18 dic. 10.30 − 12	8 gennaio 10.30 − 12
19 dic 9 − 12	9 gennaio 9 − 12
20 dic 10.30 − 12	10 gennaio 10.30 − 12

$\boxed{\text{FINE LEZIONI 10 gennaio}}$

$$\int_a^{+\infty} f(x)\,dx = \lim_{M\to+\infty} \int_a^M f(x)\,dx$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{M\to+\infty} \int_{-M}^{b} f(x)\,dx$$

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} +\infty & \text{se } \alpha \leq 1 \\ \dfrac{1}{\alpha-1} & \text{se } \alpha > 1 \end{cases}$$

Es determinare per quali α esiste finito

$$\int_2^{+\infty} \frac{1}{x(\lg x)^\alpha}\,dx = \lim_{M\to+\infty} \int_2^M \frac{1}{x(\lg x)^\alpha}\,dx$$

$$\int_2^M \frac{1}{x(\lg x)^\alpha}\,dx$$

$y = \lg x$ $\qquad$ $x = e^y$

$dx = e^y\,dy$

$x = 2 \longrightarrow y = \lg 2$

$x = M \longrightarrow y = \lg M$

$$\int_{\lg 2}^{\lg M} \frac{1}{\cancel{e^y}\, y^\alpha}\,\cancel{e^y}\,dy = \int_{\lg 2}^{\lg M} \frac{1}{y^\alpha}\,dy$$

$$\lim_{M\to+\infty} \int_{\lg 2}^{\lg M} \frac{1}{y^{\alpha}}\,dy < +\infty \iff \alpha > 1$$

$$\int \frac{1}{y^{\alpha}}\,dy = \int y^{-\alpha}\,dy =$$

$$= \frac{1}{1-\alpha} y^{1-\alpha} + c \quad \boxed{\text{se } \alpha \neq 1}$$

$$= \lg|y| + c \quad \alpha = 1$$

① $\alpha \neq 1$

$$\int_{\lg 2}^{\lg M} \frac{1}{y^{\alpha}}\,dy = \underbrace{\frac{1}{1-\alpha}(\lg M)^{1-\alpha} - \frac{1}{1-\alpha}(\lg 2)^{1-\alpha}}$$

$$\longrightarrow \frac{1}{1-\alpha}(+\infty)^{1-\alpha} - \frac{1}{1-\alpha}(\lg 2)^{1-\alpha}$$

$M \to +\infty$

$\boxed{1-\alpha > 0}$ $+\infty$

$(1-\alpha) < 0$ $\frac{1}{\alpha-1}(\lg 2)^{1-\alpha}$

② $\alpha = 1$

$$\int_{\lg 2}^{\lg M} \frac{1}{y}\,dy = \lg(\lg M) - \lg(\lg 2) \longrightarrow +\infty$$

$\lg(+\infty) = +\infty$

Quindi

$$\int_2^{+\infty} \frac{1}{x(\lg x)^\alpha}\,dx < +\infty \iff \alpha > 1$$

$$\parallel$$

$$\lim_{n\to+\infty} \int_2^{M} \frac{1}{x(\lg x)^\alpha}\,dx$$

$$\parallel$$

$$\lim_{n\to+\infty} \int_{\lg 2}^{\lg M} \frac{1}{y^\alpha}\,dy = \int_{\lg 2}^{+\infty} \frac{1}{y^\alpha}\,dy$$

$$\sum_{n=2}^{+\infty} \frac{1}{n(\lg n)^\alpha} < +\infty \iff \alpha > 1$$

Es

Determinare (se esiste finito) il valore

$$\int_3^{+\infty} \frac{1}{e^x+3}\,dx = \lim_{M\to+\infty} \boxed{\int_3^{M} \frac{1}{e^x+3}\,dx}$$

$$\int_3^{M} \frac{1}{e^x+3}\,dx = \boxed{\int_{e^3}^{e^M} \frac{1}{y+3}\cdot\frac{1}{y}\,dy} = \int_{e^3}^{e^M} \frac{1}{(y+3)y}\,dy$$

