$f : \mathbb{R} \to \mathbb{R}$ $\quad f \in C^1(\mathbb{R})$ $\quad \int_{\mathbb{R}} |f(x)|\,dx < +\infty$

we define $\underbrace{\hat{f}(x)}$:= Fourier transform $= \int_{\mathbb{R}} f(y)\, e^{ixy}\, dy =$

$$= \underbrace{\overbrace{\int_{\mathbb{R}} f(y) \cos(xy)\, dy}^{\mathrm{Re}\,\hat{f}(x)}}\; \cancel{+}\; + i \underbrace{\int_{\mathbb{R}} f(y) \sin(xy)\, dy}_{\mathrm{Im}\,\hat{f}(x)}$$

$\hat{f} : \mathbb{R} \to \mathbb{C}$ $\qquad \hat{f}(x) \in \mathbb{C}$

→ $\sup_{x \in \mathbb{R}} |\hat{f}(x)| \leq \|f\|_{L^1} = \int_{\mathbb{R}} |f(y)|\,dy$

($|e^{ixy}| = 1 \quad \forall x, y \in \mathbb{R}$

$|e^{ixy}| = \sqrt{(\cos(xy))^2 + (\sin(xy))^2} = 1$

→ $\hat{f}$ is continuous $\qquad \lim_{h \to 0} \hat{f}(x+h) = \hat{f}(x)$

→ $\lim_{|x| \to +\infty} |\hat{f}(x)| = 0$

So if $f \in L^1(\mathbb{R}) \to$ then $\hat{f} \in C_0(\mathbb{R}) := \{ g \in C(\mathbb{R})$, g is continuous in $\mathbb{R}$, $\lim_{|x| \to +\infty} |g(x)| = 0 \}$

($\hat{f} \in C_0(\mathbb{R})$

$$\operatorname{Re} \hat{f}(x) = \int_{\mathbb{R}} f(y) \cos(xy)\, dy \in C_0(\mathbb{R})$$

$$\operatorname{Im} \hat{f}(x) = \int_{\mathbb{R}} f(y) \sin(xy)\, dy \in C_0(\mathbb{R})$$

$\operatorname{Re} \hat{f} : \mathbb{R} \to \mathbb{R}$ $\quad$ $\operatorname{Im} \hat{f} : \mathbb{R} \to \mathbb{R}$

(note that $C_0(\mathbb{R})$ are in particular continuous bounded functions).

Properties of Fourier transform

1) $f, g \in L^1$ $\qquad \underbrace{\widehat{f * g}(x)} = \hat{f}(x)\, \hat{g}(x).$

2) if $\underbrace{|x|^m f(x) \in L^1(\mathbb{R})}$ $\forall m \le k$ $\left(\int_{\mathbb{R}} |x|^m |f(x)|\, dx < +\infty \quad \forall m \le k \right)$

$\downarrow$

$\hat{f} \in \mathcal{C}^k(\mathbb{R})$ $\qquad$ ($\hat{f}$ is differentiable k-times)

$\boxed{\forall m \le k}$

$$\frac{d^m}{dx^m} \hat{f}(x) = (i)^m \underbrace{\left[\widehat{y^m f(y)} \right](x)}_{= \int_{\mathbb{R}} y^m f(y)\, e^{ixy}\, dy}$$

(decay properties of f at ∞ translate into differentiability of $\hat{f}$).

3) if f is differentiable k-times, and

$$\frac{d^m}{dy^m} f(y) \in L^1(\mathbb{R}) \quad \forall m \leq k \qquad \lim_{|y| \to +\infty} \frac{d^m f(y)}{dy^m} = 0$$

$$\forall m < k$$

then

$$\left(\widehat{\frac{d^m}{dy^m} f(y)} \right)(x) = (-i)^m \, \underline{x^m \, \hat{f}(x)} \, .$$

Prop Let $a>0$ $\quad f(x) = e^{-a|x|^2} = e^{-ax^2}$

$$\hat{f}(x) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{x^2}{4a}}$$

(we proved last time for $a=1$)

