Definition of PERIMETER of a set (INTRODUCED BY R. CACCIOPPOLI, '50, studied [DE GIORGI, FEDERER])

Let $E \subseteq \mathbb{R}^n$ a measurable set, $|E| < +\infty$. $\left(\int_{\mathbb{R}^n} \chi_E(x)\,dx < +\infty \right)$

Def (CACCIOPPOLI – DE GIORGI)

E is a set of FINITE PERIMETER if $\chi_E \in BV(\mathbb{R}^n)$

E is a set of FINITE PERIMETER in U if $\chi_E \in BV(U)$.

$Per(E) = V(\chi_E, \mathbb{R}^n)$ $\qquad$ $Per(E; U) = V(\chi_E, U)$

E of class C^1 bounded

$$V(\chi_E, \mathbb{R}^n) = \sup \left\{ \int_E \operatorname{div} \Phi \quad \Phi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n), \ \|\Phi\|_\infty \le 1 \right\}$$

$$\text{(divergence thm)} = \sup \Big\{ \underbrace{\int_{\partial E} \nu_E \cdot \Phi \, d\mathcal{H}^{n-1}}_{\le \mathcal{H}^{n-1}(\partial E)}, \ \|\Phi\|_\infty \le 1 \ \Phi \in C^1_c \Big\}$$

If E is of class C^k for $k > 1$, the signed distance d_S is C^k in a neighborhood of $\partial E \to \nu_E = \nabla d_S$ is $C^{k-1} \Rightarrow$ I can choose $\Phi = \nu_E$ .. $\Rightarrow$ $V(\chi_E, \mathbb{R}^n) = Per(E) = \mathcal{H}^{n-1}(\partial E)$

Also for E of class C^1, $\mathcal{H}^{n-1}(\partial E) = \mathrm{Per}(E)$

In general though $\mathrm{Per}(E) \underset{\neq}{\leq} \mathcal{H}^{n-1}(\partial E)$!

Ex take x_i enumeration of $\mathbb{Q}^n$.

$$E = \bigcup_{i=1}^{\infty} B\left(x_i, \frac{1}{2^i}\right) \qquad |E| \leq \sum_{1}^{\infty} \omega_n \left(\frac{1}{2^i}\right)^n = \omega_n \sum_i \frac{1}{2^{in}} < +\infty$$

$\boxed{\bar{E} = \mathbb{R}^n}$ by density $\qquad \mathcal{H}^{n-1}(\partial E) = +\infty$

$$E = \lim_k \underbrace{\bigcup_{i=1}^{k} B\left(x_i, \frac{1}{2^i}\right)}_{E_k} \quad \text{in } L^1 \text{ sense} \quad \chi_{E_k} \to \chi_E$$

$$\mathrm{Per}(E) = V(\chi_E, \mathbb{R}^n) \leq \liminf_k V(\chi_{E_k}, \mathbb{R}^n) = \mathrm{Per}(E_k) =$$

$$= \mathcal{H}^{n-1}(\partial E_k) \leq \sum_{i=1}^{k} \mathcal{H}^{n-1}\left(\partial B\left(x_i, \frac{1}{2^i}\right)\right) = n\,\omega_n \sum_1^k \frac{1}{2^{i(n-1)}} < +\infty$$

E has finite perimeter. $\qquad \mathrm{Per}(E) < \mathcal{H}^{n-1}(\partial E)$

Obs Let E be a set with $\chi_E \in BV(\mathbb{R}^n)$ set of finite perimeter

$D\chi_E$ is a finite (vector valued) Radon measure

$D\chi_E << |D\chi_E|$ let $\tilde{\nu}_E$ be the density, supp $|D\chi_E| \subseteq \partial E$

$$D\chi_E = \tilde{\nu}_E \, |D\chi_E|$$

$\tilde{\nu}_E$ is the outer measure theoretic normal

if E of class C^1 $\tilde{\nu}_E = \nu_E$ (outer normal) $|D\chi_E| = \mathcal{H}^{n-1}_{|\partial E}$

$\partial^* E$ = REDUCED BDRY (De Giorgi) $= \{ x \in \partial E,\ \exists$ limit $\lim_{r \to 0} \frac{D\chi_E(B(x,r))}{|D\chi_E(B(x,r))|} = \tilde{\nu}_E(x)$ and $|\tilde{\nu}_E(x)| = 1 \}$.