$e^x+3 \to y+3$

$y = e^x$

$x = \lg y \to dx = \frac{1}{y}\,dy$

$x=3 \to y=e^3$ $\quad x=M \to y=e^M$

$$\int \frac{1}{(y+3)\cdot y}\,dy =$$

$\downarrow$

FRATTI SEMPLICI

$$\frac{0\cdot y + 1}{(y+3)\cdot y} = \frac{A}{y+3} + \frac{B}{y} = \frac{Ay + By + 3B}{(y+3)\,y}$$

$$\begin{cases} A+B=0 \\ 3B=1 \end{cases} \qquad \begin{cases} A=-\frac{1}{3} \\ B=\frac{1}{3} \end{cases}$$

$$\frac{1}{(y+3)\,y} = -\frac{1}{3}\,\frac{1}{y+3} + \frac{1}{3}\,\frac{1}{y}$$

$$\int \frac{1}{(y+3)\,y}\,dy = -\frac{1}{3}\int \frac{1}{y+3}\,dy + \frac{1}{3}\int \frac{1}{y}\,dy =$$

$$= -\frac{1}{3}\lg|y+3| + \frac{1}{3}\lg|y| + c$$

$$= \frac{1}{3}\left[-\lg|y+3| + \lg|y|\right] + c =$$

$$= \frac{1}{3}\lg\left(\frac{|y|}{|y+3|}\right) + c$$

$$\int_{e^3}^{e^M} \frac{1}{(y+3)y}\,dy = \frac{1}{3}\lg\left(\frac{e^M}{e^M+3}\right) - \frac{1}{3}\lg\left(\frac{e^3}{e^3+3}\right)$$

$$\int_3^{+\infty} \frac{1}{e^x+3}\,dx = \lim_{M\to+\infty} \left[\cancel{\frac{1}{3}\log\left(\frac{e^M}{e^M+3}\right)} - \frac{1}{3}\log\left(\frac{e^3}{e^3+3}\right)\right]$$

$$\log\left(\frac{e^M}{e^M+3}\right) = \log\left(\frac{\cancel{e^M}}{\cancel{e^M}\left(1+\frac{3}{e^M}\right)}\right) = \log\left(\frac{1}{1+\frac{3}{e^M}}\right)$$

$$\downarrow \; M\to+\infty$$

$$\log\left(\frac{1}{1+\frac{3}{+\infty}}\right) = \log(1) = 0$$

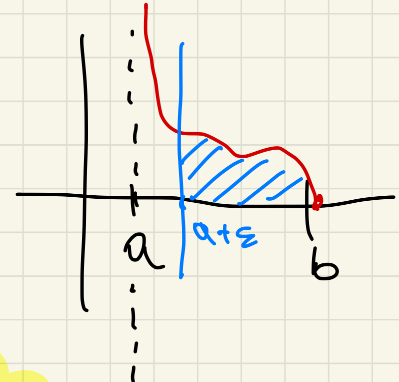
$$= -\frac{1}{3}\log\left(\frac{e^3}{e^3+3}\right) =$$

$$= \frac{1}{3}\log\left(\frac{e^3+3}{e^3}\right) = \log\left(\sqrt[3]{\frac{e^3+3}{e^3}}\right)$$

INTEGRALI GENERALIZZATI IN INTERVALLI LIMITATI (per funzioni con una singolarità)

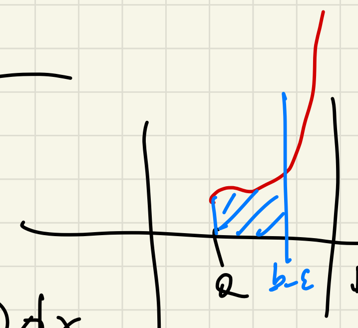
$f : (a, b] \to \mathbb{R}$ CONTINUA

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} f(x)\,dx$$



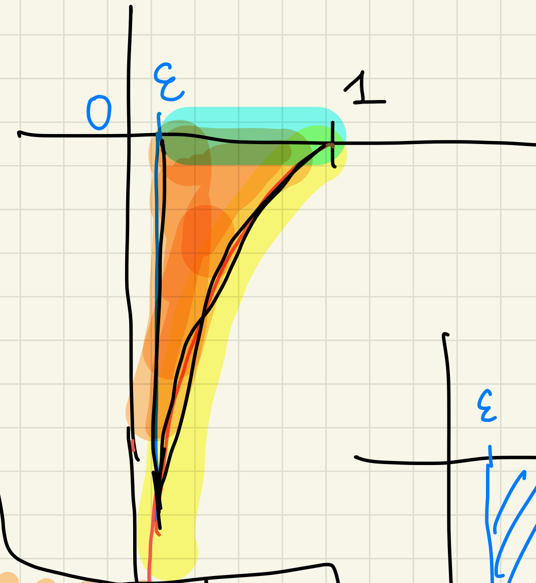
$f : [a, b) \to \mathbb{R}$ continua

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_{a}^{b-\varepsilon} f(x)\,dx$$