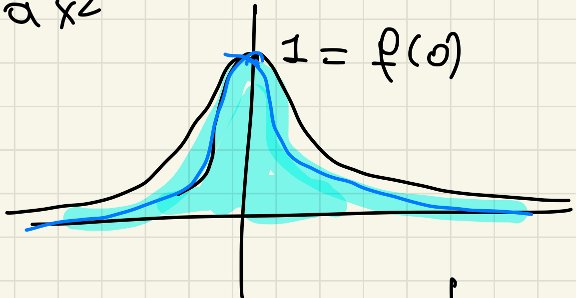
$$\hat{f}(0) = \int_{\mathbb{R}} e^{-a|y|^2} \cdot e^{0}\, dy = \int_{\mathbb{R}} e^{-a|y|^2} dy \overset{z=\sqrt{a}\,y}{=} \frac{1}{\sqrt{a}} \int_{\mathbb{R}} e^{-|z|^2} dz$$

$$= \sqrt{\frac{\pi}{a}}$$

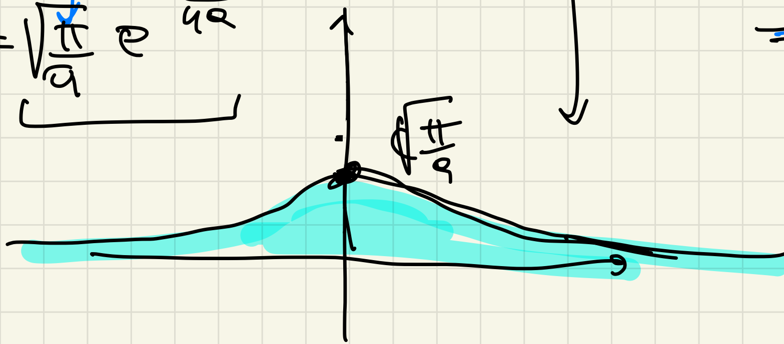
$$\frac{d}{dx}\hat{f}(x) \underset{\text{by (2)}}{=} i \int_{\mathbb{R}} y\, e^{-ay^2} e^{ixy}\, dy \overset{\text{by parts}}{=} \dots = \frac{-x}{2a}\hat{f}(x)$$

$f = e^{-ax^2}$

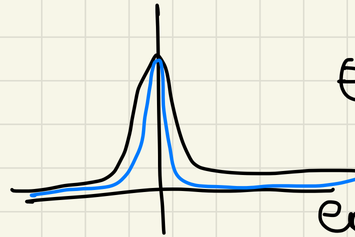
$a > 0$



$\hat{f} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{x^2}{4a}}$



Note that as $a \to +\infty$



e^{-ax^2} is becoming more and more concentrated at 0.

as $a \to +\infty$ $\hat{f} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{x^2}{4a}}$ is becoming more and more flat $\left(\sqrt{\frac{\pi}{a}} = f(0) \to 0\right)$

INDETERMINATION PRINCIPLE : f and $\hat{f}$ cannot be both concentrated (f POSITION of a particle $\to \hat{f}$ MOMENTUM of a particle)

Schwartz class

$S = \{ f : \mathbb{R} \to \mathbb{R}$ such that

f is differentiable infinite times,

$f \in C^\infty(\mathbb{R})$

$|x|^m f(x)$ is continuous, bounded and in $L^1(\mathbb{R})$

$\forall m \geq 0$ $\qquad \int_{\mathbb{R}} |x|^m f(x)\, dx < +\infty$

$\forall m \geq 0$

$|x|^m \frac{d^k}{dx^k} f(x)$ is continuous bdd and in $L^1(\mathbb{R})$ $\}$

$\forall k, m$.

$\boxed{f(x) = e^{-a x^2} \qquad a > 0 \qquad f \in S}$ $\Rightarrow$ $\boxed{f \in S \Rightarrow \hat{f} \in S}$

Prop if $f \in C_0(\mathbb{R})$ (f is continuous and $\lim_{|x| \to +\infty} f(x) = 0$)

$\forall k \in \mathbb{N}$ $\exists f_k \in S$ such that

$$\boxed{\max_{x \in \mathbb{R}} |f_k(x) - f(x)| \leq \frac{1}{k}}$$

$$\Downarrow$$

$\forall x \in \mathbb{R}$

$$f_k(x) - \frac{1}{k} \leq f(x) \leq \frac{1}{k} + f_k(x)$$

(the proof is by convolution $\to$ take $\begin{cases} g_k \in S, \text{ with } \int_{\mathbb{R}} g_k = 1 \\ g_k \geq 0 \end{cases}$

