PLATEAU PROBLEM : existence and qualitative properties of the surface (in $\mathbb{R}^3$) with minimal area among all surfaces sharing the same boundary. (given by a Jordan curve)

(posed by Lagrange in 1760)

Joseph Plateau (physician from Belgium ~ 19th century

wires in saponated water (experimental)

↓ observation and description of the basic laws:

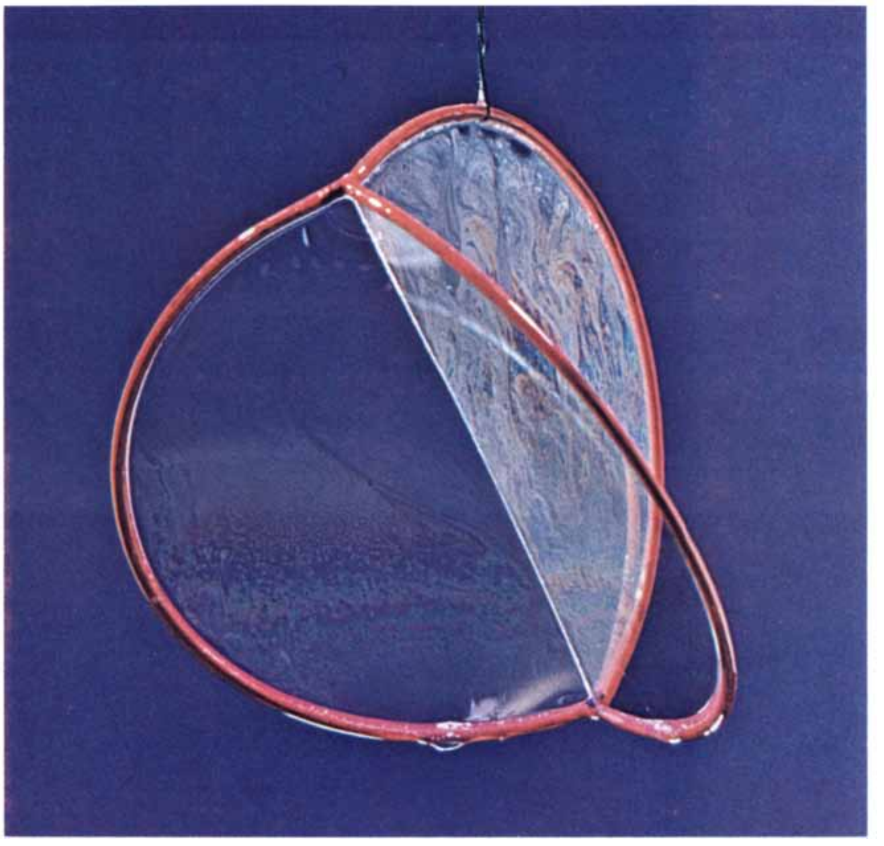
- soap films are unbroken surfaces
- films meet in only 2 ways:
  - → 3 surfaces along a smooth curve (with angles 120°)
  - → 6 surfaces with 4 curves meets at a vertex

