Criterio del confronto asintotico

① se $\lim_{x\to+\infty} \dfrac{f(x)}{g(x)} = l \neq 0, \ \neq \infty$ f, g continue positive

$$\longrightarrow \int_k^{+\infty} f(x)\,dx < +\infty \iff \int_k^{+\infty} g(x)\,dx < +\infty$$

$\downarrow$

applicazione   se per $x \to +\infty$ $f(x) = \dfrac{1}{x^\alpha} \cdot [\ell(x)]$, $\ell(x) \to l \neq 0, \ \neq \infty$

allora $\displaystyle\int_1^{+\infty} f(x)\,dx < +\infty \iff \alpha > 1$

(2) se $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L \neq 0, \neq \infty$ $\quad f, g$ continue e positive in $(a, b]$

allora $\int_a^b f(x)\,dx < +\infty \iff \int_a^b g(x)\,dx < +\infty$

applicazione Tipica è

per $x \to 0^+$ $\quad f(x) = \frac{1}{x^\alpha} \left[\overbrace{\ell(x)}^{\to L \neq 0, \neq \infty} \right]$

allora $\int_0^1 f(x)\,dx < +\infty \iff \alpha < 1$

CRITERIO del CONFRONTO

se $\quad 0 \leq f(x) \leq g(x) \quad \forall x \geq k > 0$

$$0 \leq \underbrace{\int_k^{+\infty} f(x)\,dx} \leq \underbrace{\int_k^{+\infty} g(x)\,dx}$$

QUINDI

se $\displaystyle\int_k^{+\infty} g(x)\,dx < +\infty$ allora $\displaystyle\int_k^{+\infty} f(x)\,dx < +\infty$

se $\displaystyle\int_k^{+\infty} f(x)\,dx = +\infty$ allora $\displaystyle\int_k^{+\infty} g(x)\,dx = +\infty$

FUNZIONE GAMMA (di EULERO)

$\Gamma : (0, +\infty) \to \mathbb{R}$ continua, positiva

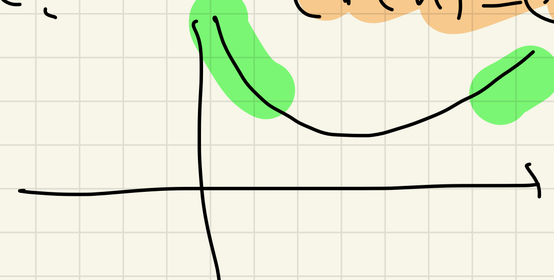
gamma maiuscola in greco

$\lim_{x \to 0^+} \Gamma(x) = +\infty$ $\lim_{x \to +\infty} \Gamma(x) = +\infty$

$\Gamma(1) = 1 = \Gamma(2)$

$\Gamma(n+1) = n!$ $(\Gamma(n) = (n-1)!)$

$\Gamma(5) = 4!$



definizione $a \in \mathbb{R}$

$$\int_0^{+\infty} e^{-x} x^{a-1}\, dx = \int_0^1 e^{-x} x^{a-1}\, dx + \int_1^{+\infty} e^{-x} x^{a-1}\, dx$$

per quali $a \in \mathbb{R}$ questo integrale È FINITO?

① per quali $a \in \mathbb{R}$ è finito

$$\int_0^1 e^{-x} x^{a-1}\, dx \;?$$

$a \geq 1$ ok $\quad x^{a-1}$

$\to 1$ per $x \to 0^+$

$$f(x) = e^{-x} \cdot x^{a-1} = e^{-x} \frac{1}{x^{-(a-1)}} = e^{-x} \frac{1}{x^{-a+1}}$$

per CONFRONTO ASINTOTICO INTEGRALE È finito $\iff -a+1 < 1 \iff \boxed{a > 0}$

$$\int_0^1 e^{-x} x^{a-1}\, dx < +\infty \iff a>0$$

~ · ~

per quali $a>0$ è finito integrale

$$\int_1^{+\infty} \underbrace{e^{-x} x^{a-1}}\, dx = ?$$

$$c^b = \frac{1}{c^{-b}}$$

$$f(x) = e^{-x} x^{a-1}$$

$$\lim_{x\to+\infty} e^{-x} x^{a-1} = \lim_{x\to+\infty} \frac{x^{a-1}}{e^x} = 0$$