$f_k = \underbrace{f * g_k} \in C^\infty \ldots$

FOURIER INVERSION THEOREM

$f \in L^1(\mathbb{R})$ $\quad \check{f}(x) =$ anti - Fourier transform of $f =$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} f(y) e^{-ixy} dy = \frac{1}{2\pi} \hat{f}(-x)$$

$$\hat{f}(x) = \int_{\mathbb{R}} f(y) e^{ixy} dy$$

$\boxed{\text{theorem}}$ Let $f \in L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$ $\quad \left[\int_{\mathbb{R}} |\hat{f}(x)| dx < +\infty \right]$

then

$$f(x) = \check{\hat{f}}(x) = \hat{\check{f}}(x) \quad \forall x$$

$\left[\begin{array}{l} \operatorname{Re} \hat{f} \in L^1(\mathbb{R}) \\ \operatorname{Im} \hat{f} \in L^1(\mathbb{R}) \end{array} \right.$

→ In particular since f is the Fourier transform of a L^1 function $\Rightarrow$ $f \in C_0(\mathbb{R})$

(because Fourier transform of L^1 functions are continuous functions going to 0 at $\pm\infty$).

→ Moreover if $f, g \in L^1(\mathbb{R})$ $\underline{\hat{f}(x) = \hat{g}(x)}$ $\forall x$

$\Rightarrow f(x) = g(x) \quad \forall x \in \mathbb{R}$.

(FOURIER TRANSFORM IS INJECTIVE)

$$\left(f, g \in L^1 \Rightarrow f - g \in L^1(\mathbb{R}) \qquad \widehat{f-g}(x) = \hat{f}(x) - \hat{g}(x) \equiv 0 \in L^1(\mathbb{R})\right)$$

by Theorem $(f-g)(x) = \check{\widehat{(f-g)}}(x) = \check{0}(x) = 0$

→ Finally $\forall f \in S$ $\quad \check{\hat{f}}(x) = f(x) = \hat{\check{f}}(x)$

$$\boxed{\hat{\ } : S \longrightarrow S \text{ is a bijection}}$$

proof $f, \underbrace{\hat{f} \in L^1(\mathbb{R})}$

$$\check{f}(x) = \frac{1}{2\pi} \int f(y) e^{-ixy} dy$$

$$\underline{2\pi \check{\hat{f}}(x)} = \int_{\mathbb{R}} \underbrace{\hat{f}(y) e^{-ixy}} dy = \lim_{a \to 0^+} \int_{\mathbb{R}} \underbrace{\hat{f}(y) e^{-ixy}} \underline{e^{-ay^2}} dy$$

$a \to 0^+ \quad e^{-ay^2} \to 1$ (I can take the limit inside the integral by the Lebesgue dominated convergence thm)

$$1 = \lim_{a \to 0^+} e^{-ay^2}$$

$$= \lim_{a \to 0^+} \int_{\mathbb{R}} \boxed{\int_{\mathbb{R}} f(z) \underline{e^{iyz}} \, \underline{dz}} \, \underline{e^{-ixy} e^{-ay^2}} \, dy =$$

$=$ I exchange the order of integration (FUBINI TONELLI THEOREM)

$$= \lim_{a\to 0^+} \int_{\mathbb{R}} \int_{\mathbb{R}} f(z)\, \underbrace{e^{iyz} \cdot e^{-ixy}\, e^{-ay^2}}\, dy\, dz$$

$$= \lim_{a\to 0^+} \int_{\mathbb{R}} f(z) \boxed{\int_{\mathbb{R}} \underbrace{e^{-ay^2}}\, e^{iy(z-x)}\, dy} =$$

$$e^{iyz}\, e^{-ixy} = e^{iyz - ixy}$$

$$= \lim_{a\to 0^+} \int_{\mathbb{R}} f(z) \underbrace{\widehat{\left(e^{-ay^2}\right)} (z-x)}\, dz =$$

$$= \lim_{a\to 0^+} \int_{\mathbb{R}} f(z)\, \frac{\sqrt{\pi}}{\sqrt{a}} \cdot e^{-\frac{(z-x)^2}{4a}}\, dz =$$