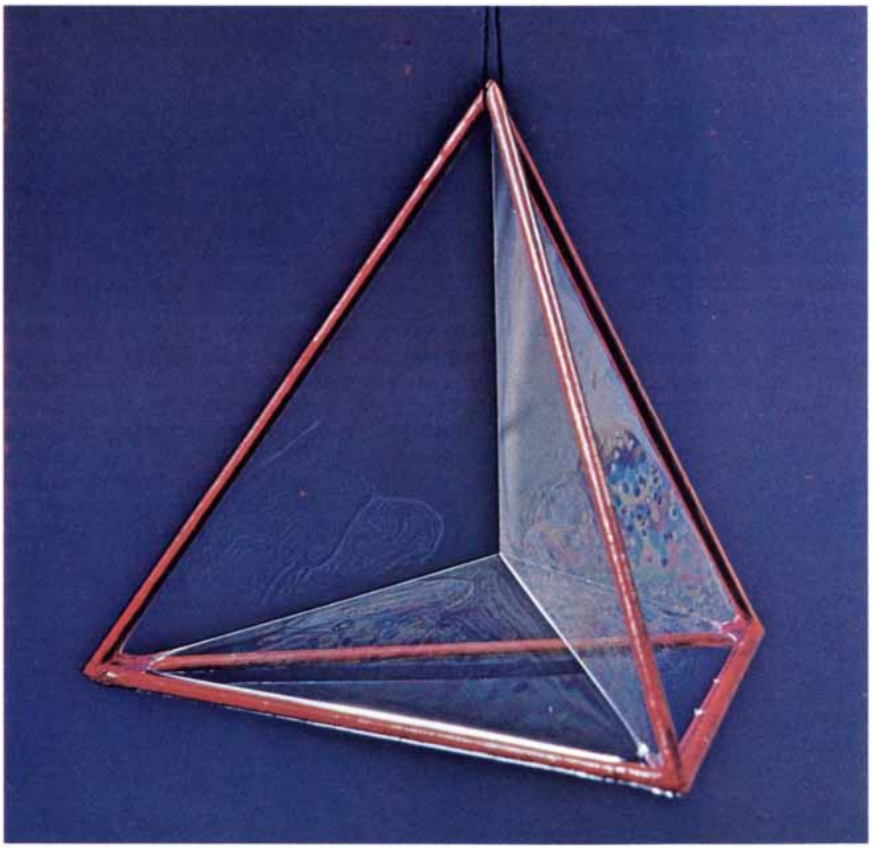
The basic physical principle governing the geometry of soap bubbles and soap films is simply stated: A physical system will remain in a certain configuration only if it cannot readily change to a configuration with less energy. In any liquid at a constant temperature and at rest, whether it is a bulk liquid in a container or a thin layer of liquid in a soap bubble, the relevant components of the energy are the gravitational potential energy, the compressional energy of any volumes of trapped air and the surface energy (often expressed in terms of the surface tension). The surface energy arises as a result of the attractive forces between molecules, which are unbalanced at the surface of the liquid [*see top illustration on page 85*]. The existence of these unbalanced forces means that in the absence of gravity and differences in air pressure the surface of a liquid will act as an elastic membrane, tending to minimize its surface area and thereby its surface energy. These unbalanced forces are much stronger between polar molecules, such as those of water, than between nonpolar molecules, such as those of gasoline. (A polar molecule is one with an asymmetrical distribution of electric charge.)

A soap or detergent molecule is somewhat peculiar, consisting typically of a long, slender nonpolar hydrocarbon chain with a highly polar oxygen-rich group attached at one end. When such molecules are added to water, they tend to migrate to the surface and orient themselves so that their nonpolar ends are sticking out [*see middle illustration on page 85*]. The surface of the water is thus partly or entirely covered with a nonpolar layer that reduces the surface tension drastically. The addition of the soap to the water has two important effects on the formation of a film. First, the surface of the liquid acquires additional stabilizing elastic properties: the stretching of the layer of soap molecules on the surface results not only in an increase of the total surface area but also in an increase in the surface energy per unit surface area (in other words, the surface tension), owing to the decrease in the number of soap molecules per unit area. The effect is attributable to the fact that the new surface created by stretching must be made up of the water molecules that were just below the surface until other soap molecules can diffuse to the surface; it may not take long for this process to happen, but the time involved is adequate to provide a critically important restoring force for small perturbations.

