COMPUTABILITY (17/12/2024)

2^{nd} RECURSION THEOREM

let $f : \mathbb{N} \to \mathbb{N}$ computable total extensional

$$P \longrightarrow \boxed{f} \longrightarrow P'$$

$\forall e, e' \in \mathbb{N} \quad \varphi_e = \varphi_{e'}$

$$\varphi_{f(e)} = \varphi_{f(e')}$$

by Myhill - Shepherdson's Theorem there is a (unique)

recursive functional $\Phi : \mathcal{F}(\mathbb{N}) \to \mathcal{F}(\mathbb{N})$

$$\forall e \in \mathbb{N} \qquad \Phi(\varphi_e) = \varphi_{f(e)}$$

then by the 1^{st} recursion theorem Φ has a least fixed point

$f_\Phi : \mathbb{N} \to \mathbb{N}$ computable

$$\begin{cases} \Phi(f_\Phi) = f_\Phi \\ \exists e_0 \in \mathbb{N} \quad \text{s.t.} \quad f_\Phi = \varphi_{e_0} \end{cases}$$

$$\varphi_{e_0} = f_\Phi = \Phi(f_\Phi) = \Phi(\varphi_{e_0}) = \varphi_{f(e_0)}$$

In summary

if $f : \mathbb{N} \to \mathbb{N}$ total computable $\boxed{\cancel{\text{extensional}}}$

then there is $e_0 \in \mathbb{N}$ s.t.

$$\varphi_{e_0} = \varphi_{f(e_0)}$$

without this hypothesis

2^{nd} recursion theorem

$$P \longrightarrow \boxed{f} \longrightarrow P'$$

2^{nd} RECURSION THEOREM

Let $f: \mathbb{N} \to \mathbb{N}$ be total computable function

Then there is $e_0 \in \mathbb{N}$ s.t. $\varphi_{e_0} = \varphi_{f(e_0)}$

<u>proof</u>

let $f: \mathbb{N} \to \mathbb{N}$ total computable

observe $\quad x \longmapsto f(\varphi_x(x)) = f(\Psi_U(x,x))$ $\quad$ computable

define

$$g(x,y) = \varphi_{f(\varphi_x(x))}(y) = \Psi_U(f(\varphi_x(x)), y) = \Psi_U(f(\Psi_U(x,x)), y)$$ computable

convention $\varphi_\uparrow = \uparrow$

By smn theorem there is $s: \mathbb{N} \to \mathbb{N}$ total computable s.t.

$$\varphi_{s(x)}(y) = g(x,y) = \varphi_{f(\varphi_x(x))}(y) \qquad \forall x,y$$

Since s is computable there is $m \in \mathbb{N}$ s.t. $s = \varphi_m$, hence

$$\varphi_{\varphi_m(x)}(y) = \varphi_{f(\varphi_x(x))}(y) \qquad \forall x, y$$

For $x = m$

$$\varphi_{\varphi_m(m)}(y) = \varphi_{f(\varphi_m(m))}(y) \qquad \forall y$$

$$\hookrightarrow \varphi_{\varphi_m(m)} = \varphi_{f(\varphi_m(m))}$$

If we let $e_0 = \varphi_m(m)$ (note $\varphi_m(m) = s(m)\downarrow$ hence e_0 is a number)

$$\varphi_{e_0} = \varphi_{f(e_0)}$$

$\square$

<u>idea</u>: for $h : \mathbb{N} \to \mathbb{N}$ computable

$$\begin{array}{ccccc} \varphi_0 & \varphi_1 & \varphi_2 & \varphi_3 & \cdots \\ & & \big\downarrow h & & \\ \varphi_{h(0)} & \varphi_{h(1)} & \varphi_{h(2)} & \varphi_{h(3)} & \end{array}$$

you can do the above for all computable functions $h = \varphi_i \quad i \in \mathbb{N}$

$$\begin{array}{lccccc} E_0 & \varphi_{\varphi_0(0)} & \varphi_{\varphi_0(1)} & \varphi_{\varphi_0(2)} & \varphi_{\varphi_0(3)} & \cdots \\ E_1 & \varphi_{\varphi_1(0)} & \varphi_{\varphi_1(1)} & \varphi_{\varphi_1(2)} & \varphi_{\varphi_1(3)} & \cdots \\ E_2 & \varphi_{\varphi_2(0)} & \varphi_{\varphi_2(1)} & \varphi_{\varphi_2(2)} & \varphi_{\varphi_2(3)} & \cdots \end{array}$$

