n(n) = \Big(\tfrac{2k_1-k_2}{5}(-\tfrac{1}{3})^{n+1} + \tfrac{3k_1+k_2}{5}(\tfrac{1}{2})^{n+1}\Big)\, 1_0(n) \ .$$

FOUNDATIONS OF SIGNALS AND SYSTEMS
20.6 Homework assignment
Prof. T. Erseghe

Exercises 20.6

Solve the following problems on difference equations:

1. Consider a discrete-time system with transfer function

$$H(z) = \frac{1}{(z^{-1}+1)(z^{-1}+3)} \ .$$

   We want to know: a) which are the poles; b) whether the system is BIBO stable or not, and in case it is not identify a limited input signal that provides an unlimited output; c) the input signal if the output is $y(n) = -(-\frac{1}{3})^{n+1} 1_0(n)$ with zero initial conditions.

2. Consider a discrete-time system with impulse response and input

$$h(n) = (1+2n)\,(-1)^n\, 1_0(n) + \tfrac{1}{2}\,(-\tfrac{1}{2})^n\, 1_0(n) \ , \quad x(n) = \tfrac{1}{3}\,(-\tfrac{1}{3})^n\, 1_0(n) \ ,$$

   and with zero initial conditions $y(n) = 0$ for $n < 0$. We want to know: a) if the system is BIBO stable; b) the difference equation that describes the system; and c) the natural and forced responses.

Solutions.

1. By inspection the poles are $p_1 = \frac{1}{-1} = -1$ and $p_2 = \frac{1}{-3}$. Since $P - 1$ lays in the unit circle, the system is not BIBO stable. A limited input providing an unlimited output is one that stimulates the unstable pole, for example

$$x(n) = \delta(n) - 2(-1)^{n+1}\,1(n) \quad \Longrightarrow \quad X(z) = 1 + \frac{2}{z^{-1}+1} = \frac{z^{-1}+3}{z^{-1}+1}$$

to have a forced response of the form

$$Y_f(z) = H(z)X(z) = \frac{1}{(z^{-1}+1)^2}$$

that is

$$y_f(n) = (n+1)\,(-1)^{n+2}\,1(n)\;.$$

In the case the initial conditions are zero, then the output is the forced response, that is we have

$$Y(z) = Y_f(z) = H(z)X(z) = \frac{X(z)}{(z^{-1}+1)(z^{-1}+3)} = \frac{1}{z^{-1}+3}\;.$$

By solving on X we get $X(z) = z^{-1} + 1$, that is $x(n) = \delta(n) + \delta(n-1)$.

2. By inspection of the impulse response, we see that it is diverging for large n, hence the system is not BIBO stable. The difference equation can be easily obtained from the transfer function, that is the Z transform of $h(n)$. To this end, we recall the Z pairs

$$\begin{aligned} g_1(n) &= -(-1)^{n+1}\,1_0(n) \quad \Longrightarrow \quad \frac{1}{z^{-1}+1} \\ g_2(n) &= (n+1)\,(-1)^{n+2}\,1_0(n) \quad \Longrightarrow \quad \frac{1}{(z^{-1}+1)^2} \\ g_3(n) &= -(-\tfrac{1}{2})^{n+1}1_0(n) \quad \Longrightarrow \quad \frac{1}{z^{-1}+2} \end{aligned}$$

which allow writing $h(n) = -g_1(n) + 2g_2(n) + g_3(n)$ so that

$$\begin{aligned} H(z) &= -\frac{1}{z^{-1}+1} + \frac{2}{(z^{-1}+1)^2} + \frac{1}{z^{-1}+2} \\ &= \frac{z^{-1}+3}{(z^{-1}+1)^2(z^{-1}+2)} = \frac{z^{-1}+3}{z^{-3}+4z^{-2}+5z^{-1}+2} \end{aligned}$$

and the equation is

$$x(n-1) + 3x(n) = y(n-3) + 4y(n-2) + 5y(n-1) + 2y(n)\;.$$

Because of zero initial conditions on $y(n)$ and $x(n)$, the natural response is $y_n(n) = 0$. For the forced response, instead we can work in the (unilateral) Z domain to have

$$Y_f(z) = H(z)X(z) = \frac{z^{-1}+3}{(z^{-1}+1)^2(z^{-1}+2)}\, X(z)\ , \quad X(z) = \frac{1}{z^{-1}+3}$$

that is

$$Y_f(z) = \frac{1}{(z^{-1}+1)^2(z^{-1}+2)} = -\frac{1}{z^{-1}+1} + \frac{1}{(z^{-1}+1)^2} + \frac{1}{z^{-1}+2}$$

so that

$$\begin{aligned} y_f(n) &= -g_1(n) + g_2(n) + g_3(n) \\ &= (2+n)\,(-1)^n\,1_0(n) + \tfrac{1}{2}\,(-\tfrac{1}{2})^n\,1_0(n)\ . \end{aligned}$$