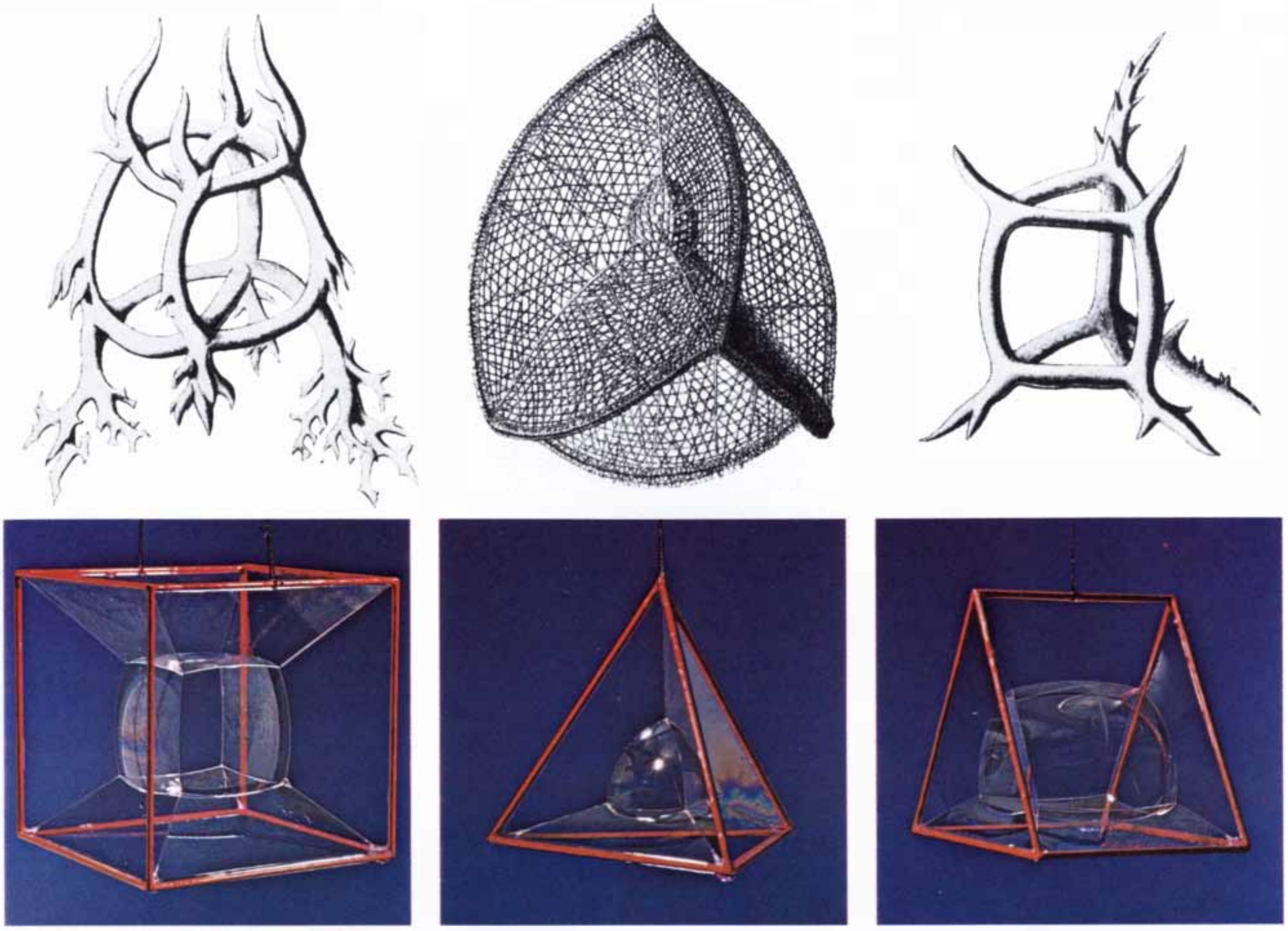
The second effect of soap on the formation of a film is that it seems to limit the minimum thickness of soap films to the length of two soap molecules stacked end to end, one for each side of the film [*see bottom illustration on page 85*]. Moderately thick films can be shown mathematically to be self-healing with respect to small punctures, but this healing ability diminishes greatly as the films become thinner. Hence without something to limit the thinness of the films, they probably could not form.

