COMPUTABILITY (07/01/2025)

Computability
Jan 19 2022

definitions
proofs
small variations

Exercise 1

a. Provide the definition of reducibility, i.e., given sets $A, B \subseteq \mathbb{N}$ define what it means that $A \leqslant_m B$.

b. Show that if A is not recursive and $A \leqslant_m B$ then B is not recursive.

c. Show that if A is recursive then $A \leqslant_m \{1\}$.

constructions of PR / R
diagonalisation
smn

Exercise 2

Is there a non-computable total function $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = f(x+1)$ on infinitely many inputs x, i.e., such that the set $\{x \in \mathbb{N} \mid f(x) = f(x+1)\}$ is infinite? Provide an example or show that such a function cannot exist.

classify sets (recursive, r.e.), saturatedness

Exercise 3

Say that a function $f : \mathbb{N} \to \mathbb{N}$ is quasi-total if it is undefined on a finite number of inputs, i.e., $\overline{dom(f)}$ is finite. Classify the set $A = \{x \in \mathbb{N} \mid \varphi_x \text{ quasi-total}\}$ from the point of view of recursiveness, i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 4

Classify the set $B = \{x \in \mathbb{N} \mid \exists y > 2x.\ y \in E_x\}$ from the point of view of recursiveness, i.e., establish whether B and $\bar{B}$ are recursive/recursively enumerable.

Note: Each exercise contributes with the same number of points (8) to the final grade.

Exercise 1

a. Provide the definition of reducibility, i.e., given sets $A, B \subseteq \mathbb{N}$ define what it means that $A \leqslant_m B$.

b. Show that if A is not recursive and $A \leqslant_m B$ then B is not recursive.

c. Show that if A is recursive then $A \leqslant_m \{1\}$.

(a) Given sets $A, B \subseteq \mathbb{N}$ we write $A \leq_m B$ when there is a reduction function i.e. a total computable function $f : \mathbb{N} \to \mathbb{N}$ s.t. $\forall x \in \mathbb{N}$

$$x \in A \quad \text{iff} \quad f(x) \in B$$

(b) We prove the countermominal i.e.

if $A \leq_m B$ and B is recursive then A is recursive

assume $A \leq_m B$ and let $f : \mathbb{N} \to \mathbb{N}$ be the reduction function, total computable s.t.

$$\forall x \quad x \in A \quad \text{iff} \quad f(x) \in B \qquad (*)$$

B recursive i.e.

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases} \quad \text{is computable} \qquad (**)$$

The characteristic function of A is

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \Leftrightarrow f(x) \in B \\ 0 & \text{if } x \notin A \Leftrightarrow f(x) \notin B \end{cases} \overset{(**)}{=} \chi_B(f(x))$$

by $(*)$

Hence χ_A, composition of computable functions χ_B and f is computable, i.e. A recursive.

(c) if A recursive then $A \leq_m \{1\}$

if A is recursive then

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$ is computable (& total)

Note that $\forall x$

$x \in A$ iff $\chi_A(x) = 1$ iff $\chi_A(x) \in \{1\}$

i.e. χ_A is a reduction function for $A \leq_m \{1\}$

EXTRA QUESTION: Does the converse hold? i.e.

if $A \leq_m \{1\}$ then A is recursive ?

yes: $\{1\}$ is finite hence recursive

since $A \leq_m \{1\}$ then A recursive

more directly:

let $f : \mathbb{N} \to \mathbb{N}$ be a reduction function for $A \leq_m \{1\}$

i.e. f is total computable and

$\forall x \in \mathbb{N}$ $x \in A$ iff $f(x) \in \{1\}$ iff $f(x) = 1$

hence

$$\chi_A(x) = \begin{cases} 1 & \text{if } f(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