(per confronto tra infiniti)

$$\int_1^{+\infty} e^{-x} x^{a-1}\,dx$$

per $x \ge 1$

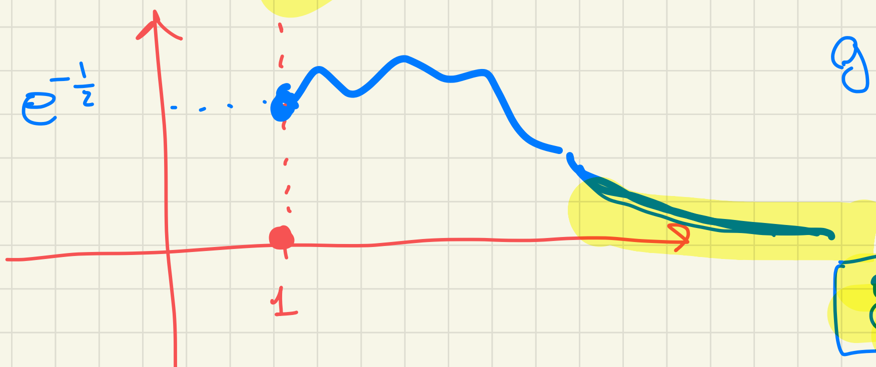
$$0 \le e^{-x} x^{a-1} \le e^{-\frac{x}{2}} \boxed{e^{-\frac{x}{2}} x^{a-1}} \le e^{-\frac{x}{2}} \left[\overset{C}{\overset{\parallel}{\max g}}\right]$$

$$\boxed{g(x) = e^{-\frac{x}{2}} x^{a-1}}$$

$$g(1) = e^{-\frac{1}{2}} \cdot 1 = \frac{1}{\sqrt{e}}$$

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} e^{-\frac{x}{2}} x^{a-1} = \lim_{x\to+\infty} \frac{x^{a-1}}{e^{\frac{x}{2}}} = 0$$

$\frac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$



g è continua

$\Downarrow$ g ha un massimo

$(\exists\, x_0 \in [1,+\infty))$

$$\boxed{g(x) \le g(x_0) = C} \quad \forall x \in [1,+\infty)$$

quindi $\quad 0 \leq e^{-x} x^{a-1} \leq C \cdot e^{-\frac{x}{2}}$

$C = \max g$

per confronto

$$0 \leq \underbrace{\int_1^{+\infty} e^{-x} x^{a-1}\, dx} \leq \underbrace{C \int_1^{+\infty} e^{-\frac{x}{2}}\, dx} = \underbrace{2Ce^{-\frac{1}{2}}}.$$

$$\int_1^{+\infty} e^{-\frac{x}{2}}\, dx = \lim_{M \to +\infty} \int_1^{M} e^{-\frac{x}{2}}\, dx$$

$$\int e^{-\frac{x}{2}}dx = \frac{1}{-\frac{1}{2}} e^{-\frac{1}{2}x} + c \qquad \int e^{\alpha x} = \frac{1}{\alpha} e^{\alpha x} + c$$

$$= \underbrace{-2e^{-\frac{x}{2}}} + c$$

$$\int_1^M e^{-\frac{x}{2}}dx = -2e^{-\frac{M}{2}} - \left(-2e^{-\frac{1}{2}}\right) =$$

$$= -2e^{-\frac{M}{2}} + 2e^{-\frac{1}{2}}$$

$$\lim_{M\to+\infty} \int_1^M e^{-\frac{x}{2}}dx = \lim_{M\to+\infty} \cancel{-2e^{-\frac{M}{2}}} + 2e^{-\frac{1}{2}} = 2e^{-\frac{1}{2}}$$

$$-2e^{-\infty} = 0$$

per ogni a $\quad \underbrace{\int_1^{+\infty} e^{-x} x^{a-1}\,dx < +\infty}$.

Riassumendo $\forall\, a>0$ esiste finito

$$\boxed{\Gamma(a) := \int_0^{+\infty} e^{-x} x^{a-1}\,dx}$$

per $a \to 0^+$ $\quad \int_0^{+\infty} e^{-x} x^{a-1}\,dx \to \int_0^{+\infty} e^{-x} x^{-1}\,dx =$

$$= \int_0^{+\infty} e^{-x} \underbrace{\frac{1}{x}}\,dx = +\infty$$

$$\Gamma(a) = \int_0^{+\infty} e^{-x} x^{a-1} dx$$

$$\Gamma(1) = \int_0^{+\infty} e^{-x} \underbrace{x^{1-1}}_{\substack{\parallel \\ x^0 = 1}} dx = \int_0^{+\infty} e^{-x} dx =$$

$$= \lim_{M \to +\infty} \int_0^{M} e^{-x} dx = \lim_{M \to +\infty} \left[-e^{-M} - (-e^{-0}) \right] =$$

$$\int e^{-x} dx = -e^{-x} + c$$

$$= \lim_{M \to +\infty} -e^{-M} + 1 = -\underbrace{\cancel{e^{-\infty}}}_{\substack{\parallel \\ 0}} + 1 = 1$$

$$\Gamma(2) = \int_0^{+\infty} e^{-x} x^{2-1}\,dx = \int_0^{+\infty} e^{-x} x\,dx = \lim_{M\to+\infty} \int_0^{M} e^{-x} x\,dx$$

