COMPUTABILITY (18/12/2024)

A typical exam ...

Computability
February 2, 2024

definitions
proofs
small variations

Exercise 1

a. Provide the definition of a recursive set.

b. Provide the definition of a recursively enumerable (r.e.) set.

c. Show that given $A, B \subseteq \mathbb{N}$, if A is recursive and $B = A \cap \mathbb{P}$ then B is recursive (here $\mathbb{P}$ denotes the set of even numbers). Does the converse hold? I.e., is it the case that if $B = A \cap \mathbb{P}$ is recursive then A is recursive?

constructions of PR / R
diagonalisation
smn

Exercise 2

State the s-m-n theorem and use it to prove that there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $W_{s(x)} = \mathbb{P}$ and $E_{s(x)} = \{z \in \mathbb{N} \mid z \geqslant x\}$ (where again $\mathbb{P}$ is the set of even numbers).

classify sets $\left(\begin{array}{c}\text{recursive} \\ \text{r.e.}\end{array}\right)$, saturatedness

Exercise 3

Classify the following set from the point of view of recursiveness

$$A = \{x \mid W_x = E_x\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 4

Classify the following set from the point of view of recursiveness

$$B = \{x \in \mathbb{N} \mid \exists z.\ \varphi_x(z) > x\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable. Also establish if B is saturated.

Note: Each exercise contributes with the same number of points (8) to the final grade.

$\boxed{\text{ORAL EXAM}}$: optional, needed for distinction (lode)
focused on theory/proofs
range: +/- 4

Exercise 1

a. Provide the definition of a recursive set.

b. Provide the definition of a recursively enumerable (r.e.) set.

c. Show that given $A, B \subseteq \mathbb{N}$, if A is recursive and $B = A \cap \mathbb{P}$ then B is recursive (here $\mathbb{P}$ denotes the set of even numbers). Does the converse hold? I.e., is it the case that if $B = A \cap \mathbb{P}$ is recursive then A is recursive?

(a) A set $A \subseteq \mathbb{N}$ is <u>recursive</u> if the characteristic function

$\chi_A : \mathbb{N} \to \mathbb{N}$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad \text{is computable}$$

(b) A set $A \subseteq \mathbb{N}$ is r.e. if semi-characteristic function

$SC_A : \mathbb{N} \to \mathbb{N}$

$$SC_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases} \quad \text{computable}$$

(c) Let $A \subseteq \mathbb{N}$ be recursive set, $\mathbb{P} = \{ n \in \mathbb{N} \mid n \text{ is even} \}$

Then we want show that $B = A \cap \mathbb{P}$ is recursive

In fact

$\mathbb{P}$ is recursive in fact $\chi_{\mathbb{P}}(x) = \overline{sg}(rm(2,x))$

is computable

and thus $B = A \cap \mathbb{P}$ intersection of two recursive sets

is recursive

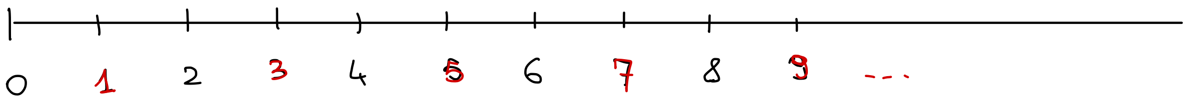
explicitly, $\chi_B(x) = \chi_A(x) \cdot \chi_{\mathbb{P}}(x)$

computable since it is the

composition of computable functions

* Does the converse hold? I.e. is it the case that

$A \cap \mathbb{P}$ recursive $\Rightarrow$ A recursive ? NO



$A = \{ 2x+1 \mid x \in K \}$ is intuitively "equivalent" to K

more precisely $K \leq_m A$

the reduction function $f : \mathbb{N} \to \mathbb{N}$ total computable s.t

$\forall x$ $x \in K$ iff $f(x) \in A$

can be $f(x) = 2x+1$

Since K is not recursive , A is not recursive

However

$A \cap \mathbb{P} = \emptyset$ is recursive

Exercise 2

(a) State the s-m-n theorem and use it (b) to prove that there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $W_{s(x)} = \mathbb{P}$ and $E_{s(x)} = \{z \in \mathbb{N} \mid z \geqslant x\}$ (where again $\mathbb{P}$ is the set of even numbers).