The actual formation of a soap film as a simple wire frame is lifted from a bowl





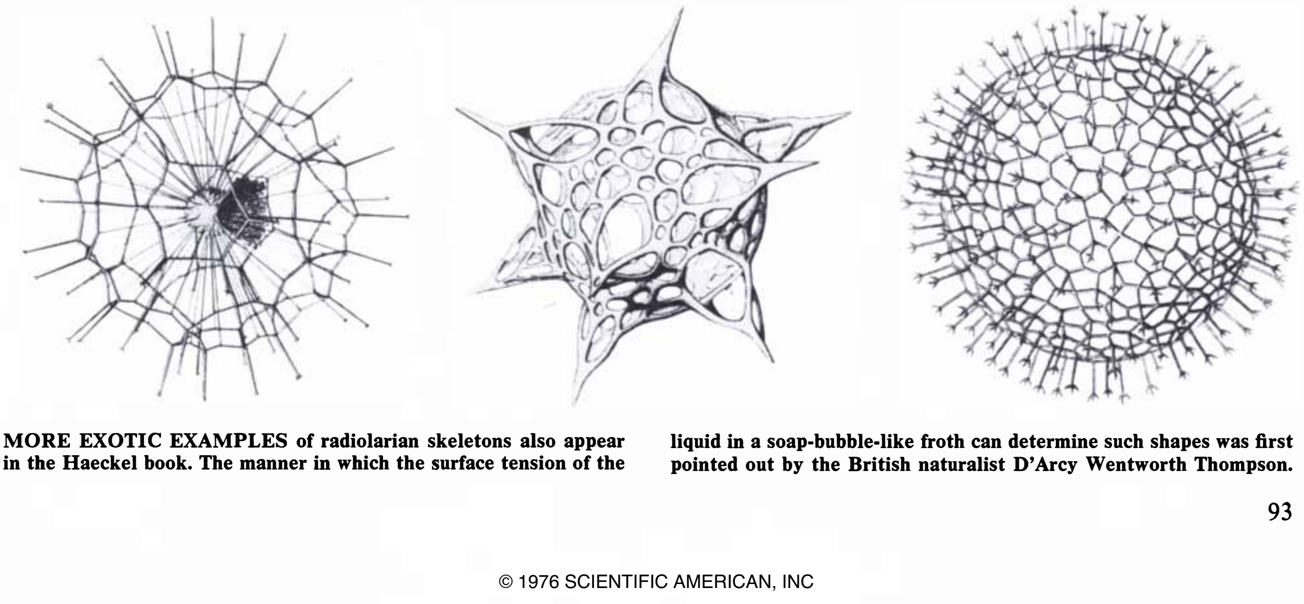
THE TWO WAYS soap bubbles or soap films can meet are demonstrated here with the aid of flat soap films formed on wire frames. The photograph at top shows three flat surfaces meeting along a straight line segment at angles of 120 degrees. The photograph at bottom shows six flat surfaces meeting three at a time along four straight line segments that in turn meet at a vertex at angles of about 109 degrees. It has long been observed and has now been proved mathematically that branchings of these two types (and their curved versions) are the only possible ones.



SOME EXAMPLES DRAWN FROM NATURE provide a striking visualization of the mathematical principles outlined in this article. The drawings at the top, reproduced from Ernst Haeckel's *Report on the Scientific Results of the Voyage of the HMS Challenger during the Years 1873–1876,* portray the microscopic skeletons left after the death and decay of several types of radiolarian, a tiny marine organism that in life consists of a small mass of protoplasm surrounded by a soap-bubble-like froth of cells. As the color photographs under the drawings show, the branchings of the radiolarian skeletons, where the silica-bearing fluid in the interfaces of the froth tends to accumulate, bear a strong resemblance to the branchings of soap bubbles blown into centers of soap-film configurations formed on wire frames.

dies, everything decays except the skeleton. These skeletons provide a striking visualization of the branchings of the froth: the pieces of curves and the vertex points. Some of them also show the shapes of the cell walls.