$$\int e^{-x} x\,dx = -e^{-x} x - \int (-e^{-x}) \cdot 1\,dx =$$

$$f(x) = e^{-x} \to F(x) = -e^{-x}$$

$$g(x) = x \to g'(x) = 1$$

$$= -e^{-x} \cdot x + \underbrace{\int e^{-x}\,dx} = -e^{-x} x \underbrace{- e^{-x}} + c$$

$$\lim_{M\to+\infty} \int_0^{M} e^{-x} \cdot x\,dx = \lim_{M\to+\infty} -e^{-M} \cdot M - e^{-M} - (-\underset{0}{\cancel{e^0 \cdot 0}} - e^0)$$

$$= \lim_{M\to+\infty} -\underset{0}{\cancel{\frac{M}{e^M}}} - \underset{0}{\cancel{\frac{1}{e^M}}} + 1 = \boxed{1 = \Gamma(2)}$$

$$\Gamma(1) = 1 \qquad \Gamma(2) = 1$$

$$\Gamma(n) = \underbrace{\int_0^{+\infty} e^{-x} x^{n-1} dx} = \lim_{M \to +\infty} \int_0^M e^{-x} x^{n-1} dx$$

$$\int e^{-x} \cdot x^{n-1} = -e^{-x} \cdot x^{n-1} - \int (-e^{-x}) (n-1) x^{n-2} dx =$$

$$f(x) = e^{-x} \Rightarrow F(x) = -e^{-x}$$

$$g(x) = x^{n-1} \Rightarrow g'(x) = (n-1) x^{n-2}$$

$$(x^k)' = k x^{k-1}$$

$$= \underbrace{\underbrace{-e^{-x} x^{n-1}} + (n-1) \int e^{-x} x^{n-2} dx}$$

$$\int_0^M e^{-x} x^{n-1} dx = -e^{-M} M^{n-1} - \cancel{(-e^0 0^{n-1})} + (n-1) \int_0^M e^{-x} x^{n-2} dx$$

$$\int_0^M e^{-x} x^{n-1}\, dx = -e^{-M} M^{n-1} + 0 + (n-1) \int_0^M e^{-x} x^{n-1-1}\, dx$$

$$M \to +\infty \quad \downarrow$$

$$\int_0^{+\infty} e^{-x} x^{n-1}\, dx = \cancel{\underbrace{\lim_{M \to +\infty} -e^{-M} M^{n-1}}} + (n-1) \int_0^{+\infty} e^{-x} x^{n-1-1}\, dx$$

$$\lim_{M \to +\infty} - \frac{M^{n-1}}{e^M} = 0$$

$$\underbrace{\int_0^{+\infty} e^{-x} x^{n-1}\, dx}_{\|} = (n-1) \int_0^{+\infty} e^{-x} x^{n-1-1}\, dx$$

$$\boxed{\Gamma(n) = (n-1)\, \Gamma(n-1)}$$

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$

$$\Gamma(1) = 1 = \Gamma(2)$$

$$\Gamma(3) = 2\,\Gamma(2) = 2\cdot 1 = 2$$

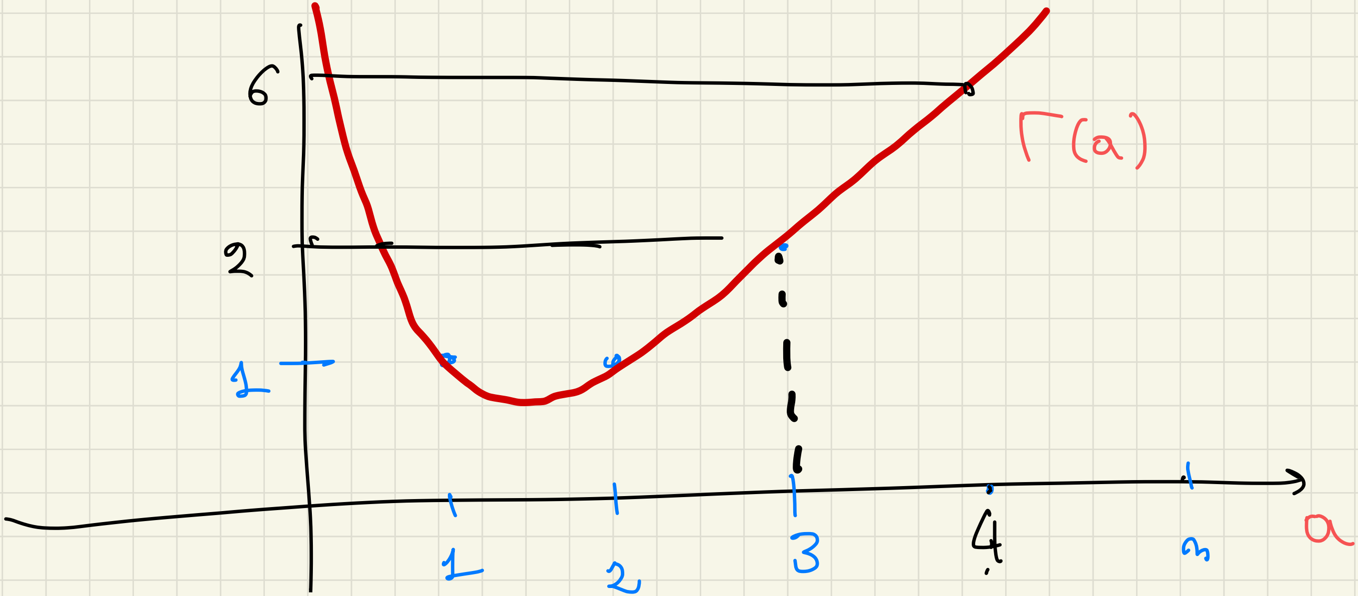
$$\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2\cdot 1$$

