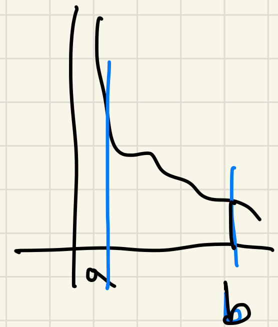
$$\int_a^{+\infty} f(x)\,dx = \lim_{M \to +\infty} \int_a^M f(x)\,dx$$

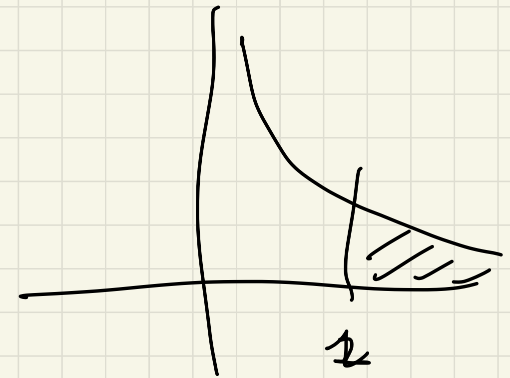
se $\lim_{x \to a^+} f(x) = \pm\infty$



$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$$

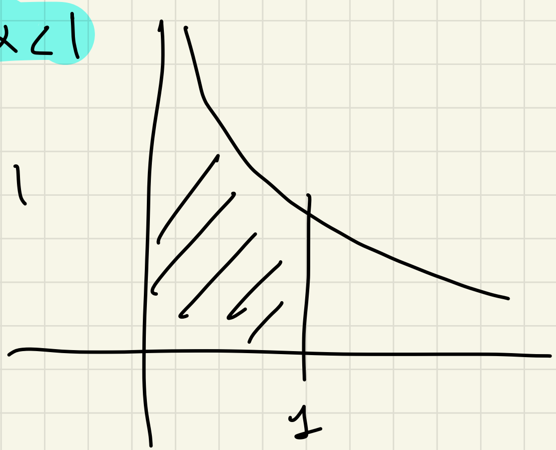
(1)

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \dfrac{1}{\alpha-1} & \text{se } \alpha>1 \\ +\infty & \alpha \le 1 \end{cases}$$



(2)

$$\int_0^{1} \frac{1}{x^\alpha}\,dx = \begin{cases} \dfrac{1}{1-\alpha} & \text{se } \alpha<1 \\ +\infty & \alpha \ge 1 \end{cases}$$



CRITERIO DEL CONFRONTO ASINTOTICO

Si può utilizzare solo per FUNZIONI POSITIVE

1 INTERVALLI ILLIMITATI

$f, g \geq 0$ $\quad f, g : [a, +\infty) \to \mathbb{R}$ continue

tali che $\displaystyle\lim_{x \to +\infty} \frac{f(x)}{g(x)} = L \neq 0, \neq \infty$

$f(x) = g(x)\ [\dots]$ per $x \to +\infty$

$L \neq 0, \infty$

ALLORA $\displaystyle\int_a^{+\infty} f(x)\,dx < +\infty \iff \int_a^{+\infty} g(x)\,dx < +\infty$

$\displaystyle\int_a^{+\infty} f(x)\,dx = +\infty \iff \int_a^{+\infty} g(x)\,dx = +\infty$

Applicazione tipica: $f \geq 0$

$$\underbrace{f(x)} = \frac{1}{x^{\alpha}} \cdot (\underbrace{h(x)}_{\substack{\downarrow \\ L \neq 0 \\ \neq \infty}}) \qquad x \to +\infty$$

$$\int_1^{+\infty} f(x)\,dx < +\infty \iff \alpha > 1$$

$$\left(\int_1^{+\infty} \frac{1}{x^{\alpha}}\,dx < +\infty\right)$$

Es: Dire per quali $k > 0$ esiste finito

$$\int_4^{+\infty} \frac{1}{\underbrace{x^{k} + 2x - 3}}\,dx$$

(e calcolarlo per $k = 2$ se possibile)