In summary, we have undertaken to demonstrate how a few observations concerning the way in which soap films and soap bubbles are free to change to decrease their energy form the basis of a mathematical model of soap-film-like and soap-bubble-like configurations of surfaces. By mathematical analysis alone one can then show the existence of such soap-film-like and soap-bubble-like configurations corresponding to various frames and enclosed volumes and prove that they must precisely obey the three principles, first formulated by Plateau, that govern the observed geometry of real soap films and soap bubbles. Indeed, the area-minimizing principle alone is sufficient to account for the overall geometry of soap films and soap bubbles.



MORE EXOTIC EXAMPLES of radiolarian skeletons also appear in the Haeckel book. The manner in which the surface tension of the liquid in a soap-bubble-like froth can determine such shapes was first pointed out by the British naturalist D'Arcy Wentworth Thompson.

Douglas (1930) solved the Plateau problem

– existence of minimal surfaces in dim $\mathbb{R}^3$
with given bdry (a Jordan curve) –

with a very involved proof (he won Field medal)

based on harmonic analysis, Riemann mapping
theorem, necessary conditions for a surface to be
of minimal area among possible perturbations ...

ALMGREN – TAYLOR ('76)

the rules formulated by Plateau (and other conditions not contained there but that have to be satisfied by soap film) can be explained and described by using a mathematical model based on an

area minimization principle

↓

soap films are stable configurations (with minimal energy)

energy:
- potential energy (negligible at first glance)
- air pressure inside the volume
- SURFACE TENSION → proportional to SURFACE AREA

(J. Taylor 1976, Annals of Math, complete classification of minimal singular cones in dim $\mathbb{R}^3$).

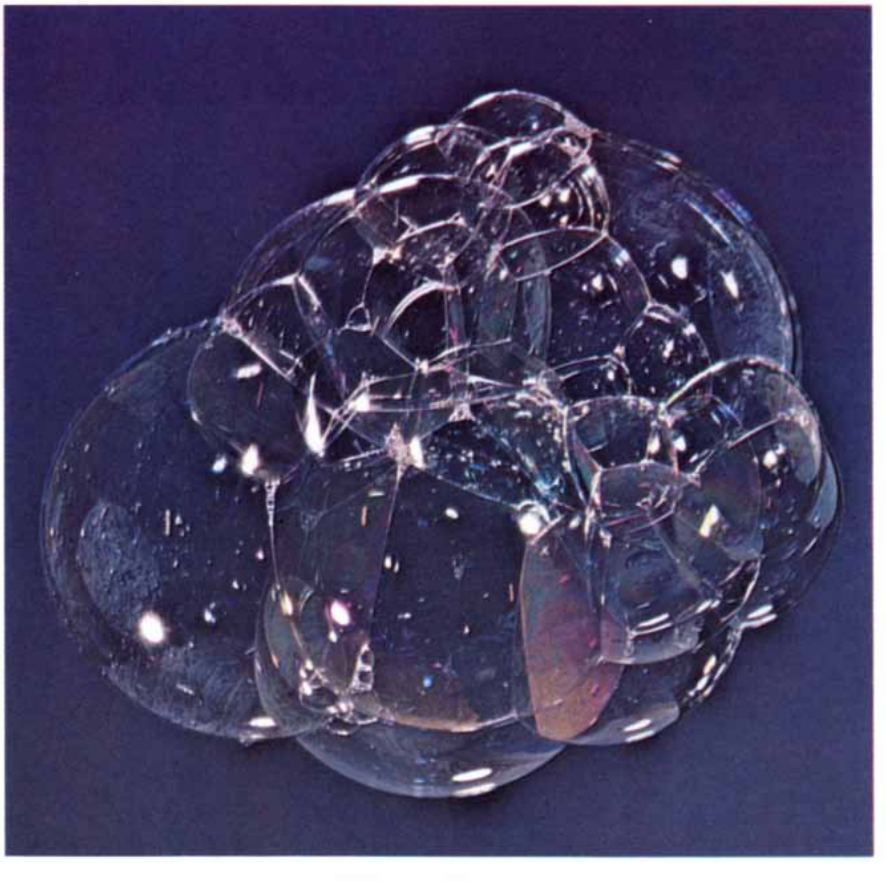
The Geometry of Soap Films and Soap Bubbles