$$\Gamma(5) = 4\cdot\Gamma(4) = 4\cdot 3\cdot 2$$

$$\boxed{\Gamma(n) = (n-1)!}$$

$$\forall a > 0$$

$$\boxed{\Gamma(a+1) = a\cdot\Gamma(a)}$$



$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} e^{-x}\, x^{\frac{1}{2}-1}\, dx = \int_0^{+\infty} \underbrace{e^{-x}\, x^{-\frac{1}{2}}}\, dx =$$

$$= \lim_{M \to +\infty} \int_0^{M} \underbrace{e^{-x}\, \frac{1}{\sqrt{x}}}\, dx = \int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

$$\int_0^M e^{-x}\frac{1}{\sqrt{x}}\,dx \qquad y=\sqrt{x} \qquad x=y^2 \qquad dx=2y\,dy$$

$$\int_0^{\sqrt{M}} e^{-y^2}\frac{1}{\cancel{y}}\,2\cancel{y}\,dy$$

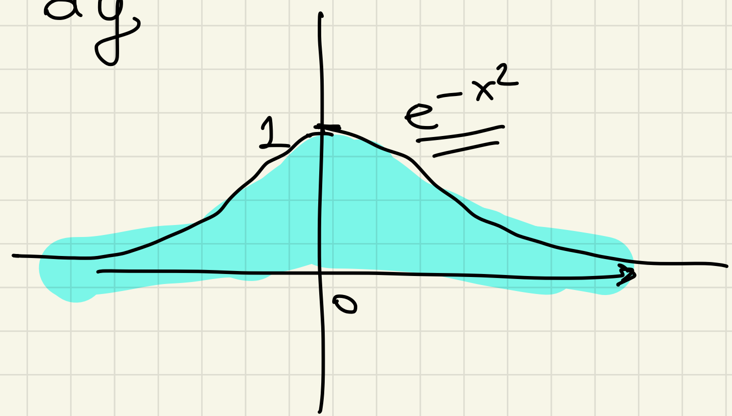
$x=0 \rightarrow y=\sqrt{0}=0$

$x=M \rightarrow y=\sqrt{M}$

$$=\int_0^{\sqrt{M}} 2e^{-y^2}\,dy = 2\int_0^{\sqrt{M}} e^{-y^2}\,dy$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{+\infty} e^{-x^2}\,dx$$

$$=\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$



Es Determinare per quali α è finito

$$\int_1^{+\infty} \underbrace{\operatorname{arctg}\left(\frac{1}{x^\alpha}\right)}\, dx$$

e calcolarlo per $\alpha = 2$ se possibile

sicuramente $\alpha > 0$ (altrimenti $\operatorname{arctg}\left(\frac{1}{x^\alpha}\right) \not\to 0$)

$\alpha > 0$ $\quad x \to +\infty \quad \frac{1}{x^\alpha} \to 0$

$$\operatorname{arctg}\left(\frac{1}{x^\alpha}\right) = \frac{1}{x^\alpha} + o\left(\frac{1}{x^\alpha}\right) = \frac{1}{x^\alpha}\left(1 + o(1)\right)$$

↓

pol. di Taylor

integrale è FINITO $\iff$ $\alpha > 1$

$\alpha = 2$ $\quad \displaystyle\int_1^{+\infty} \operatorname{arctg}\left(\frac{1}{x^2}\right) dx = \lim_{M \to +\infty} \int_1^{M} \operatorname{arctg}\left(\frac{1}{x^2}\right) dx$