$$f(x) = \frac{1}{x^{k} + 2x^{1} - 3}$$

$x \to +\infty$

dato che $x \to +\infty$ devo raccogliere la x elevata al grado massimo

$k > 1$

$$\frac{1}{x^{k}\left[1 + \underbrace{\frac{2x}{x^{k}}}_{\to 0} - \underbrace{\frac{3}{x^{k}}}_{\to 0}\right]} = \frac{1}{x^{k}} \cdot \frac{1}{\left(1 + \frac{2x}{x^{k}} - \frac{3}{x^{k}}\right)} \to \frac{1}{1} = 1$$

l'integrale è finito $\iff k > 1$

se $k \leq 1$

$$\frac{1}{x^k + 2x - 3} = \frac{1}{x\left[\frac{x^k}{x} + 2 - \frac{3}{x}\right]} =$$

$\frac{x^k}{x} \to$ 0 se $k<1$, 1 se $k=1$

$\frac{3}{x} \to 0$

$$= \frac{1}{x}\left[\dots\right]$$

$[\dots] \to$ $\frac{1}{2}$ se $k<1$

$\frac{1}{3}$ se $k=1$

l'integrale è INFINITO

dato che $f(x) = \frac{1}{x} \cdot (\dots)$

$(\dots) \to L \neq 0$

$$\int_2^{+\infty} \frac{1}{x}\,dx = +\infty$$

Calcolo integrale per $k=2$

$$\int_{4}^{+\infty} \frac{1}{x^2+2x-3}\,dx = \lim_{M\to+\infty} \underbrace{\int_{4}^{M} \frac{1}{x^2+2x-3}\,dx}$$

$$\int \frac{1}{x^2+2x-3}\,dx \qquad x^2+2x-3=0. \qquad x_{1,2}=\begin{cases} 1 \\ -3 \end{cases}$$

$$x^2+2x-3 = (x-1)(x-(-3)) = (x-1)(x+3)$$

$$\frac{0\cdot x+1}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3} = \frac{Ax+3A+Bx-B}{(x-1)(x+3)}$$

$$\begin{cases} A+B=0 \\ 3A-B=1 \end{cases} \quad \begin{cases} B=-A \\ 3A-(-A)=1 \end{cases} \quad \begin{cases} B=-A \\ 4A=1 \end{cases} \quad \begin{cases} B=-\frac{1}{4} \\ A=\frac{1}{4} \end{cases}$$

$$\frac{1}{x^2+2x-3} = \frac{\frac{1}{4}}{x-1} + \frac{\left(-\frac{1}{4}\right)}{x+3} = \frac{1}{4}\cdot\frac{1}{x-1} - \frac{1}{4}\,\frac{1}{x+3}$$

$$\int \frac{1}{x^2+2x-3}\,dx = \frac{1}{4}\int\frac{1}{x-1}\,dx - \frac{1}{4}\int\frac{1}{x+3}\,dx =$$

$$= \frac{1}{4}\lg|x-1| - \frac{1}{4}\lg|x+3| + c$$

$$= \frac{1}{4}\lg\left|\frac{x-1}{x+3}\right| + c$$

$$\int_4^M \frac{1}{x^2+2x-3}\,dx = \frac{1}{4}\lg\left(\frac{M-1}{M+3}\right) - \frac{1}{4}\lg\left(\frac{4-1}{4+3}\right) =$$

$$= \frac{1}{4}\lg\left(\frac{\cancel{M}\left(1-\frac{1}{M}\right)}{\cancel{M}\left(1+\frac{3}{M}\right)}\right) - \frac{1}{4}\lg\left(\frac{3}{7}\right)$$

$$\lim_{M\to+\infty} \frac{1}{4}\lg\left(\frac{1-\frac{1}{M}}{1+\frac{3}{M}}\right) - \frac{1}{4}\lg\left(\frac{3}{7}\right) = \cancel{\frac{1}{4}\lg 1}^{\,=\,0} - \frac{1}{4}\lg\frac{3}{7}$$

$$\frac{1-0}{1+0} = 1$$

$$= \frac{1}{4}\lg\frac{7}{3}$$

Es Dire per quali valori di $k \in \mathbb{R}$ esiste finito l'integrale

$$\int_{\frac{1}{\pi}}^{+\infty} x^k \left[\frac{1}{x} - \sin\left(\frac{1}{x}\right) \right] dx$$

e se possibile calcolare l'integrale per

$k = -4$.