(a) smn theorem: For $m, n \geq 1$ there is $s_{m,n} : \mathbb{N}^{m+1} \to \mathbb{N}$ total and computable such that $\forall e \in \mathbb{N}$ $\forall \vec{x} \in \mathbb{N}^m$ $\forall \vec{y} \in \mathbb{N}^n$

$$\varphi_e^{(m+n)}(\vec{x}, \vec{y}) = \varphi^{(n)}_{s_{m,n}(e,\vec{x})}(\vec{y})$$

(b) Define $g : \mathbb{N}^2 \to \mathbb{N}$

$$g(x, y) = \begin{cases} x + y/2 & \text{if } y \text{ is even} \\ \uparrow & \text{otherwise} \end{cases}$$

$$= x + qt(2,y) + \underbrace{\mu z . \underbrace{rm(2,y)}_{\substack{0 \text{ if } y \text{ even} \\ 1 \text{ if } y \text{ odd}}}}_{\substack{0 \text{ if } y \text{ even} \\ \uparrow \text{ if } y \text{ odd}}}$$

g is computable (composition and minimalisation of computable functions)

Hence, by (the corollary of) the smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total and computable such that $\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} x + y/2 & \text{if } y \text{ is even} \\ \uparrow & \text{otherwise} \end{cases}$$

Now we argue that s is the desired function

→ total and computable

$\rightarrow \quad W_{s(x)} = \mathbb{P}$ by construction

$$\rightarrow \quad E_{s(x)} = \{ \varphi_{s(x)}(y) \mid y \in \underset{\overset{\|}{\mathbb{P}}}{W_{s(x)}} \}$$

$$= \{ x + \tfrac{y}{2} \mid y \in \mathbb{P} \}$$

$$= \{ x + \tfrac{2z}{2} \mid z \in \mathbb{N} \}$$

$$= \{ x + z \mid z \in \mathbb{N} \}$$

$$= \{ z' \mid z' \geqslant x \}$$

as desired.

Exercise 3

Classify the following set from the point of view of recursiveness

$$A = \{x \mid W_x = E_x\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

<u>Conjecture</u> : A not r.e. W_x, E_x infinite

$\subseteq$ $\forall y$ $\boxed{\text{if } \varphi_x(y)\downarrow \text{ the } \exists z \ \varphi_x(z) = y}$

$\bar{A}$ not r.e.

We observe that A is saturated

$$A = \{x \mid \varphi_x \in \mathcal{A}\}$$

with $\mathcal{A} = \{f \mid dom(f) = cod(f)\}$

We use Rice-Shapiro for showing that A and $\bar{A}$ are not r.e.

- <u>A not r.e.</u>

$\mathbb{1}$ (constant one $\mathbb{1}(x) = 1$ $\forall x$) then $dom(\mathbb{1}) = \mathbb{N} \neq \{1\} = cod(\mathbb{1})$

hence $\mathbb{1} \notin \mathcal{A}$

and $\vartheta = \emptyset$ finite, $\vartheta \subseteq \mathbb{1}$, $\vartheta \in \mathcal{A}$

always undefined

in fact $dom(\vartheta) = \emptyset = cod(\vartheta)$

hence by Rice-Shapiro A not r.e.

- $\bar{A}$ is not r.e.

$\bar{\mathcal{A}} = \{ f \mid dom(f) \neq cod(f) \}$

we look for a function $f \notin \bar{\mathcal{A}}$ with $\vartheta \subseteq f$, ϑ finite, $\vartheta \in \bar{\mathcal{A}}$

consider $pred : \mathbb{N} \to \mathbb{N}$ $\quad pred(x) = x \dot{-} 1$

- $pred \notin \bar{\mathcal{A}}$ (i.e. $pred \in \mathcal{A}$)

  since $\quad dom(pred) = \mathbb{N} = cod(\mathbb{N})$

- define

$$\vartheta(x) = \begin{cases} pred(x) & x \leq 1 \\ \uparrow & \text{otherwise} \end{cases} = \begin{cases} 0 & \text{if } x \leq 1 \\ \uparrow & \text{otherwise} \end{cases}$$

$\vartheta \subseteq pred$, finite

$\vartheta \in \bar{\mathcal{A}}$ $\quad dom(\vartheta) = \{0, 1\} \neq \{0\} = cod(\vartheta)$

hence by Rice-Shapiro $\bar{A}$ not r.e.