$h(x) = \varphi_x(x)$

in particular if you consider

$$h(x) = f(\varphi_x(x)) = f(\psi_U(x,x)) = \varphi_m(x) \quad \text{computable}$$

$$\begin{array}{lccccc} E_0 & \varphi_{\varphi_0(0)} & \varphi_{\varphi_0(1)} & \varphi_{\varphi_0(2)} & \varphi_{\varphi_0(3)} & \cdots \\ E_1 & \varphi_{\varphi_1(0)} & \varphi_{\varphi_1(1)} & \varphi_{\varphi_1(2)} & \varphi_{\varphi_1(3)} & \cdots \\ E_2 & \varphi_{\varphi_2(0)} & \varphi_{\varphi_2(1)} & \varphi_{\varphi_2(2)} & \varphi_{\varphi_2(3)} & \cdots \\ & & & & = \varphi_x(x) & \\ & & & & & \varphi_{\varphi_m(m)} \\ E_m & \varphi_{f(\varphi_0(0))} & \varphi_{f(\varphi_1(1))} & \cdots & \varphi_{f(\varphi_m(m))} & \end{array}$$

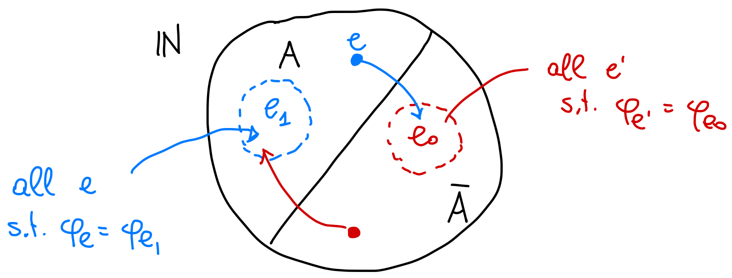
(up to some small detail i.e. $\varphi_m(m)$ could be undefined)

Rice's Theorem

Let $A \subseteq \mathbb{N}$ saturated $A \neq \emptyset$, $A \neq \mathbb{N}$ then A not recursive

proof (alternative)

Let $A \subseteq \mathbb{N}$ be saturated $A \neq \emptyset$, $A \neq \mathbb{N}$



$A \neq \emptyset$

$A \neq \mathbb{N}$

Assume by contradiction that A is recursive. Then

$$f(x) = \begin{cases} e_0 & \text{if } x \in A \\ e_1 & \text{if } x \notin A \end{cases}$$

$$= e_0 \, \chi_A(x) + e_1 \cdot \chi_{\bar{A}}(x)$$

$$\left(\begin{array}{lllll} x \in A & e_0 \cdot 1 & + & e_1 \cdot 0 & = e_0 \\ x \notin A & e_0 \cdot 0 & + & e_1 \cdot 1 & = e_1 \end{array} \right)$$

since A recursive, $\chi_A, \chi_{\bar{A}}$ computable, hence f computable

Moreover f is total

but for all $e \in \mathbb{N}$ $\quad \varphi_e \neq \varphi_{f(e)}$

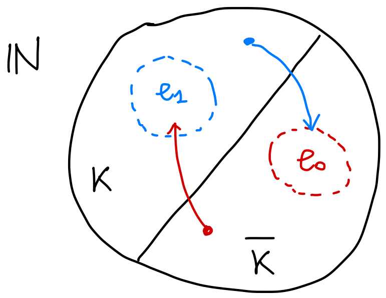
- $e \in A$ then $f(e) = e_0 \in \bar{A}$ $\quad \varphi_e \neq \varphi_{e_0} = \varphi_{f(e)}$ since A saturated
- $e \notin A$ then $f(e) = e_1 \in A$ $\quad \varphi_e \neq \varphi_{e_1} = \varphi_{f(e)}$ " " "

This contradicts the 2nd recursion theorem $\leadsto$ A not recursive.

$\square$

Proposition: The halting set $K = \{x \in \mathbb{N} \mid \varphi_x(x)\downarrow\}$ not recursive

proof (alternative)



$\forall e_0 \in \mathbb{N} \quad \varphi_{e_0}(x)\uparrow \quad \forall x$

$\Rightarrow \quad e_0 \in \overline{K}$

$\forall e_1 \in \mathbb{N}$ s.t. $\varphi_{e_1} = \mathbb{1}$ (constant 1)

$e_1 \in K$

define $f : \mathbb{N} \to \mathbb{N}$

$$f(x) = \begin{cases} e_0 & \text{if } x \in K \\ e_1 & \text{if } x \notin K \end{cases} = e_0 \, \chi_K(x) + e_1 \, \chi_{\overline{K}}(x)$$

if, by contradiction, K is recursive, $\chi_K, \chi_{\overline{K}}$ computable

then f is computable

but f is also total

by construction $\forall e \in \mathbb{N} \quad \varphi_e \neq \varphi_{f(e)}$ in fact