per $x \to +\infty$ $\quad \frac{1}{x} \to 0$

[polinomio di Taylor per $x \to 0$

$\sin x = x - \frac{1}{6} x^3 + o(x^3)$

$x \to +\infty$

$$\sin\left(\frac{1}{x}\right) = \underbrace{\frac{1}{x} - \frac{1}{6}\frac{1}{x^3} + o\left(\frac{1}{x^3}\right)}$$

$$x^k\left[\frac{1}{x} - \sin\left(\frac{1}{x}\right)\right] = x^k\left[\frac{1}{x} - \left(\frac{1}{x} - \frac{1}{6}\frac{1}{x^3} + o\left(\frac{1}{x^3}\right)\right)\right] =$$

$$= x^k\left[\cancel{\frac{1}{x}} - \cancel{\frac{1}{x}} + \frac{1}{6}\frac{1}{x^3} + o\left(\frac{1}{x^3}\right)\right] = x^k \cdot \frac{1}{x^3}\left[\frac{1}{6} + o(1)\right] =$$

$$= \frac{1}{x^{3-k}} \left[\frac{1}{6} + o(1)\right] \quad \frac{1}{6} \neq 0$$

$$f(x) = x^k \left[\frac{1}{x} - \sin\frac{1}{x} \right] = \frac{1}{x^{3-k}} \left[\frac{1}{6} + o(1) \right]$$

$x \to +\infty$

comportamento asintotico

l'integrale $\displaystyle\int_{1/\pi}^{+\infty} f(x)\,dx < +\infty$

$\iff \quad 3 - k > 1$

$-k > 1 - 3 = -2$

$$\boxed{k < 2}$$

L'integrale è finito se e solo se $\boxed{k < 2}$.

Calcolo integrale per $k = -4$ (< 2)

$$\int_{\frac{1}{\pi}}^{+\infty} x^{-4}\left[\frac{1}{x} - \sin\frac{1}{x}\right] dx = \lim_{M\to+\infty} \int_{\frac{1}{\pi}}^{M} \frac{1}{x^4}\left[\frac{1}{x} - \sin\frac{1}{x}\right] dx$$

$$\int_{\frac{1}{\pi}}^{M} \frac{1}{x^4}\left[\frac{1}{x} - \sin\left(\frac{1}{x}\right)\right] dx \qquad y = \frac{1}{x} \qquad x = \frac{1}{y}$$

$$dx = \left(-\frac{1}{y^2}\right) dy$$

$$= \int_{\pi}^{\frac{1}{M}} y^4 \left[y - \sin y\right]\left(-\frac{1}{y^2}\right) dy$$

$x = \frac{1}{\pi} \longrightarrow y = \pi$

$x = M \longrightarrow y = 1/M$

$$= -\int_{\pi}^{\frac{1}{M}} \cancel{y^4}^{2} \cdot [y - \sin y] \cdot \frac{1}{\cancel{y^2}} \, dy$$

$$= \int_{\frac{1}{M}}^{\pi} \underbrace{y^2 \cdot [y - \sin y]} \, dy = \int_{\frac{1}{M}}^{\pi} \underbrace{y^3 - y^2 \sin y} \, dy =$$

$$= \underbrace{\int_{\frac{1}{M}}^{\pi} y^3 \, dy} - \underbrace{\int_{\frac{1}{M}}^{\pi} y^2 \sin y \, dy}$$

$$\int y^3\,dy = \frac{1}{4}y^4 + c$$

$$\int_{\frac{1}{M}}^{\pi} y^3\,dy = \frac{1}{4}\pi^4 - \frac{1}{4}\left(\frac{1}{M}\right)^4$$