De Giorgi: $\partial^* E$ is a $(n-1)$ rectifiable set

$$Per(E) = \mathcal{H}^{n-1}(\partial^* E)$$

Integration by part formula on $\partial^* E$

$$\int_E \operatorname{div} \Phi \, dx = \int_{\partial^* E} \Phi \cdot \tilde{\nu}_E(x) \, d\mathcal{H}^{n-1}(x) \qquad \forall \Phi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$$

Properties of perimeter

$U \subseteq \mathbb{R}^n$ open set

1) $Per(E, U) = Per(\mathbb{R}^n \setminus E, U)$ (quite obvious)

2) $Per(E \cap F, U) + Per(E \cup F, U) \leq Per(E, U) + Per(F, U)$ subadd. (sheet 6)

3) If $E_n \to E$ in measure in U ($\iff \chi_{E_n} \to \chi_E$ in $U \iff$
$\iff |E_n \Delta E \cap U| = |(E_n \setminus E) \cup (E \setminus E_n) \cap U| \to 0$)
$\liminf_n Per(E_n, U) \geq Per(E, U)$ (proposition for BV functi)

4) COAREA FORMULA (Fleming - Rishel) (1960)

Let U open set, $f \in BV(U)$

for a.e $t \in \mathbb{R}$ $\{x \mid f(x) > t\} = A(t)$ has finite perimeter and

$$V(f, U) = \int_{-\infty}^{+\infty} Per(\{x \mid f(x) > t\}, U)\, dt$$

$(\forall\, U' \subseteq U$ open $V(f, U') = \int_{-\infty}^{+\infty} Per(\{x \mid f(x) > t\}, U')\, dt$

(ex: take $f(x) = |x|$ in $B(0,R) = U$)

$$V(f, B(0,R)) = \| |Df| \|_{L^1} = \int_{B(0,R)} dx = \int_0^R Per\{|x| > t\}\, dt =$$

$$= \int_0^R \int_{\partial B(0,t)} d\mathcal{H}^{n-1}(y)\, dt \;)$$

BY COAREA and DENSITY in STRICT SENSE and SARD theorem.

$\forall E$ of finite perimeter, $\exists E_n$ smooth $\left[\begin{array}{l} E_n \to E \text{ in measure} \\ Per(E_n) \to Per(E) \end{array}\right.$

5) ISOPERIMETRIC INEQUALITY: $\exists\ C_n \in (0,1]$

$\forall E$ of finite perimeter

(Iso) $\|\chi_E\|_{L^{1^*}} = |E|^{\frac{n-1}{n}} \leq C_n \operatorname{Per}(E) = C_n V(\chi_E, \mathbb{R}^n)$

(GNS) inequality $\left(\|f\|_{L^{n/n-1}} \leq V(f, \mathbb{R}^n)\right)$

Iso invariant by rescaling $\lambda > 0$ $\lambda E = E_\lambda$ $\left[\begin{array}{l} |\lambda E| = \lambda^n |E| \\ \operatorname{Per}(\lambda E) = \lambda^{n-1} \operatorname{Per}(E) \end{array}\right.$

$\Rightarrow \forall E$ of finite perimeter with $|E| = 1$ $\operatorname{Per}(E) \geq \frac{1}{C_n}$.

(ISOPERIMETRIC PROBLEM: among sets of fixed volume, which is the minimal perimeter?

$$C_n = \frac{|B|^{\frac{n-1}{n}}}{\operatorname{Per}(B)} = \frac{\omega_n^{\frac{n-1}{n}}}{n\,\omega_n} = \frac{1}{n(\omega_n)^{1/n}} \sim \frac{1}{\sqrt{2\pi e n}}$$

(De Giorgi in full generality 1958, by adaptation of the Steiner symmetrization technique, ball is the unique solution)

Poincaré inequality U bold of class C^1 and CONNECTED

$\exists\, C>0$ s.that $\forall f \in BV(U)$

$$\left\| f - \frac{1}{|U|}\int_U f(y)\,dy \right\|_{L^{\frac{n}{n-1}}} \leq C\, V(f, U).$$

proof Assume not true (proof by contradiction)