$$= \lim_{a\to 0^+} \sqrt{\frac{\pi}{a}} \int_{\mathbb{R}} f(z)\, e^{-\frac{(x-z)^2}{4a}}\, dz =$$

$$\xi = \frac{z-x}{2\sqrt{a}} \qquad d\xi = \frac{dz}{2\sqrt{a}}$$

$$z = 2\sqrt{a}\cdot\xi + x$$

$$= \lim_{a\to 0^+} \sqrt{\frac{\pi}{\cancel{a}}} \int_{\mathbb{R}} f(x+2\sqrt{a}\xi)\cdot e^{-\xi^2}\, 2\cancel{\sqrt{a}}\, d\xi =$$

$$= \lim_{a\to 0^+} 2\sqrt{\pi} \int_{\mathbb{R}} f(x+\cancelto{0}{2\sqrt{a}\xi})\, e^{-\xi^2} d\xi$$

= I can take the limit inside the integral =

$$= 2\sqrt{\pi} \int_{\mathbb{R}} \underbrace{f(x)}_{} e^{-\xi^2} d\xi = f(x)\, 2\sqrt{\pi} \underbrace{\int_{\mathbb{R}} e^{-|\xi|^2} d\xi}_{} =$$

$$= f(x) \cdot 2\sqrt{\pi}\sqrt{\pi} = f(x)\, 2\pi$$

$$\cancel{2\pi}\, \check{\hat{f}}(x) = f(x)\, \cancel{2\pi} \implies \check{\hat{f}}(x) = f(x)$$

(same argument to show $\hat{\check{f}}(x) = f(x)$]

An important corollary of this theorem is the following "convergence result".

Theorem: Let $f_n \in L^1(\mathbb{R})$, $f \in L^1(\mathbb{R})$

Assume

1) $\forall x \in \mathbb{R}$ $\quad \lim_n \hat{f}_n(x) = \hat{f}(x)$ $\quad$ ($\hat{f}_n \to \hat{f}$ pointwise)

2) $\exists\, C > 0$ $\quad \|f_n\|_{L^1} = \int_{\mathbb{R}} |f_n(x)|\, dx \leq C$

(INDIPENDENT of n).

Then $\forall g \in C_0(\mathbb{R})$ $\quad \lim_{n \to +\infty} \int_{\mathbb{R}} f_n(x) g(x)\, dx = \int_{\mathbb{R}} f(x) g(x)\, dx$

(Note that saying that $\forall g \in C_0(\mathbb{R})$

$$\lim_n \int_{\mathbb{R}} f_n(x)\, g(x)\, dx = \int_{\mathbb{R}} f(x) g(x)\, dx$$

DOES NOT IMPLY that $\int_{\mathbb{R}} |f_n(x) - f(x)|\, dx \longrightarrow 0$
(convergence in L^1 sense).

This is a weaker notion of convergence (which is VAGUE CONVERGENCE / WEAK* CONVERGENCE).

ex. $\lim_{n \to +\infty} \int_{\mathbb{R}} \underbrace{\cos}\,(nx)\, g(x)\, dx = 0 \quad \forall g \in C_0(\mathbb{R})$

$\cos(nx)$ converges vaguely to 0
(Riemann Lebesgue lemma)

proof $\underbrace{g \in C_0(\mathbb{R})}$

we want to prove that

$$\lim_{n \to +\infty} \int_{\mathbb{R}} [f_n(x) - f(x)]\, g(x)\, dx = 0$$

$$\left[\lim_n \int_{\mathbb{R}} f_n(x) g(x)\, dx = \int_{\mathbb{R}} f(x) g(x)\, dx\right].$$

$\forall k \in \mathbb{N} \quad \exists g_k \in S$ such that $|g(x) - g_k(x)| \leq \frac{1}{k} \quad \forall x \in \mathbb{R}$

If I can prove that

$$\boxed{\lim_{n \to +\infty} \int_{\mathbb{R}} [f_n(x) - f(x)]\, h(x)\, dx = 0 \quad \forall h \in S}$$

it will imply the conclusion

Indeed if $\lim_{n\to+\infty} \int_{\mathbb{R}} (f_n(x)-f(x))\, h(x)\,dx = 0 \quad \forall h \in S$