$F(y)g(y) - \int F(y)\,g'(y)\,dy$

$$\int y^2 \sin y\,dy = -(\cos y)\,y^2 - \int (-\cos y)\cdot 2y\,dy =$$

$f(y) = \sin y \Rightarrow F(y) = -\cos y$

$g(y) = y^2 \rightarrow g'(y) = 2y$

$f(y) = \cos y \rightarrow F(y) = \sin y$

$g(y) = y \rightarrow g'(y) = 1$

$$= -(\cos y)\,y^2 + 2\int \cos y \cdot y\,dy =$$

$$= -\cos y\;y^2 + 2\left[\sin y \cdot y - \int \sin y \cdot 1\,dy\right] =$$

$$= -\cos y \, y^2 + 2 \sin y \cdot y - 2 \underbrace{\int \sin y \, dy}_{} =$$

$$= -\cos y \cdot y^2 + 2 \sin y \cdot y - 2(-\cos y) + c =$$

$$= \underbrace{-\cos y \cdot y^2 + 2 \sin y \cdot y + 2 \cos y + c}_{}$$

$$\int_{\frac{1}{M}}^{\pi} y^2 \sin y \, dy = -(\overbrace{\cos \pi}^{=-1})(\pi)^2 + 2 \cancel{\sin(\pi)}^{0} \cdot \pi + 2 \overbrace{\cos \pi}^{=-1} +$$

$$- \left[-\cos\left(\frac{1}{M}\right) \frac{1}{M^2} + 2 \sin\left(\frac{1}{M}\right) \cdot \frac{1}{M} + 2 \cos \frac{1}{M} \right]$$

$$= -(-1)\pi^2 + 2(-1) + \cos\left(\frac{1}{M}\right) \cdot \frac{1}{M^2} - 2 \sin \frac{1}{M} \cdot \frac{1}{M} - 2 \cos \frac{1}{M}$$

$$= \pi^2 - 2 + \cos \frac{1}{M} \frac{1}{M^2} - 2 \sin \frac{1}{M} \cdot \frac{1}{M} - 2 \cos \frac{1}{M}$$

$$\lim_{M\to+\infty} \int_{1/\pi}^{M} \frac{1}{x^4}\left[\frac{1}{x} - \sin\frac{1}{x}\right] dx =$$

$$= \lim_{M\to+\infty} \left[\underbrace{\int_{\frac{1}{M}}^{\pi} y^3\, dy} - \int_{\frac{1}{M}}^{\pi} y^2 \sin y \, dy \right] =$$

$$= \lim_{M\to+\infty} \frac{1}{4}\pi^4 - \frac{1}{4}\left(\frac{1}{M}\right)^4 \to 0 - \left[\pi^2 - 2 + \cancel{\cos\frac{1}{M}\cdot\frac{1}{M^2}} - \cancel{2\sin\frac{1}{M}\,\frac{1}{M}} + \right.$$

$\cos 0 \cdot 0$ $\quad$ $\sin 0 \cdot 0$

$$\left. -2\underbrace{\cos\frac{1}{M}}_{\cos 0 = 1} \right] = \frac{1}{4}\pi^4 - \pi^2 + 2 + 2 = \frac{1}{4}\pi^4 - \pi^2 + 4.$$

CRITERIO DEL CONFRONTO ASINTOTICO SU INTERVALLI LIMITATI

$f, g \geq 0$ $\quad f, g : (a, b] \to \mathbb{R}$

$$\lim_{x \to a^+} f(x) = +\infty = \lim_{x \to a^+} g(x)$$

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L \neq 0, \quad \neq \infty$$

allora $\displaystyle\int_a^b f(x)\,dx < +\infty \iff \int_a^b g(x)\,dx < +\infty$

$$\int_a^b f(x)\,dx = +\infty \iff \int_a^b g(x)\,dx = +\infty$$

applicazione tipica

se $f \geq 0$ per $x \to 0^+$ $f(x) = \frac{1}{x^\alpha} \cdot [h(x)]$, $h(x) \to L \neq 0$ per $x \to 0^+$

allora $\int_0^1 f(x)\,dx < +\infty \iff \alpha < 1$

Es determinare per quali $k \in \mathbb{R}$ esiste finito

$$\int_0^1 \frac{\log(1+x) - x}{x^k}\,dx$$

(calcolarlo se possibile per $k = -1$)

