Functional Analysis- teacher A. Cesaroni

Exam- Janauary 13, 2025- (90 minutes)

Exercise 1.

(1) Give the definition of absolutely continuous measure and of singular measure (with respect to the Lebesgue measure in $\mathbb{R}$).
(2) Consider the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 2 - e^{-x} & x \geq 0 \end{cases}$$

and let μ_F the Borel measure associated to this function. What is the singular part of this measure? What is the density of the absolutely continuous part of this measure?

Exercise 2.
Let $H = M^2(\Omega, \mathbb{P}, \mathcal{F})$ the space of random variables with bounded second moment. For $\mathcal{G}$ a σ-algebra in Ω strictly contained in $\mathcal{F}$, we define $V = M^2(\Omega, \mathbb{P}, \mathcal{G})$ the closed subspace of H which contains $\mathcal{G}$-measurable random variables with bounded second moment.

(1) State the orthogonal projection theorem in a general Hilbert space H. Using the notion of orthonormal basis, state the formula to compute the orthogonal projection of a generic element $h \in H$ to a closed subspace $V \subseteq H$.
(2) Let $\mathcal{G} = \{\emptyset, \Omega\}$ and compute the orthogonal projection of a random variable $X \in H$ in the closed subspace $M^2(\Omega, \mathbb{P}, \mathcal{G})$.
(3) Let $A \in \mathcal{F}$, and $\mathcal{G} = \{\emptyset, \Omega, A, \Omega \setminus A)$. Compute the orthogonal projection of a random variable $X \in H$ in the closed subspace $M^2(\Omega, \mathbb{P}, \mathcal{G})$.
Hint: the space $M^2(\Omega, \mathbb{P}, \mathcal{G}) = \{a\chi_A + b$, with $a, b \in \mathbb{R}\}$, where $\chi_A(\omega)$ is the random variable which assumes values 1 for $\omega \in A$ and 0 elsewhere.

Exercise 3.

(1) Recall the definition of Fourier transform of a function f and its main properties.
(2) Let

$$f(x) = \begin{cases} 1 & x \in [-c, c] \\ 0 & x \in (-\infty, -c) \cup (c, +\infty). \end{cases}$$

Compute the Fourier transform of f.

Sketch of solutions

Solution 1.

2 F is continuous in $\mathbb{R}\backslash 0$. Moreover $F'(x) = f(x) = e^{-x}\chi_{(0,+\infty)}$ and F is constant in $x < 0$. Therefore the singular part of μ_F is given by δ_0 and the absolutely continuous part has density $f(x)$.

Solution 2.

2 $M^2(\Omega, \mathbb{P}, \mathcal{G})$ is the space of constant random variables, so the projection of X is given by $\mathbb{E}(X)$ since $X - \mathbb{E}(X)$ is orthogonal to all the constant random variables.

3 $M^2(\Omega, \mathbb{P}, \mathcal{G})$ is a 2 dimensional space generated by $1, \chi_A$. We orthonormalize this basis, and obtain $X_1 = 1$, $X_2 = \frac{\chi_A - \mathbb{E}(\chi_A)}{\sqrt{\mathbb{E}[(\chi_A - \mathbb{E}(\chi_A))^2]}}$. Since $\mathbb{E}(\chi_A) = \mathbb{P}(A)$, we obtain

$$X_2 = \frac{\chi_A - \mathbb{P}(A)}{\sqrt{\mathbb{P}(A)(1 - \mathbb{P}(A)}}.$$

So the projection of a random variable X is given by

$$\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X1) + \mathbb{E}(XX_2)X_2 = \mathbb{E}(X) + \frac{\mathbb{E}(X\chi_A) - \mathbb{E}(X)\mathbb{P}(A)}{\mathbb{P}(A)(1 - \mathbb{P}(A))}(\chi_A - \mathbb{P}(A)).$$

Solution 3.

2 By definition,

$$\hat{f}(x) = \int_{\mathbb{R}} f(y)e^{ixy}dy = \int_{-c}^{c} \cos xy dy + i\int_{-c}^{c} \sin xy dy = 2\int_{0}^{c} \cos xy dy = \begin{cases} \frac{2}{x}\sin cx & x \neq 0 \\ 2c & x = 0. \end{cases}$$