$\boxed{\text{PER PARTI}}$

$$\int 1 \cdot \operatorname{arctg}\left(\frac{1}{x^2}\right) dx$$

$f(x) = 1 \longrightarrow F(x) = x$

$g(x) = \operatorname{arctg}\left(\frac{1}{x^2}\right)$

$$g'(x) = \left(\operatorname{arctg}\left(\frac{1}{x^2}\right)\right)' = \frac{1}{1+\left(\frac{1}{x^2}\right)^2} \cdot \left(\frac{1}{x^2}\right)'$$

$$\left(\frac{1}{x^2}\right)' = \left(x^{-2}\right)' = -2x^{-2-1} = -2x^{-3} = -2\frac{1}{x^3}$$

$$g'(x) = \frac{1}{1+\left(\frac{1}{x^2}\right)^2} \cdot \left(-\frac{2}{x^3}\right) = \frac{1}{1+\frac{1}{x^4}} \cdot \left(-\frac{2}{x^3}\right) =$$

$$= \frac{1}{\frac{x^4+1}{x^4}} \left(-\frac{2}{x^3}\right) = -2 \cdot \frac{x^{\cancel{4}}}{x^4+1} \frac{1}{\cancel{x^3}} =$$

$$= \frac{-2x}{x^4+1}$$

$$\int \operatorname{arctg}\left(\frac{1}{x^2}\right) dx = x \cdot \operatorname{arctg}\left(\frac{1}{x^2}\right) - \int \frac{x \cdot (-2x)}{x^4+1} dx =$$

$f(x) = 1 \quad \rightarrow F(x) = x$

$g(x) = \operatorname{arctg} \frac{1}{x^2} \rightarrow g'(x) = \frac{-2x}{x^4+1}$

$$= x \operatorname{arctg} \frac{1}{x^2} + 2 \int \frac{x^2}{x^4+1}\, dx$$

$$x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$$

$$x^4 + 0x^3 + 0x^2 + 0x + 1$$

per teorema fondam. algebra

ogni polinomio a coefficienti reali si scrive come prodotto di polinomio a coeffic. reali di grado 1 o 2.

$$\begin{array}{l} x^3 \\ x^2 \\ x \\ 1 \end{array} \begin{cases} a + c = 0 \\ ac + b + d = 0 \\ ad + bc = 0 \\ bd = 1 \end{cases} \qquad \begin{cases} a = -c \\ -a^2 + b + \frac{1}{b} = 0 \\ ad - a\frac{1}{d} = 0 \\ b = 1/d \end{cases}$$

$$d^2 = 1$$

$$c = -\sqrt{2} \qquad a = \sqrt{2} \qquad d = 1 \qquad b = 1$$

$$x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$$

$$\frac{x^2}{x^4+1} = \frac{x^2}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} =$$

$$= \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1} \quad \ldots$$

(Esempio si risolve con fratti semplici)

Es Dire per quali k è finito

$$\int_0^1 \frac{\lg\left(1+\sqrt[3]{x}\right) - \sqrt[3]{x}}{x^k}\, dx$$

e calcolarlo per $k = \frac{2}{3}$.

$$f(x) = \frac{\log\left(1+\sqrt[3]{x}\right) - \sqrt[3]{x}}{x^k}$$

$x \to 0^+ \quad \sqrt[3]{x} \to 0$

$$\log(1+x) \underset{x\to 0}{=} x - \frac{1}{2}x^2 + o(x^2)$$

$$\log\left(1+\sqrt[3]{x}\right) = \sqrt[3]{x} - \frac{1}{2}\left(\sqrt[3]{x}\right)^2 + o\left(\sqrt[3]{x}\right)^2$$

$$f(x) = \frac{\cancel{\sqrt[3]{x}} - \frac{1}{2}x^{\frac{2}{3}} + o\left(x^{2/3}\right) \cancel{- \sqrt[3]{x}}}{x^k} = \frac{x^{\frac{2}{3}}\left(-\frac{1}{2} + o(1)\right)}{x^k}$$

$$= \frac{\left(-\frac{1}{2} + o(1)\right)}{x^{k-\frac{2}{3}}}$$

INTEGRALE è finito

$$\boxed{k - \frac{2}{3} < 1}$$

$$k < 1 + \frac{2}{3} = \frac{5}{3}$$

$$K = \frac{2}{3} < \frac{5}{3} \quad \int_0^1 \frac{\log(1+\sqrt[3]{x}) - \sqrt[3]{x}}{x^{2/3}}\,dx =$$

$$= \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{\log(1+\sqrt[3]{x}) - \sqrt[3]{x}}{x^{2/3}}\,dx$$

$$\int_\varepsilon^1 \frac{\log(1+\sqrt[3]{x}) - \sqrt[3]{x}}{x^{2/3}}\,dx$$

$$y = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$x = y^3 \qquad dx = 3y^2\,dy$$

$$= \int_{\sqrt[3]{\varepsilon}}^1 \frac{\log(1+y) - y}{\cancel{y^2}}\, 3\cancel{y^2}\,dy$$