The possible configurations they can form are governed by a few elementary rules that have been known for more than a century. A new mathematical model provides a sound basis for those rules

by Frederick J. Almgren, Jr., and Jean E. Taylor

Soap bubbles and soap films evoke a special fascination. Their shifting iridescence, their response to a puff of air, their fragility—all contribute to their charm. More captivating still is the exquisite perfection of their geometry, the absolute smoothness of their forms. What are the principles that enable soap bubbles and soap films to exist in certain geometric configurations and not in others? What are the possible shapes they can assume?

Three tentative principles are likely to suggest themselves to anyone who looks carefully at a number of different bubbles and films. First, a compound soap bubble or a soap film spanning a wire frame consists of flat or smoothly curved surfaces smoothly joined together. Second, the surfaces meet in only two ways: Either exactly three surfaces meet along a smooth curve or six surfaces (together with four curves) meet at a vertex. Third, when surfaces meet along curves or when curves and surfaces meet at points, they do so at equal angles. In particular, when three surfaces meet along a curve, they do so at angles of 120 degrees with respect to one another, and when four curves meet at a point, they do so at angles of close to 109 degrees.



FLOATING CLUSTER of soap bubbles is suggestive of the infinite variety of configurations that can be formed by compound soap bubbles or soap films spanning a wire frame. In spite of the apparent complexity of such shapes the separate pieces of flat or smoothly curved surfaces can come together in only two ways. The color photographs accompanying this article were made by Fritz Goro using a homemade bubble solution consisting of roughly equal amounts of water and commercial dishwashing detergent, with a small admixture of glycerin to stabilize the films.

The first person to have studied the geometry of soap bubbles and soap films systematically and to have recorded these simple general rules governing their possible forms appears to have been the Belgian physicist Joseph A. F. Plateau, who conducted his research more than a century ago. In his honor an entire range of mathematical questions that deal with the geometry of soap-bubble-like and soap-film-like surfaces is referred to as Plateau's problem.

Recently we have been able to show that the three basic rules governing the geometry of soap bubbles and soap films are a mathematical consequence of a simple area-minimizing principle. Other principles of this general type have been studied in connection with soap bubbles and soap films for at least two centuries by many well-known mathematicians but never before with the present simplicity of formulation or with the present success in revealing the true geometric underpinning of Plateau's rules. Here we shall describe both the area-minimizing principle itself and explain some of the mathematics necessary for its use.

The three basic rules cited above do not completely characterize the geometry of soap bubbles and soap films. Indeed, there are a number of subtler conditions that such configurations must satisfy; for example, it has long been known that the mean curvature of each separate piece of surface must be constant everywhere on that piece. This condition and as far as we know all such conditions are also satisfied by our mathematical model. (In addition Plateau's laws do not explain how a soap film attaches to a wire frame.)

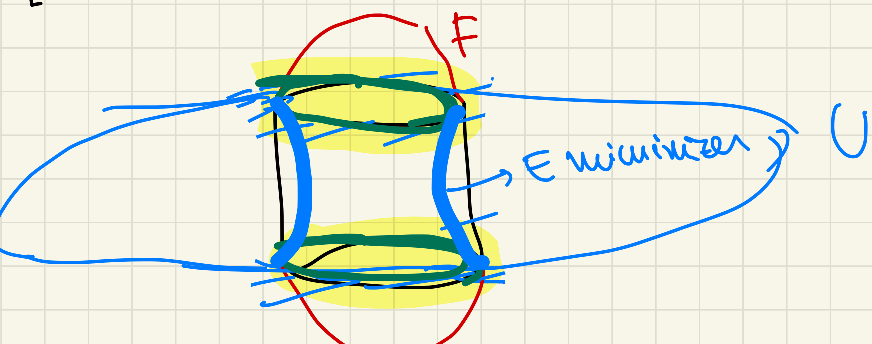
How to state the Plateau problem in a mathematical framework?