$\to$ it is true also for $h = g_k \in S$

$\Rightarrow \lim_{n\to+\infty} \int_{\mathbb{R}} [f_n(x)-f(x)]\, g_k(x)\,dx = 0 \quad \forall k \in \mathbb{N}$

Now I know that $\forall x \in \mathbb{R}$

$$g_k(x) - \frac{1}{k} \leq g(x) \leq g_k(x) + \frac{1}{k}$$

$$\int_{\mathbb{R}} [f_n(x)-f(x)]\, g(x)\,dx \leq \int_{\mathbb{R}} [f_n(x)-f(x)]\, g_k(x) + \boxed{\frac{1}{k}\int_{\mathbb{R}} f_n(x)-f(x)\,dx}$$

$$\int_{\mathbb{R}} [f_n(x)-f(x)]\, g(x)\,dx \geq \int_{\mathbb{R}} [f_n(x)-f(x)]\, g_k(x) - \frac{1}{k}\int_{\mathbb{R}} (f_n(x)-f(x))\,dx$$

$$\left| \frac{1}{k} \int_{\mathbb{R}} f_n(x) - f(x) \, dx \right| \leq \frac{1}{k} \int_{\mathbb{R}} |f_n(x) - f(x)| \, dx$$

$$\leq \frac{1}{k} \Big[\underbrace{\int_{\mathbb{R}} |f_n(x)| dx}_{} + \int_{\mathbb{R}} |f(x)| dx \Big] \leq$$

by assumption

$$\leq \frac{1}{k} \left[C + \|f\|_{L^1} \right] = \frac{\bar{C}}{k} \qquad \bar{C} = C + \|f\|_{L^1}$$

$$\underbrace{\int_{\mathbb{R}} [f_n(x) - f(x)] g_k(x) dx}_{\to 0} - \frac{\bar{C}}{k} \leq \int_{\mathbb{R}} [f_n(x) - f(x)] g(x) \, dx \leq \int_{\mathbb{R}} [f_n(x) - f(x)] g_k(x) + \frac{\bar{C}}{k}$$

$$\downarrow n \to +\infty \qquad \downarrow n \to +\infty$$

$$-\frac{\bar{C}}{k} \leq \lim_{n \to +\infty} \int_{\mathbb{R}} [f_n(x) - f(x)] g(x) dx \leq \frac{\bar{C}}{k} \qquad \forall k \in \mathbb{N}$$

$$\Longleftrightarrow \quad \lim_{n\to+\infty} \int_{\mathbb{R}} [f_n(x) - f(x)]\, g(x)\, dx = 0 .$$

So to prove the theorem we are reduced to prove that

$$\lim_{n} \int_{\mathbb{R}} [f_n(x) - f(x)]\, h(x)\, dx = 0 \qquad \forall\, h \in S'$$

$$h \in S \Longrightarrow h(x) = \hat{\check{h}}(x) \qquad \text{(by the inversion theorem)}$$

$$h(x) = \hat{\check{h}}(x) = \int_{\mathbb{R}} \check{h}(y)\, e^{ixy}\, dy$$

$$\int_{\mathbb{R}} [f_n(x) - f(x)]\, h(x)\, dx = \int_{\mathbb{R}} (f_n(x) - f(x)) \int_{\mathbb{R}} \check{h}(y)\, e^{ixy}\, dy\, dx$$

= I exchange the order of integration =

$$= \int_{\mathbb{R}} \check{h}(y) \boxed{\int_{\mathbb{R}} [f_n(x) - f(x)]\, e^{ixy}\, dx}\, dy$$

$$= \int_{\mathbb{R}} \underbrace{\check{h}(y)}_{\text{integrable and bdd}} \underbrace{[\hat{f}_n(y) - \hat{f}(y)]}\, dy \xrightarrow{n \to +\infty} 0$$

integrable
and bdd

$\hat{f}_n(y) \to \hat{f}(y)$ for all y by ass.