$$x^{2/3} = (x^{1/3})^2 = y^2$$

$$x = 1 \to y = \sqrt[3]{1} = 1$$

$$x = \varepsilon \to y = \sqrt[3]{\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} \int_{\sqrt[3]{\varepsilon}}^{1} \left[\log(1+y) - y\right] \cdot 3 \, dy =$$

$$= 3\int_0^1 \left[\log(1+y) - y\right] dy = 3\int_0^1 \log(1+y)\,dy \overset{-\frac{3}{2}}{-3\int_0^1 y\,dy}$$

$$3\int_0^1 y\,dy = 3\left[\frac{1}{2}1^2 - \frac{1}{2}0^2\right] = \frac{3}{2} \qquad \int y\,dy = \frac{1}{2}y^2$$

$$\int_0^1 \log(1+y)\,dy$$

$$\int 1\cdot \lg(1+y)\,dy = \overset{F}{(y+1)}\cdot \overset{g}{\lg(y+1)} - \int \overset{F}{\cancel{(y+1)}}\,\overset{g'}{\frac{1}{\cancel{y+1}}}\,dy =$$

$$f(y) = 1 \longrightarrow F(y) = y+1$$

$$g(y) = \lg(1+y) \qquad g'(y) = \frac{1}{1+y}$$

$$= (y+1)\lg(y+1) - \int 1\,dy = (y+1)\lg(y+1) - y + C$$

$$\int_0^1 \lg(1+y)\,dy = (1+1)\lg(1+1) - 1 - [\underbrace{(0+1)\lg(0+1)}_{\lg(1)=0} - 0]$$

$$= \underline{2\lg 2 - 1}$$

$$3\int_0^1 \lg(1+y) - y\,dy = 3(2\lg 2 - 1) - \frac{3}{2} = 6\lg 2 - 3 - \frac{3}{2}$$

Es Calcolare se esiste finito

$$\int_2^{+\infty} \frac{e^x}{e^{2x}+3e^x-4}\,dx = \lim_{M\to+\infty} \int_2^M \frac{e^x}{e^{2x}+3e^x-4}\,dx$$

$$\int_2^M \frac{e^x}{\underbrace{e^{2x}+3e^x-4}}\,dx$$

$$e^{2x}=(e^x)^2 \qquad x=2 \to y=e^2$$

$$x=M \to y=e^M$$

$$y=e^x$$

$$e^{2x}+3e^x-4 \to y^2+3y-4$$

$$y=e^x \quad x=\log y \quad dx=\frac{1}{y}\,dy$$

$$\int_{e^2}^{e^M} \frac{\cancel{y}}{y^2+3y-4}\,\frac{1}{\cancel{y}}\,dy$$

$$\int_{e^2}^{e^4} \frac{1}{y^2+3y-4}\,dy$$

$$y^2+3y-4=0 \qquad y_{1 2} = \begin{cases} 1 \\ -4 \end{cases}$$

$$y^2+3y-4 = (y-1)(y-(-4)) = (y-1)(y+4)$$

$$\frac{0\cdot y+1}{y^2+3y-4} = \frac{A}{y-1} + \frac{B}{y+4} = \frac{Ay+4A+By-B}{(y-1)(y+4)}$$

$$\begin{cases} A+B=0 \\ 4A-B=1 \end{cases} \quad \begin{cases} -B=A \\ 5A=1 \end{cases} \quad \begin{cases} B=-1/5 \\ A=1/5 \end{cases}$$

$$\frac{1}{y^2+3y-4} = \frac{1/5}{y-1} + \frac{\left(-\frac{1}{5}\right)}{y+4} =$$

$$= \frac{1}{5}\,\frac{1}{y-1} - \frac{1}{5}\,\frac{1}{y+4}$$

$$\int \frac{1}{y^2+3y-4}\,dy = \frac{1}{5}\int \frac{1}{y-1}\,dy - \frac{1}{5}\int \frac{1}{y+4}\,dy =$$

$$= \frac{1}{5}\lg|y-1| - \frac{1}{5}\lg|y+4| + c = \frac{1}{5}\lg\frac{|y-1|}{|y+4|} + c$$

$$\int_{e^2}^{e^M} \frac{1}{y^2+3y-4}\,dy = \frac{1}{5}\log\left(\frac{e^M-1}{e^M+4}\right) - \frac{1}{5}\log\left(\frac{e^2-1}{e^2+4}\right)$$

$$= \frac{1}{5}\log\left(\frac{\cancel{e^M}\left(1-\frac{1}{e^M}\right)}{\cancel{e^M}\left(1+\frac{4}{e^M}\right)}\right) - \frac{1}{5}\log\left(\frac{e^2-1}{e^2+4}\right)$$

$$\lim_{M\to+\infty} \int_{e^2}^{e^M} \frac{1}{y^2+3y-4}\,dy = \lim_{M\to+\infty} \frac{1}{5}\log\left(\frac{1-\frac{1}{e^M}}{1+\frac{4}{e^M}}\right) - \frac{1}{5}\log\left(\frac{e^2-1}{e^2+4}\right)$$