Fix the boundary → fix $F \subseteq \mathbb{R}^n$ and U open bdd of class C^1

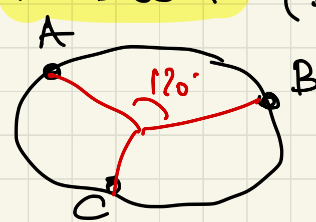
$F \cap \partial U$ is the given boundary

Let $U \subseteq \mathbb{R}^n$ open bdd of class C^1, $F \subseteq \mathbb{R}^n$ set of finite perimeter in $\mathbb{R}^n$. Consider the following minimization problem:

$$\min \{ Per(E) \, , \; E \subseteq \mathbb{R}^n \text{ such that } E \setminus U = F \setminus U \}.$$



in dim 2 it is the STEINER PROBLEM (bdry = points).



Helly's theorem

$U \subseteq \mathbb{R}^n$ bdd open set of class C^1

$BV(U) \hookrightarrow L^p(U)$ compactly $\forall\ p \in [1, 1^*)$ $\qquad 1^* = \frac{n}{n-1}$

$BV(U) \hookrightarrow L^{1^*}(U)$ continuously.

Compactness theorem for sets of finite perimeter.

Let E_k be a sequence of measurable sets such that $Per(E_k, U) \leq C$ for some U bdd open of class C^1.

Then $\exists\ E$ a measurable set such that, up to passing to a subsequence,

$$\begin{cases} E_k \cap U \to E \cap U & \text{in measure} \\ Per(E, U) \leq \liminf_k Per(E_k, U). \end{cases}$$

proof $f_k = \chi_{E_k}$ $\qquad \|f_k\|_{BV(U)} = |E_k \cap U| + Per(E_k, U) \leq C$

$\to$ by Helly's thm $\exists\ f$ such that $f_{k_i} \to f$ in $L^p(U)$. $p \in [1, 1^*)$

$f_{k_i} \to f$ also a.e $\Rightarrow f(x) \in \{0,1\}$ a.e, Extend $f \equiv 0$ in $\mathbb{R}^n \setminus U \Rightarrow f = \chi_E$

$f_{k_i} \to f$ in $L^1(U) \Rightarrow E_{k_i} \cap U \to E \cap U$ in measure

$\liminf V(f_k, U) = \liminf Per(E_k, U) \geq Per(E, U) = V(f, U).$

proof (EXISTENCE of MINIMIZERS)

$c = \inf \{ Per(E), \; E \subseteq \mathbb{R}^n, \; E \setminus U = F \setminus U \}$

Let E_k be a minimizing sequence $\rightarrow$ $Per(E_k) \leq c + \frac{1}{k}$

$M_k = F \Delta E_k = (E_k \setminus F) \cup (F \setminus E_k) \subseteq U$

$|M_k| \leq |U|$ and

$Per(M_k) \leq Per(E_k \setminus F) + Per(F \setminus E_k) \leq Per(E_k) + Per(F) +$

$+ Per(F) + Per(E_k) \leq \underset{+2}{2c} + 2 Per(F)$ $(Per(E) = Per(\mathbb{R}^n \setminus E)!$

$\Rightarrow \chi_{M_k} \in BV(\mathbb{R}^n)$ and $\|\chi_{M_k}\|_{BV} \leq C \Rightarrow \|\chi_{M_k}\|_{BV(U)} \leq C$

$\chi_{M_k} \rightarrow \chi_M$ in measure $\Rightarrow$ $E_k = (M_k \cup F) \setminus (M_k \cap F)$

$M \subseteq U$ $\rightarrow (M \cup F) \setminus (M \cap F) = E$

Note that $E \setminus U = F \setminus U$ $\quad Per(E) \leq \lim_k Per(E_k) = c$

$\Rightarrow E$ is a minimizer.