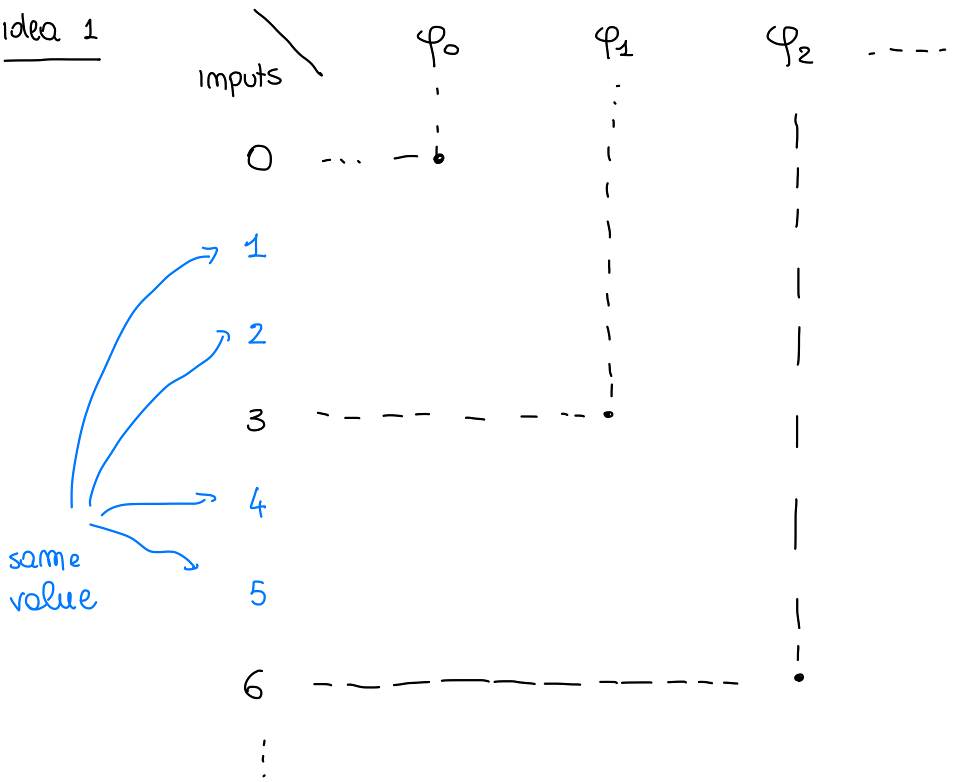
$$= \overline{sg}(|f(x) - 1|)$$

computable by composition.

Exercise 2

Is there a non-computable total function $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = f(x+1)$ on infinitely many inputs x, i.e., such that the set $\{x \in \mathbb{N} \mid f(x) = f(x+1)\}$ is infinite? Provide an example or show that such a function cannot exist.

<u>idea 1</u>



$$f(x) = \begin{cases} \varphi_y(x) + 1 & \text{if } x = 3y \text{ and } \varphi_y(x)\downarrow \\ 0 & \text{if } x = 3y \text{ and } \varphi_y(x)\uparrow \\ 0 & \text{if } x \neq 3y \ \forall y \end{cases}$$

function f is the desired one:

- f is total by definition
- f not computable since it is different from all computable functions

  $\forall y \qquad f(3y) \neq \varphi_y(3y)$

  in fact if $\varphi_y(3y)\downarrow \Rightarrow f(3y) = \varphi_y(3y)+1 \neq \varphi_y(3y)$

  if $\varphi_y(3y)\uparrow \Rightarrow f(3y) = 0 \neq \varphi_y(3y)$
- the set $\{x \mid f(x) = f(x+1)\} \supseteq \{3y+1 \mid y \in \mathbb{N}\}$ is infinite!

Idea 2 :

Consider $\chi_K : \mathbb{N} \to \mathbb{N}$ (total , not computable)

0	1	1	0	1	0	0	0
0	1	2	3	4	5	6	7

1

OBSERVATION : let $f : \mathbb{N} \to \mathbb{N}$ be a total function s.t. $cod(f) \subseteq \{0,1\}$ and there is $d \in \mathbb{N}$ s.t. $\forall x \geq d \quad f(x) \neq f(x+1)$

v_0	v_1	v_2	v_3	- - -							
0	1	1	0	1	0	1	0	1	0	1	- - -
0	1	2	3	4	- - -	$d-1$	d	$d+1$	$d+2$	$d+3$	

$$\underbrace{d \quad d+1 \quad d+2 \quad d+3 \quad \dots}_{g}$$

then f is computable.

In fact let

$$f(x) = v_x \qquad x \leq d \qquad \text{and assume } v_d = 0 \quad (\text{wlog})$$

Define $g : \mathbb{N} \to \mathbb{N}$

$$\begin{cases} g(0) = 0 \\ g(x+1) = \overline{sg}\,(g(x)) \end{cases} \qquad \text{computable}$$

Then

$$f(x) = \prod_{i=0}^{d-1} \overline{sg}\left(|x-i|\right) \cdot v_i \; + \; g(x \dot{-} d)$$

if $x = i < d$ $\qquad v_x \qquad + \qquad g(\overset{x \dot{-} d}{\overset{\shortparallel}{0}}) = 0 \quad \rightsquigarrow \quad v_x$

if $x \geq d$ $\qquad 0 \qquad\qquad g(x \dot{-} d) = f(x) \quad \rightsquigarrow \quad f(x)$

f is a composition of computable functions, hence it is computable.

Hence χ_K is a function with the desired properties

- χ_K total
- χ_K not computable
- by the observation, since $cod(\chi_K) = \{0,1\}$ the set $\{x \mid \chi_K(x) = \chi_K(x+1)\}$ is infinite (otherwise χ_K would be computable).

Exercise 3

Say that a function $f : \mathbb{N} \to \mathbb{N}$ is quasi-total if it is undefined on a finite number of inputs, i.e., $\overline{dom(f)}$ is finite. Classify the set $A = \{x \in \mathbb{N} \mid \varphi_x \text{ quasi-total}\}$ from the point of view of recursiveness, i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

conjecture : A not r.e.
$\bar{A}$ not r.e.