Since $A, \bar{A}$ not r.e. they are neither recursive.

Exercise 4

Classify the following set from the point of view of recursiveness

$$B = \{x \in \mathbb{N} \mid \exists z.\ \varphi_x(z) > x\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable. Also establish if B is saturated.

conjecture : • B r.e. , not recursive

↳ $\bar{B}$ not r.e. $\Rightarrow$ $\bar{B}$ not recursive

• B not saturated

(a) B is r.e.

$$B = \{x \in \mathbb{N} \mid \exists z.\ \varphi_x(z) > x\},$$

The semi-charactic function

$$sc_B(x) = \text{“} \mathbb{1}\Big(\mu(z,y,t).\ S(x,z,y,t)\ \&\ (\underset{\substack{\uparrow \\ y = x+1+y'}}{y} > x)\Big)\text{”}$$

$$= \mathbb{1}\Big(\mu\,\underbrace{(\overset{(\omega)_1}{z}, \overset{(\omega)_2}{y'}, \overset{(\omega)_3}{t})}_{\omega}.\ S(x, z, x+1+y', t)\Big)$$

$$= \mathbb{1}\Big(\mu\omega.\ \underbrace{S(x, (\omega)_1, x+1+(\omega)_2, (\omega)_3)}_{\text{decidable}}\Big)$$

$$= \mathbb{1}\Big(\mu\omega.\ \underbrace{|\chi_S(x, (\omega)_1, x+1+(\omega)_2, (\omega)_3) - 1|}_{\text{computable}}\Big)$$

computable by composition and minimalisation.

(show $B \leq_m K$ is an alternative, typically not convenient)

(b) <u>B is not recursive</u>

we show the above by arguing that $\boxed{K \leq_m B}$

and since K is not recursive, B is not recursive.

Define
$$g(x, y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{if } x \notin K \end{cases}$$
$$= y * sc_K(x)$$

computable, hence by smn theorem, there is $s: \mathbb{N} \to \mathbb{N}$ total and computable such that $\forall x, y$
$$\varphi_{s(x)}(y) = g(x, y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

We claim that s is the reduction function for $K \leq_m B$

* $x \in K \rightsquigarrow s(x) \in B$

if $x \in K$ then $\varphi_{s(x)}(y) = g(x,y) = y \quad \forall y$

in particular forall $z > s(x)$ e.g. $z = s(x)+1$

$\varphi_{s(x)}(z) = z > s(x)$ hence $s(x) \in B$

* $x \notin K \rightsquigarrow s(x) \notin B$

if $x \notin K$ then $\varphi_{s(x)}(y) = g(x,y) \uparrow \quad \forall y$ hence

$\nexists z$ s.t. $\varphi_{s(x)}(z) > s(x)$. Thus $s(x) \notin B$

Thus $K \leq_m B$, hence B is not recursive

Since B is r.e. and not recursive, $\bar{B}$ is not r.e.

(otherwise if $B, \bar{B}$ were r.e. B would be recursive).

Since $\bar{B}$ is not r.e. then it is not recursive.

(c) Is B saturated? NO $B = \{x \in \mathbb{N} \mid \exists z.\ \varphi_x(z) > x\}$, !

We show that there is $e \in \mathbb{N}$ s.t.

$$\boxed{\varphi_e(x) = e+1 \qquad \forall x} \qquad (*)$$

This is sufficient to conclude that B is not saturated

* $e \in B$ $\qquad \varphi_e(0) = e+1 > e$
* there are infinitely many e' s.t. $\varphi_{e'} = \varphi_e$.

  Take $e' > e$ s.t. $\varphi_{e'} = \varphi_e$

* $e' \notin B$ $\qquad \varphi_{e'}(x) = \varphi_e(x) = e \not> e' \qquad \forall x$

We show $(*)$. Define

$$g(x, y) = x+1 \qquad \text{computable}$$

By smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total computable s.t. $\forall x, y$

$$\varphi_{s(x)}(y) = g(x, y) = x+1$$

Since s is total computable, by 2nd recursion theorem there is $e \in \mathbb{N}$ s.t. $\varphi_e = \varphi_{s(e)}$. Thus

$$\varphi_e(y) = \varphi_{s(e)}(y) = g(e, y) = e+1 \qquad \forall y$$

Hence B not saturated.