$\sup_y |\hat{f}_n(y) - \hat{f}(y)| \le C$

Example of application of this theorem

Let consider $f(x) \geq 0$ $\quad f : \mathbb{R} \to [0, +\infty)$ such that

$$\int_{\mathbb{R}} f(x)\,dx = 1, \quad \int_{\mathbb{R}} x f(x)\,dx = 0, \quad \int_{\mathbb{R}} x^2 f(x)\,dx = 1$$

$$f^{*n}(x) = f * f * \dots * f \,(x) \qquad n\text{-times}$$

$$f_n(x) := \sqrt{n}\, f^{*n}(\sqrt{n}\, x) = \sqrt{n}\, \underbrace{f * f * \dots * f}_{n \text{ times}}(\sqrt{n}\, x)$$

Then $\forall g \in C_0(\mathbb{R})$ $\displaystyle \lim_n \int_{\mathbb{R}} f_n(x) g(x)\,dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} g(x)\,dx$

(f_n is converging VAGUELY to $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Comments

$f \geq 0 \qquad \int_{\mathbb{R}} f(x)\,dx = 1 \qquad \int_{\mathbb{R}} x f(x)\,dx = 0$

$\int_{\mathbb{R}} x^2 f(x)\,dx = 1$

f is the density of an absolutely continuous variable X such that $\mathbb{E}(X) = \int_{\mathbb{R}} x f(x)\,dx = 0$

$\mathbb{E}(X^2) = \int_{\mathbb{R}} x^2 f(x)\,dx = 1$

X has mean $= 0$ and variance $= 1$

X_1, X_2 2 independent a.c. random variables which are identically distributed with the same density function f

Therefore X_1+X_2 is a a.c. random variable with $E(X_1+X_2)=0$ $\quad E((X_1+X_2)^2)=2$ (since X_1 and X_2 are independent) and X_1+X_2 has density $f*f$

$Z=\dfrac{X_1+X_2}{\sqrt{2}}$ is a random variable with $E(Z)=0$

$E(Z^2)=1$

the density of Z is $\quad f_Z(x)=\sqrt{2}\,(f*f)(\sqrt{2}x)$

$$P(\omega \mid Z(\omega)\le a)=\int_{-\infty}^{a} f_Z(x)\,dx$$

$$P(\omega \mid (X_1+X_2)(\omega)\le a\sqrt{2})=\int_{-\infty}^{a\sqrt{2}}(f*f)(x)\,dx=$$

$$=\left(\frac{x}{\sqrt{2}}=y\right)=\int_{-\infty}^{a}(f*f)(\sqrt{2}y)\,\sqrt{2}\,dy$$

If I have $X_1, X_2, \dots X_n$ all independent absolutely continuous random variables, all with mean 0 and variance 1, and identically distributed (so every X_i has the same density f)

$$\frac{X_1 + \dots + X_n}{\sqrt{n}} = Z$$ is a r.v. with mean 0 and variance 1

and density (arguing as before)

$$f_Z(x) = \sqrt{n}\,(\underbrace{f * \dots * f}_{n \text{ times}})(\sqrt{n}\,x)$$

So if I have $X_1 \dots X_n$ a.c. random variable independent and identically distributed (all with the same density function f) then the density function f_n of

$$\frac{X_1 + \dots + X_n}{\sqrt{n}}$$

is converging vaguely to the function $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ (which is the density function of the Gaussian distribution with mean 0 and variance 1).

$\Rightarrow \frac{X_1 + \dots + X_n}{\sqrt{n}}$ is converging in DISTRIBUTION to the

Gaussian with mean 0 and variance 1.

(A.C. version of the central limit theorem)

~ · ~

proof of the result.

$f \geq 0 \qquad \int_{\mathbb{R}} f(x)\,dx = 1 \qquad \int_{\mathbb{R}} x f(x)\,dx = 0 \qquad \int_{\mathbb{R}} x^2 f(x)\,dx = 1$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\underbrace{f \in L^1 \qquad |x| f \in L^1 \qquad x^2 f \in L^1}$$

$$\hat{f} \in \mathcal{C}^2(\mathbb{R})$$

$$\hat{f}(0) = \int_{\mathbb{R}} f(y) e^{0} dy = \int_{\mathbb{R}} f(y) dy = 1$$

$$\frac{d}{dx}\hat{f}(x) = (i)^1 \int_{\mathbb{R}} y f(y) \underbrace{e^{-ixy}}_{} dy$$

$$\frac{d}{dx}\hat{f}(0) = i^1 \underbrace{\int_{\mathbb{R}} y f(y) dy}_{} = 0$$

$$\frac{d^2}{dx^2}\hat{f}(x) = (i)^2 \int_{\mathbb{R}} y^2 f(y) e^{ixy} dy$$

$$\frac{d}{dx^2}\hat{f}(0) = (-1) \cdot \underbrace{\int_{\mathbb{R}} y^2 f(y) dy}_{} = (-1) 1 = -1$$