Es $\lg x \quad x \in (0,1]$

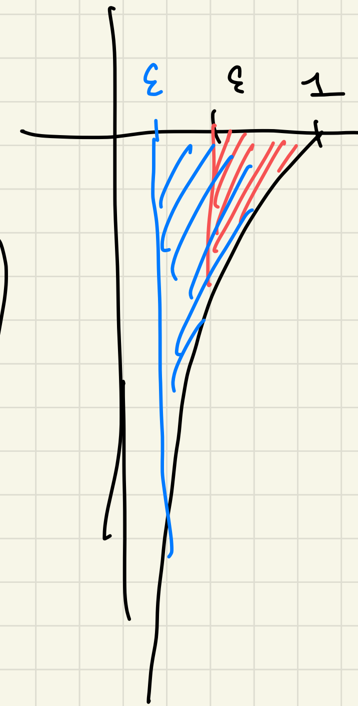
$$\lim_{x \to 0^+} \lg x = -\infty$$



$$\int_0^1 \lg x \, dx = \lim_{\varepsilon \to 0^+} \boxed{\int_\varepsilon^1 \lg x \, dx}$$

$\varepsilon > 0$

$$\int_\varepsilon^1 \lg x \, dx = ?$$



$$\int_{\varepsilon}^{1} \lg x \, dx \Big|_{\varepsilon}$$

$$\int \lg x \, dx = \underbrace{x \cdot \lg x - x}_{} + c$$

per parti

$f(x) = 1 \longrightarrow F(x) = x$

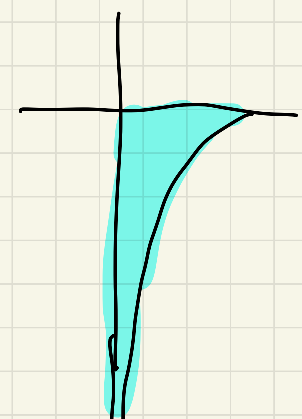
$g(x) = \lg x \longrightarrow g'(x) = \frac{1}{x}$

$$\int \lg x \, dx = x \cdot \lg x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$$

$$\int_{\varepsilon}^{1} \lg x \, dx = \overbrace{1 \cdot \lg 1}^{=0} - 1 - (\varepsilon \lg \varepsilon - \varepsilon) =$$

$$0 - 1 - \varepsilon \lg \varepsilon + \varepsilon$$

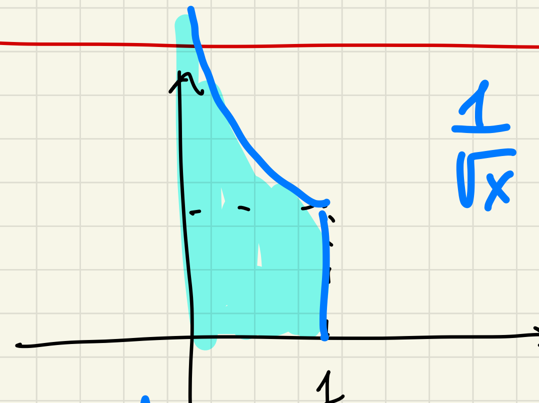
$$\int_{0}^{1} \lg x \, dx = \lim_{\varepsilon \to 0^+} -1 - \underbrace{\varepsilon \lg \varepsilon}_{\to 0} + \cancel{\varepsilon}^{\nearrow 0} = -1$$



$$\boxed{\lim_{\varepsilon \to 0^+} \varepsilon \lg \varepsilon = 0}$$

$$\lim_{x \to 0^+} x \lg x = 0$$

Es $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\, dx$



$f(x) = \dfrac{1}{\sqrt{x}}$ è

continua in $(0, 1]$

$$\lim_{x \to 0^+} \frac{1}{\sqrt{x}} = +\infty$$

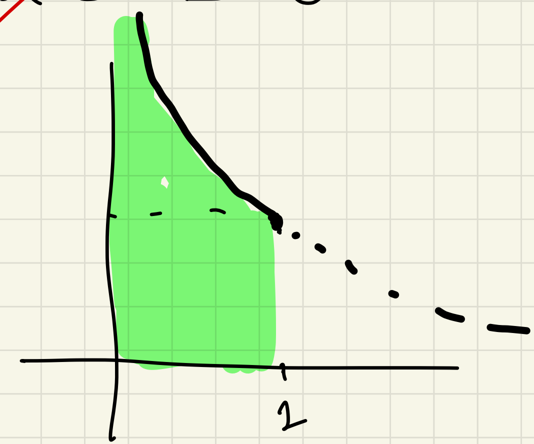
$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{\varepsilon \to 0^+} \boxed{\int_\varepsilon^1 \frac{1}{\sqrt{x}}\, dx}$$