$\frac{1}{e^M} \to 0$, $\frac{4}{e^M} \to 0$

$\frac{1}{5}\log 1 = 0$

$$= -\frac{1}{5}\log\left(\frac{e^2-1}{e^2+4}\right) = \frac{1}{5}\log\left(\frac{e^2+4}{e^2-1}\right)$$

Es

$$\int_9^{+\infty} \frac{1}{x^k(x-\sqrt{x}-2)}\,dx$$

per quali k integrale è finito e calcolare integrale per $k=\frac{1}{2}$ (se possibile)

$x \to +\infty$

$$f(x) = \frac{1}{x^k(x-\sqrt{x}-2)} = \frac{1}{x^k\, x\left[1-\frac{1}{\sqrt{x}}-\frac{2}{x}\right]} =$$

$$= \frac{1}{x^{k+1}} \cdot \frac{1}{\left(1-\frac{1}{\sqrt{x}}-\frac{2}{x}\right)} \to \frac{1}{1-0-0}=1$$

per confronto asint. integrale è FINITO

$k+1>1$

$$\boxed{k>0}$$

$$\int_9^{+\infty} \frac{1}{x^{\frac{1}{2}}(x-\sqrt{x}-2)}\,dx = \lim_{M\to+\infty} \int_9^{M} \frac{1}{\sqrt{x}\,[x-\sqrt{x}-2]}\,dx$$

$$\int_9^{M} \frac{1}{\sqrt{x}\,(\underbrace{x-\sqrt{x}-2})}\,dx$$

$$x = (\sqrt{x})^2$$

$$y = \sqrt{x} \qquad x = y^2$$

$$\int_3^{\sqrt{M}} \frac{1}{\cancel{y}\,(y^2-y-2)}\,2\cancel{y}\,dy$$

$$x = y^2$$

$$dx = 2y\,dy$$

$$x = 9 \longrightarrow y = \sqrt{9} = 3$$

$$x = M \longrightarrow y = \sqrt{M}$$

$$= \int_3^{\sqrt{M}} \frac{2}{y^2-y-2}\,dy$$

$$y^2-y-2=0 \qquad y=\begin{cases}-1\\ 2\end{cases}$$

$$y^2-y-2=(y-(-1))(y-2)=(y+1)(y-2)$$

$$\frac{2}{y^2-y-2} = \frac{A}{y+1}+\frac{B}{y-2} = \frac{Ay-2A+By+B}{(y+1)(y-2)}$$

$$\begin{cases}A+B=0\\ -2A+B=2\end{cases} \qquad \begin{cases}A=-B\\ 2B+B=2\end{cases} \qquad \begin{cases}A=-2/3\\ B=2/3\end{cases}$$

$$\frac{2}{y^2-y-2} = \frac{\left(-2/3\right)}{y+1} + \frac{\left(2/3\right)}{y-2} = -\frac{2}{3}\,\frac{1}{y+1} + \frac{2}{3}\,\frac{1}{y-2}$$

$$\int \frac{2}{y^2-y-2}\,dy = -\frac{2}{3}\log|y+1| + \frac{2}{3}\log|y-2| + c$$

$$= \frac{2}{3}\log\frac{|y-2|}{|y+1|} + c$$

$$\lim_{M\to+\infty}\int_3^{\sqrt{M}} \frac{2}{y^2-y-2}\,dy = \lim_{M\to+\infty} \overset{0}{\cancel{\frac{2}{3}\log\left(\frac{\sqrt{M}-2}{\sqrt{M}+1}\right)}} - \frac{2}{3}\log\left(\frac{3-2}{3+1}\right)$$

$$\frac{2}{3}\log\left(\frac{\sqrt{M}}{\sqrt{M}}\,\frac{\left(1-2/\sqrt{M}\right)}{\left(1+1/\sqrt{M}\right)}\right) \to \frac{3}{2}\log 1 = 0$$

$$= -\frac{2}{3}\log\left(\frac{1}{4}\right) = \frac{2}{3}\log 4$$

es $\displaystyle\int_0^{\pi/2} \frac{\operatorname{sen} x(\cos x - 1)}{(\cos x)^2 + 2}\,dx = \int_0^{\pi/2} \frac{(\cos x - 1)\operatorname{sen} x\,dx}{(\cos x)^2 + 2}$

1ª SCELTA

$y = \cos x \rightarrow$ $\boxed{\begin{array}{l} dy = -\sin x\,dx \\ (-1)\cdot dy = \sin x\,dx \end{array}}$