- if $e \in K$ then $f(e) = e_0$ and $\varphi_e(e)\downarrow \neq \varphi_{f(e)}(e) = \varphi_{e_0}(e)\uparrow$
- if $e \notin K$ then $f(e) = e_1$ and $\varphi_e(e)\uparrow \neq \varphi_{f(e)}(e) = \varphi_{e_1}(e) = 1\downarrow$

contradiction $\Rightarrow$ K not recursive

$\square$

* <u>K is not saturated</u>

$K = \{ x \in \mathbb{N} \mid \varphi_x(x)\downarrow \}$

We want to show K not saturated there are $e, e' \in \mathbb{N}$

$$\boxed{\begin{array}{ll} \varphi_e = \varphi_{e'} & \\ e \in K & e' \in \bar{K} \end{array}}$$

* Assume that there is $e \in \mathbb{N}$ s.t.

$$\varphi_e(x) = \begin{cases} 0 & \text{if } x = e \\ \uparrow & \text{otherwise} \end{cases} \qquad (*)$$

then

- $e \in K$ since $\varphi_e(e) = 0 \downarrow$
- there is $e' \neq e$ $\varphi_{e'} = \varphi_e$
- $e' \notin K$ since $\varphi_{e'}(e') = \varphi_e(e') \uparrow$
  $e' \neq e$

* We want $e \in \mathbb{N}$

$$\varphi_e(x) = \begin{cases} 0 & \text{if } x = e \\ \uparrow & \text{otherwise} \end{cases}$$

Kleene.py

```
def P(x) .
    if x = "________"
        then return 0
        else loop
```

read ("Kleene.py")

program we are defining

formally define

$$g(x, y) = \begin{cases} 0 & \text{if } y = x \\ \uparrow & \text{otherwise} \end{cases} = \mu z . |y - x|$$

computable

by smn theorem there is $s: \mathbb{N} \to \mathbb{N}$ total computable s.t.

$\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} 0 & \text{if } y = x \\ \uparrow & \text{otherwise} \end{cases}$$

By 2nd recursion theorem there is $e \in \mathbb{N}$ s.t. $\varphi_e = \varphi_{s(e)}$

hence

$$\varphi_e(y) = \varphi_{s(e)}(y) = g(e,y) = \begin{cases} 0 & \text{if } y = e \\ \uparrow & \text{otherwise} \end{cases}$$

hence (*) is true !

$\Rightarrow$ K is not SATURATED

$\square$

EXERCISE : RANDOM NUMBERS (from early ages)

→ $n \in \mathbb{N}$ is random if all programs generating n in output are "larger" than n

two points :

→ there are infinitely many random numbers

→ the property of being random is undecidable

Try again

→ size of a program : $|\varphi_e| = e$

→ define $n \in \mathbb{N}$ random if

for all $e \in \mathbb{N}$ s.t. $\varphi_e(0) = n$ then $e > n$

EXERCISE:

Let $f: \mathbb{N} \to \mathbb{N}$ be a function

and consider $B_f = \{ e \in \mathbb{N} \mid \varphi_e = f \}$

Are B_f, $\overline{B_f}$ recursive / r.e. ?

(1) f not computable

$B_f = \emptyset$ $\qquad \overline{B_f} = \mathbb{N}$ $\qquad$ recursive

(r.e.)

(2) f computable

B_f saturated

$B_f \neq \emptyset$ (since f computable there is $e \in \mathbb{N}$ s.t. $f = \varphi_e \Rightarrow e \in B_f$)

$B_f \neq \mathbb{N}$ (if $g \neq f$ g computable e' s.t. $\varphi_{e'} = g$ then $e' \notin B_f$)

$\hookrightarrow$ by Rice's theorem B_f, $\overline{B_f}$ not recursive

What about r.e. ?

$f = \emptyset$ $\quad$ ($\emptyset(x)\uparrow \quad \forall x$)

$$\overline{B_f} = \{ e \mid \varphi_e \neq \emptyset \}$$

$$= \{ e \mid \underbrace{\exists y. \underbrace{\varphi_e(y)\downarrow}_{\text{semi dec.}}}_{\text{semi dec.}} \}$$

$$sc_{\overline{B_f}}(x) = \mathbb{1}(\mu \omega. \ H(x, (\omega)_1, (\omega)_2)) \text{ computable}$$

complete the exercise!