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,dx = \frac{1}{-\frac{1}{2}+1}\, x^{-\frac{1}{2}+1} + c = 2x^{\frac{1}{2}} + c$$

$$= 2\sqrt{x} + c$$

$$\int_{\varepsilon}^{1} \frac{1}{\sqrt{x}}\,dx = 2\cdot\sqrt{1} - 2\sqrt{\varepsilon}$$

$$\int_{0}^{1} \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon \to 0^+} 2\cdot\sqrt{1} - \cancel{2\sqrt{\varepsilon}}^{\,=\,0} = 2$$



ES

$$\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{x}\,dx =$$

$$= \lim_{\varepsilon \to 0^+} \left(\cancel{\lg 1} - \lg \varepsilon \right) =$$

$$= \lim_{\varepsilon \to 0^+} -\lg \varepsilon = -(-\infty) = +\infty .$$

Per quali α $\displaystyle\int_0^1 \frac{1}{x^\alpha}\,dx < +\infty$?

$\boxed{\alpha = 1 \quad \text{NO}}$ appena visto $\alpha=1$ $\displaystyle\int_0^1 \frac{1}{x}\,dx$

$\alpha = \frac{1}{2}$ sì appena visto. $\alpha=\frac{1}{2}$ $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$\alpha \neq 1$

$$\int_0^1 \frac{1}{x^\alpha}\,dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{x^\alpha}\,dx =$$

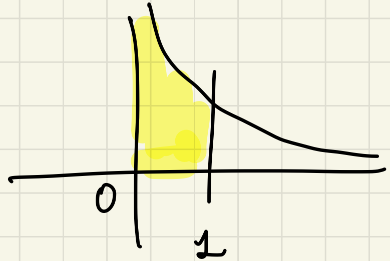
$$= \lim_{\varepsilon\to 0^+} \left[\frac{1}{-\alpha+1} 1^{-\alpha+1} - \frac{1}{-\alpha+1} \varepsilon^{-\alpha+1} \right] < +\infty$$

$$\Updownarrow$$

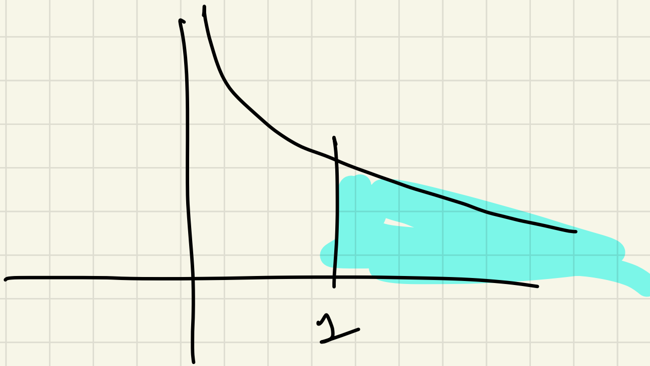
$$-\alpha+1>0$$

$\alpha<1$

$$\int_0^1 \frac{1}{x^\alpha}\,dx < +\infty \iff \alpha < 1$$



$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx < +\infty \iff \alpha > 1$$



ES Calcolare se esiste finito

$$\int_3^{+\infty} \frac{1}{3x^2-2x-1}\,dx = \lim_{M\to+\infty} \int_3^M \frac{1}{3x^2-2x-1}\,dx$$

es 1) calcolare $\int_3^5 \frac{1}{3x^2-2x-1}\,dx$

2) dire se esiste finito e calcolare $\int_3^{+\infty} \frac{1}{3x^2-2x+1}\,dx$

$$\int_3^M \frac{1}{3x^2-2x-1}\,dx$$

$$\int \frac{1}{3x^2-2x-1}\,dx$$

$$3x^2-2x-1=0$$

$$x_{1,2}=\frac{2\pm\sqrt{4+12}}{6}=\frac{2\pm 4}{6} \begin{cases} 1 = x_1 \\ -\frac{1}{3} = x_2 \end{cases}$$