$\forall n \quad \exists f_n \in BV(U) \quad \left\| f - \frac{1}{|U|}\int_U f(y)\,dy \right\|_{L^{\frac{n}{n-1}}} \geq n\, V(f_n, U)$

$$\Rightarrow v_n = \frac{f_n - \frac{1}{|U|}\int_U f_n}{\left\| f_n - \frac{1}{|U|}\int_U f_n \right\|_{L^{\frac{n}{n-1}}}} \qquad V(v_n, U) = \frac{V(f_n, U)}{\left\| f_n - \frac{1}{|U|} f_n \right\|_{L^{\frac{n}{n-1}}}}.$$

$\|v_n\|_{L^{\frac{n}{n-1}}} = 1 \qquad \boxed{V(v_n, U) \leq \frac{1}{n} < 1} \xrightarrow{\text{by HELLY}} \exists f \quad v_{n_i} \to f \text{ in } L^1$

$\Downarrow \|v_n\|_{L^1} \leq C$ (Hölder)

$0 = \liminf_{n_j} V(v_n, U) \geq V(f, U)$

$V(f, U) = 0 \Rightarrow \frac{\partial}{\partial x_i} T_f \equiv 0 \quad \forall i \Rightarrow f$ is constant $\square$

6) (Poincaré) $\rightarrow$ LOCAL ISOPERIMETRIC INEQUALITY : $\exists\, \tilde{C}_n = \tilde{C}$

for all $E \subseteq \mathbb{R}^n$ measurable such that $\chi_E \in BV_{loc}(\mathbb{R}^n)$

$$\forall\, x \in \mathbb{R}^n \quad Per(E, B(x,1)) \geq \tilde{C} \min \left[|E \cap B(x,1)|^{\frac{n-1}{n}}, |B(x,1) \setminus E|^{\frac{n-1}{n}} \right]$$

proof apply Poincaré inequality to $\chi_E \in BV(B(x,1))$.

$$\left\| \chi_E - \frac{1}{|B(x,1)|} \int_{B(x,1)} \chi_E(y)\, dy \right\|_{L^{n/n-1}} \leq C\, Per(E, B(x,1))$$

$$= \left[\int_{B(x,1) \cap E} \left| 1 - \frac{|E \cap B(x,1)|}{\omega_n} \right|^{\frac{n}{n-1}} + \int_{B(x,1) \setminus E} \left| \frac{|E \cap B(x,1)|}{\omega_n} \right|^{\frac{n}{n-1}} \right]^{\frac{n-1}{n}}$$

where we used $|B(x,1)| = \omega_n$. Let $t = |B(x,1) \cap E| / \omega_n$.

$$= \left[(1-t)^{\frac{n}{n-1}} \overbrace{|B(x,1) \cap E|}^{t\omega_n} + \overbrace{|B(x,1) \setminus E|}^{(1-t)\omega_n} \cdot t^{\frac{n}{n-1}} \right]^{\frac{n-1}{n}}$$

$$= \left[(1-t)^{\frac{n}{n-1}} t\omega_n + (1-t)\omega_n t^{n/n-1} \right]^{\frac{n-1}{n}}$$

without loss of generality $t \geq (1-t)$ $(t \geq \frac{1}{2})$

$$\geq \left[\underbrace{(1-t)^{\frac{n}{n-1}}}_{\geq 0} (1-t)\omega_n + (1-t)\omega_n \underbrace{t^{n/n-1}}_{\geq 1/2} \right]^{\frac{n-1}{n}} \geq (1-t)^{\frac{n-1}{n}} \omega_n^{\frac{n-1}{n}} \frac{1}{2}$$

$$C \, \mathrm{Per}(E, B(x,1)) \geq \frac{1}{2} \, \omega_n^{\frac{n-1}{n}} \left[1 - \frac{|E \cap B(x,1)|}{\omega_n} \right]^{\frac{n-1}{n}}$$

$$\geq \frac{1}{2} \, |B(x,1) \setminus E|^{\frac{n-1}{n}} =$$

$$= \frac{1}{2} \min \left[|E \cap B(x,1)|^{\frac{n-1}{n}}, \; |B(x,1) \setminus E|^{\frac{n-1}{n}} \right].$$