2ª SCELTA

$x = \arccos y$

$dx = (\arccos y)'\,dy = -\dfrac{1}{\sqrt{1-y^2}}\,dy$

LE 2 SCELTE SONO EQUIVALENTI

1° scelta

$$\int_0^{\pi/2} \frac{(\cos x - 1)}{(\cos x)^2 + 2} \sin x \, dx$$

$y = \cos x$

$dy = -\sin x \, dx$

$(-1)\, dy = \sin x \, dx$

$$\int_1^0 \frac{y-1}{y^2+2} (-1)\, dy$$

$x = 0 \rightarrow y = \cos 0 = 1$

$x = \frac{\pi}{2} \rightarrow y = \cos \frac{\pi}{2} = 0$

$$-\int_1^0 \frac{y-1}{y^2+2} \, dy = \boxed{\int_0^1 \frac{y-1}{y^2+2} \, dy}$$

2ª scelta

$$\int_0^{\pi/2} \frac{\sin x\,(\cos x - 1)}{(\cos x)^2 + 2}\,dx =$$

$\cos x = y$

$x = \arccos y$

$dx = -\frac{1}{\sqrt{1-y^2}}\,dy$

$x = 0 \to y = \cos 0 = 1$

$x = \frac{\pi}{2} \to y = \cos\frac{\pi}{2} = 0$

$$= \cancel{\int_1^0} \frac{\cancel{\sqrt{1-y^2}}\,(y-1)}{y^2+2}\left(\cancel{-}\frac{1}{\cancel{\sqrt{1-y^2}}}\right)dy$$

$$\sin x = \sin(\arccos y) = \sqrt{1 - [\cos(\arccos y)]^2} = \sqrt{1-y^2}$$

$\sin \alpha = \sqrt{1-\cos^2\alpha}$

in entrambi i casi arrivo all'integrale

$$\int_0^1 \frac{y-1}{y^2+2}\,dy = \int_0^1 \frac{y}{y^2+2}\,dy - \underbrace{\int_0^1 \frac{1}{y^2+2}\,dy}$$

$$\int \frac{y}{y^2+2}\,dy = \begin{array}{l} z = y^2+2 \\ y = \sqrt{z-2} \\ dy = \dfrac{1}{2\sqrt{z-2}}\,dz \end{array} = \int \frac{\cancel{\sqrt{z-2}}}{z} \cdot \frac{1}{2\cancel{\sqrt{z-2}}}\,dz$$

$$= \frac{1}{2}\int \frac{1}{z}\,dz = \frac{1}{2}\lg|z| + c = \underbrace{\frac{1}{2}\lg(y^2+2)+c}$$

$$\int \frac{y}{y^2+2}\,dy = \frac{1}{2}\lg(y^2+2) + c$$

$$\int_0^1 \frac{y}{y^2+2}\,dy = \frac{1}{2}\lg(1+2) - \frac{1}{2}\lg(0+2) =$$

$$= \frac{1}{2}\lg 3 - \frac{1}{2}\lg 2 = \frac{1}{2}\lg\left(\frac{3}{2}\right)$$

$$\int_0^1 \frac{1}{y^2+2}\,dy$$

$$\int \frac{1}{ay^2+b}\,dy = \frac{1}{\sqrt{ab}}\,\mathrm{arctg}\left(\sqrt{\frac{a}{b}}\,y\right)+c$$

$$\int \frac{1}{y^2+2}\,dy = \frac{1}{\sqrt{2}}\,\mathrm{arctg}\left(\frac{1}{\sqrt{2}}\,y\right) + c$$

$$\int_0^1 \frac{1}{y^2+2}\,dy = \frac{1}{\sqrt{2}}\,\mathrm{arctg}\,\frac{1}{\sqrt{2}} - \overset{0}{\cancel{\frac{1}{\sqrt{2}}\,\mathrm{arctg}\,0}}$$

$$\int_0^1 \frac{y-1}{y^2+2}\,dy = \frac{1}{2}\lg\left(\frac{3}{2}\right) - \frac{1}{\sqrt{2}}\arctg\left(\frac{1}{\sqrt{2}}\right).$$

~ . ~

Calcolare $\displaystyle\int_e^{e^2} \frac{1}{x \lg x}\,dx$

$$\int_e^{e^2} \frac{1}{x \lg x}\,dx$$

$y = \lg x$

$x = e^y$

$x = e \rightarrow \lg e = 1$

$x = e^2 \rightarrow \lg e^2 = 2$

$\underline{dx = e^y\,dy}$

$$\int_1^2 \frac{1}{\cancel{e^y} \cdot y} \cdot \cancel{e^y}\,dy = \int_1^2 \frac{1}{y}\,dy = \lg 2 - \cancel{\lg 1}$$