$$3x^2-2x-1=3\cdot(x-1)\left(x-\left(-\frac{1}{3}\right)\right)=$$

$$=3(x-1)\left(x+\frac{1}{3}\right)$$

$$ax^2+bx+c=a(x-x_1)(x-x_2)$$

$$\frac{1}{3x^2-2x-1} = \frac{0\cdot x+1}{3(x-1)(x+\frac{1}{3})} = \frac{1}{3}\left[\frac{A}{x-1}+\frac{B}{x+\frac{1}{3}}\right]$$

$$= \frac{1}{3}\,\frac{Ax+\frac{1}{3}A+Bx-B}{(x-1)(x+\frac{1}{3})} = \frac{Ax+Bx+\frac{1}{3}A-B}{3(x-1)(x+\frac{1}{3})}$$

$$\begin{cases} A+B=0 \\ \frac{1}{3}A-B=1 \end{cases} \quad \begin{cases} A=-B \\ \frac{1}{3}(-B)-B=1 \end{cases} \quad \begin{cases} A=-B \\ -\frac{1}{3}B-1=1 \end{cases} \quad \begin{cases} A=-B \\ -\frac{4}{3}B=1 \end{cases}$$

$$\begin{cases} A=+\frac{3}{4} \\ B=-\frac{3}{4} \end{cases}$$

$$\frac{1}{3x^2-2x-1} = \frac{1}{3}\left[\frac{3/4}{(x-1)} + \frac{(-3/4)}{(x+\frac{1}{3})}\right] =$$

$$= \frac{1}{3}\cdot\left[\frac{3}{4}\cdot\frac{1}{(x-1)} - \frac{3}{4}\,\frac{1}{(x+\frac{1}{3})}\right] =$$

$$= \frac{1}{4}\cdot\frac{1}{(x-1)} - \frac{1}{4}\,\frac{1}{(x+\frac{1}{3})}$$

$$\int \frac{1}{3x^2-2x-1}\,dx = \frac{1}{4}\int\frac{1}{x-1}\,dx - \frac{1}{4}\int\frac{1}{x+\frac{1}{3}}\,dx =$$

$$= \frac{1}{4}\lg|x-1| - \frac{1}{4}\lg\left|x+\frac{1}{3}\right| + c = \frac{1}{4}\lg\frac{|x-1|}{|x+\frac{1}{3}|} + c$$

$$\int_3^M \frac{1}{3x^2-2x-1}\,dx = \frac{1}{4}\lg\frac{|M-1|}{|M+\frac{1}{3}|} - \frac{1}{4}\lg\frac{|3-1|}{|3+\frac{1}{3}|}$$

$$= \frac{1}{4}\lg\left(\frac{M-1}{M+\frac{1}{3}}\right) - \frac{1}{4}\lg\left(\frac{2}{\frac{10}{3}}\right) =$$

$$= \frac{1}{4}\lg\left(\frac{M-1}{M+\frac{1}{3}}\right) - \frac{1}{4}\lg\left(\frac{3}{5}\right)$$

$$\lim_{M\to+\infty}\int_3^M \frac{1}{3x^2-2x-1}\,dx = \lim_{M\to+\infty}\frac{1}{4}\lg\left(\frac{M-1}{M+\frac{1}{3}}\right) - \frac{1}{4}\lg\left(\frac{3}{5}\right)$$

$$= \lim_{M\to+\infty}\frac{1}{4}\lg\left(\frac{\cancel{M}\left(1-\frac{1}{M}\right)}{\cancel{M}\left(1+\frac{1}{3M}\right)}\right) - \frac{1}{4}\lg\frac{3}{5} = \cancel{\frac{1}{4}\lg 1}^{\,0} - \frac{1}{4}\lg\frac{3}{5}$$

$$= \left(\tfrac{1}{4}\lg 5/3\right)$$

Es Calcolare se esiste finito

$$\int_0^{1/4} \frac{1}{\sqrt{x}\cdot(\sqrt{x}+1)}\,dx = \lim_{\varepsilon\to 0^+} \boxed{\int_\varepsilon^{\frac{1}{4}} \frac{1}{\sqrt{x}\cdot(\sqrt{x}+1)}\,dx}$$

$$\int_\varepsilon^{\frac{1}{4}} \frac{1}{\sqrt{x}\,(\sqrt{x}+1)}\,dx$$

$$y=\sqrt{x}$$

$$x = y^2 \qquad dx = 2y\,dy$$

$$x=\varepsilon \to y=\sqrt{\varepsilon}$$

$$x=\frac{1}{4} \to y = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\int_{\sqrt{\varepsilon}}^{\frac{1}{2}} \frac{1}{y\,(y+1)}\,2y\,dy$$