A is saturated

$A = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{A} \}$

$\mathcal{A} = \{ f \mid f \text{ is quasi-total} \} = \{ f \mid \overline{dom(f)} \text{ is finite} \}$

* A not r.e.

observe that $id \in \mathcal{A}$ since $\overline{dom(id)} = \overline{\mathbb{N}} = \emptyset$ finite

$\forall \theta \subseteq id$, θ finite $\theta \notin \mathcal{A}$ since

$dom(\theta)$ finite hence $\overline{dom(\theta)}$ infinite

then by Rice Shapiro A not r.e. (thus neither recursive)

* $\bar{A}$ not r.e. ($\bar{\mathcal{A}} = \{ f \mid \overline{dom(f)} \text{ not finite} \}$
$= \{ f \mid \overline{dom(f)} \text{ infinite} \}$

note that $id \notin \bar{\mathcal{A}}$

and $\theta = \emptyset \subseteq id$ and $\theta \in \bar{\mathcal{A}}$

since $\overline{dom(\theta)} = \overline{\emptyset} = \mathbb{N}$

is infinite

hence, by Rice Shapiro $\bar{A}$ not r.e. (thus neither recursive)

Exercise 4

$\exists z$ s.t. $\varphi_x(z) > 2x$

Classify the set $B = \{x \in \mathbb{N} \mid \exists y > 2x.\ y \in E_x\}$ from the point of view of recursiveness, i.e., establish whether B and $\bar{B}$ are recursive/recursively enumerable.

conjecture : B r.e. , not recursive

(so $\bar{B}$ not r.e. , hence not recursive)

* <u>B r.e.</u>

in fact

$$y = d + 2x + 1$$

$$sc_B(x) = 1\!\!1\left(\mu(z,y,t).\ S(x,z,y,t) \wedge y > 2x\right)$$
$$= 1\!\!1\left(\mu(z,d,t).\ S(x,z,d+2x+1,t)\right)$$
$$= 1\!\!1\left(\mu w.\ S(x,(w)_1,(w)_2+2x+1,(w)_3)\right)$$
$$= 1\!\!1\left(\mu w.\ |\chi_S(x,(w)_1,(w)_2+2x+1,(w)_3) - 1|\right)$$

minimalisation of computable functions , hence sc_B is computable

hence B is r.e.

* <u>B is not recursive</u>

We show that $K \leq_m B$, i.e. there is a total computable $s: \mathbb{N} \to \mathbb{N}$ s.t. $\forall x$

$$x \in K \quad \text{iff} \quad s(x) \in B$$

$\exists z.\ \varphi_{s(x)}(z) > 2s(x)$

define

$$g(x,y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{if } x \notin K \end{cases}$$

$$= y \cdot sc_K(x) \qquad \text{computable}$$

by the smn theorem there is $s: \mathbb{N} \to \mathbb{N}$ s.t. $\forall x, y$

$$\varphi_{s(x)} = g(x,y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

We claim that s is the reduction function for $K \leq_m B$

* if $x \in K \rightsquigarrow s(x) \in B$

if $x \in K$ then $\varphi_{s(x)}(y) = y \quad \forall y$

In particular $\varphi_{s(x)}(2s(x)+1) = 2s(x)+1 > 2s(x)$. Thus $s(x) \in B$

* if $x \notin K \rightsquigarrow s(x) \notin B$

if $x \notin K$ then $\varphi_{s(x)}(y) \uparrow \quad \forall y$ hence $E_{s(x)} = \emptyset$

hence $\nexists y \in E_{s(x)}$ s.t. $y > 2s(x)$. Hence $s(x) \notin B$

In summary, B is r.e., not recursive. Hence $\bar{B}$ not r.e. (otherwise $B, \bar{B}$ r.e. would imply B recursive). Thus $\bar{B}$ not recursive.

EXTRA QUESTION: Is B saturated?

We believe it is not since the defining condition

$$x \in B \quad \text{iff} \quad \exists y > 2x \text{ s.t. } y \in E_x$$

refers to the program code

$$\text{iff} \quad \exists z.\ \varphi_x(z) > 2x$$

We want to show that there are $e, e' \in \mathbb{N}$ s.t.

$$e \in B \qquad e' \notin B \qquad \varphi_e = \varphi_{e'}$$

We show that there is $e \in \mathbb{N}$ s.t.

$$\varphi_e(x) = 2e+1 \qquad \forall x$$

Define $g(n, x) = 2n+1 \quad \forall x$

Function g is computable, hence by smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total computable such that $\forall n, x$

$$\varphi_{s(n)}(x) = g(n,x) = 2n+1$$

By the 2nd recursion theorem, since s is total computable, there is $e \in \mathbb{N}$ s.t. $\varphi_e = \varphi_{s(e)}$. Thus

$$\varphi_e(x) = \varphi_{s(e)}(x) = g(e,x) = 2e+1$$

Now:

- $\boxed{e \in B}$ since $\varphi_e(x) = 2e+1 > 2e$

- there are infinitely many $e' \in \mathbb{N}$ s.t. $\varphi_{e'} = \varphi_e$

  take $e' > e$ s.t. $\boxed{\varphi_e = \varphi_{e'}}$. Then $\forall x$

  $$\varphi_{e'}(x) = \varphi_e(x) = 2e+1 \leq 2e'$$

  hence $\boxed{e' \notin B}$

Thus B is not saturated.