$x \to 0^+$

$$f(x) = \frac{\log(1+x) - x}{x^k}$$

$$\log(1+x) = x - \frac{1}{2}x^2 + o(x^2)$$

$$f(x) = \frac{\cancel{x} - \frac{1}{2}x^2 + o(x^2) \cancel{- x}}{x^k} = \frac{x^2\left[-\frac{1}{2} + o(1)\right]}{x^k} =$$

$$= \frac{\left[-\frac{1}{2} + o(1)\right]}{x^{k-2}} \to -\frac{1}{2} \neq 0$$

$$\boxed{k - 2 < 1 \quad k < 3}$$

calcolo per $k=-1$ $\quad (k=-1<3)$

$$\int_0^1 \frac{\lg(1+x)-x}{x^{-1}}\,dx = \int_0^1 x\left[\lg(1+x)-x\right]dx =$$

$$= \int_0^1 x\lg(1+x) - x^2 \,dx = \int_0^1 x\lg(1+x) - \underbrace{\int_0^1 x^2\,dx}$$

$$\int_0^1 x^2\,dx = \frac{1}{3}1^3 - \frac{1}{3}0^3 = \frac{1}{3}$$

$$\int x^2\,dx = \frac{1}{3}x^3 + c$$

$$\int_0^1 x \lg(1+x)\,dx$$

$$\int x \lg(1+x)\,dx = \frac{1}{2}x^2 \cdot \lg(1+x) - \int \frac{1}{2}x^2 \frac{1}{x+1}\,dx$$

$$\begin{cases} f(x) = x \rightarrow F(x) = \frac{1}{2}x^2 \\ g(x) = \lg(1+x) \rightarrow g'(x) = \frac{1}{1+x} \cdot 1 \end{cases}$$

$$= \frac{1}{2}x^2 \lg(1+x) - \frac{1}{2}\int \frac{x^2}{x+1}\,dx$$

$$\frac{x^2}{x+1} = \frac{x^2-1+1}{x+1} = \frac{x^2-1}{x+1} + \frac{1}{x+1} =$$

$$= \frac{(x-1)\cancel{(x+1)}}{\cancel{x+1}} + \frac{1}{x+1} = x-1+\frac{1}{x+1}$$

$$\int \frac{x^2}{x+1}\,dx = \int x-1\,dx + \int \frac{1}{x+1}\,dx =$$

$$= \frac{1}{2}x^2 - x + \lg|x+1| + c$$

$$\int x \lg(x+1)\,dx = \frac{1}{2}x^2\lg(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}\,dx =$$

$$= \frac{1}{2}x^2\lg(x+1) - \frac{1}{2}\left[\frac{1}{2}x^2 - x + \lg|x+1| + c\right] =$$

$$= \frac{1}{2}x^2\lg(x+1) - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2}\lg|x+1| + c$$

$$\int_0^1 x \lg(1+x)\,dx =$$

$$= \frac{1}{2}\, 1^2 \lg(1+1) - \frac{1}{4}\, 1^2 + \frac{1}{2} \cdot 1 - \underbrace{\frac{1}{2} \lg(1+1)} +$$

$$- \left(\cancel{\frac{1}{2}\, 0^2 \lg(0+1)} - \cancel{\frac{1}{4}\, 0^2} + \cancel{\frac{1}{2} \cdot 0} - \underbrace{\cancel{\frac{1}{2} \lg(0+1)}} \right)^{0} =$$

$$= \cancel{\frac{1}{2} \lg 2} - \frac{1}{4} + \frac{1}{2} - \cancel{\frac{1}{2} \lg 2} = \frac{1}{4}$$

$$\int_0^1 x \left[\lg(1+x) - x \right] dx = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}.$$