$$\int_{\sqrt{\varepsilon}}^{\frac{1}{2}} \frac{1}{\cancel{y}(y+1)} 2\cancel{y}\, dy = 2\int_{\sqrt{\varepsilon}}^{\frac{1}{2}} \frac{1}{y+1}\, dy$$

$$\int \frac{1}{y+1}\, dy = \lg|y+1| + c$$

$$= 2\left[\lg\left(\frac{1}{2}+1\right) - \lg\left(\sqrt{\varepsilon}+1\right)\right] =$$

$$= 2\lg\left(\frac{3}{2}\right) - 2\lg\left(\sqrt{\varepsilon}+1\right)$$

$$\int_0^{1/4} \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx = \lim_{\varepsilon\to 0^+} 2\lg\frac{3}{2} - 2\underbrace{\lg(\sqrt{\varepsilon}+1)}_{\downarrow\ \lg(0+1) = \lg 1} =$$

$$= 2\lg\frac{3}{2} - \cancel{2\underbrace{\lg 1}_{=0}} = 2\lg\left(\frac{3}{2}\right)$$

$$\int_{\frac{2}{\pi}}^{+\infty} \frac{1}{x^3} \cos\left(\frac{1}{x}\right) dx = \lim_{M \to +\infty} \int_{\frac{2}{\pi}}^{M} \frac{1}{x^3} \cos\left(\frac{1}{x}\right) dx$$

$$\int_{\frac{2}{\pi}}^{M} \frac{1}{x^3} \cos\left(\frac{1}{x}\right) dx$$

$$y = \frac{1}{x} \qquad \frac{1}{x^3} = y^3$$

$$x = \frac{1}{y} \qquad dx = \left(\frac{1}{y}\right)' dy$$

$$dx = -\frac{1}{y^2}\, dy$$

$$\int_{\frac{\pi}{2}}^{1/M} y^3 \cos(y) \left(-\frac{1}{y^2}\right) dy$$

$$x = \frac{2}{\pi} \to y = \frac{\pi}{2} \qquad x = M \to y = \frac{1}{M}$$

$$\int_{\frac{\pi}{2}}^{\frac{1}{M}} \cancel{y^3} \cdot \cos(y) \cdot \left(-\frac{1}{\cancel{y^2}}\right) dy = -\int_{\frac{\pi}{2}}^{\frac{1}{M}} y \cos(y) \cdot dy =$$

$$= \int_{\frac{1}{M}}^{\frac{\pi}{2}} y \cos(y)\, dy$$

$$\int y \cos(y) dy = \text{per parti} = \sin y \cdot y - \underbrace{\int \sin(y) \cdot 1\, dy}_{} =$$

$F(y)g(y)$ $\quad$ $F(y) \cdot g'(y)$

$f(y) = \cos(y) \to F(y) = \sin y$

$g(y) = y \to g'(y) = 1$

$$= \sin(y)\, y - (-\cos y) + c$$

$$= \sin(y)\, y + \cos(y) + C$$

$$\int_{\frac{1}{M}}^{\frac{\pi}{2}} y\cos y\,dy = \overbrace{\sin\left(\frac{\pi}{2}\right)}^{=1}\cdot\frac{\pi}{2} + \cancel{\overbrace{\cos\frac{\pi}{2}}^{=0}} +$$

$$-\left[\sin\left(\frac{1}{M}\right)\cdot\frac{1}{M} + \cos\left(\frac{1}{M}\right)\right]$$

$$= \frac{\pi}{2} - \sin\left(\frac{1}{M}\right)\cdot\frac{1}{M} - \cos\left(\frac{1}{M}\right)$$

$$\int_{\frac{2}{\pi}}^{+\infty} \frac{1}{x^3}\cos\frac{1}{x}\,dx = \lim_{M\to+\infty} \frac{\pi}{2} - \sin\Big(\underbrace{\frac{1}{M}}_{\to 0}\Big)\cdot\underbrace{\frac{1}{M}}_{\to 0} - \cos\Big(\underbrace{\frac{1}{M}}_{\to 0}\Big)$$

$$= \frac{\pi}{2} - \cancel{\sin 0\cdot 0} - \cos 0 = \frac{\pi}{2} - 1$$