$\hat{f} \in \mathcal{C}^2(\mathbb{R})$

$$\hat{f}(x) = \underbrace{\hat{f}(0)}_{=1} + \cancel{\frac{d\hat{f}(0)}{dx}}^{=0} \cdot x + \frac{1}{2} \underbrace{\frac{d^2\hat{f}(0)}{dx^2}}_{=-1} \cdot x^2 + o(x^2)$$

$x \to 0$

$$\hat{f}(x) = 1 - \frac{1}{2}x^2 + o(x^2) \qquad x \to 0.$$

$$\boxed{f_n(x) = \sqrt{n}\,(\underbrace{f * \cdots * f}_{n \text{ times}})(\sqrt{n}\,x)}$$

then

$$\hat{f}_n(x) = \underbrace{\hat{f}\left(\frac{x}{\sqrt{n}}\right) \cdots \hat{f}\left(\frac{x}{\sqrt{n}}\right)}_{n\text{-times}}.$$

assume $n = 2$ $\quad \hat{f}_2(x) = \int_{\mathbb{R}} \sqrt{2}\, f * f(\sqrt{2}\,y)\, e^{iyx}\, dy =$

$$= \int_{\mathbb{R}} \sqrt{2} \int_{\mathbb{R}} f(z) f(\sqrt{2}\,y - z)\, dz\, e^{iyx}\, dy =$$

$$iyx = i\frac{x}{\sqrt{2}}(\sqrt{2}\,y - z) + i\frac{x}{\sqrt{2}} z$$

$$= \sqrt{2} \int_{\mathbb{R}} \int_{\mathbb{R}} f(z)\, e^{i\frac{x}{\sqrt{2}} z} f(\sqrt{2}\,y - z)\, e^{i\frac{x}{\sqrt{2}}(\sqrt{2}\,y - z)}\, dy\, dz =$$

$$= (t = \sqrt{2}\,y - z) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(z)\, e^{i\frac{x}{\sqrt{2}} z} f(t)\, e^{i\frac{x}{\sqrt{2}} t}\, dt\, dz = \hat{f}\left(\frac{x}{\sqrt{2}}\right) \hat{f}\left(\frac{x}{\sqrt{2}}\right)$$

$$\hat{f}_n(x) = \left[\hat{f}\left(\frac{x}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{x^2}{2n} + o\left(\frac{x^2}{2n}\right)\right]^n =$$

n large, x fixed

$$\hat{f}(x) = 1 - \frac{x^2}{2} + o(x^2) \qquad x \to 0$$

$$\hat{f}_n(x) = \left[1 - \frac{x^2}{2n} + o\left(\frac{x^2}{2n}\right)\right]^n = e^{\overbrace{n \lg\left[1 - \frac{x^2}{2n} + o\left(\frac{x^2}{2n}\right)\right]}}$$

$$= e^{n \cdot \left(-\frac{x^2}{2n} + o\left(\frac{x^2}{n}\right)\right)} = e^{-\frac{x^2}{2} + o(1)} \qquad (\text{as } n \to +\infty)$$

$$\hat{f}_n(x) \xrightarrow{n \to +\infty} \underbrace{e^{-\frac{x^2}{2}}} \qquad \forall\, x \in \mathbb{R} \text{ fixed}$$

$$e^{-\frac{x^2}{2}} = \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right]^{\wedge} = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi}\, e^{-\frac{x^2}{2}} = e^{-x^2/2}$$

$$\left(e^{-\frac{x^2}{2}}\right)^{\wedge} = \sqrt{2\pi}\, e^{-\frac{x^2}{2}}$$

$$\left(e^{-ax^2}\right)^{\wedge} = \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{x^2}{4a}} \qquad a = \frac{1}{2}$$

$$\hat{f_n}(x) \longrightarrow \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right]^{\wedge}$$

by the convergence thm $\Longrightarrow$ $\forall g \in C_0(\mathbb{R})$

$$\lim_n \int_{\mathbb{R}} f_n(x) g(x)\, dx = \int \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} g(x) dx$$