CONDITIONAL EXPECTATION of X with respect to a σ-algebra

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space $\quad M^2 = \{ X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ measurable $E(X^2) < +\infty \}$

$\mathcal{G} \subseteq \mathcal{F}$ $\quad$ $\mathcal{G}$ σ-algebra contained in $\mathcal{F}$

$$M^2_{\mathcal{G}} = \{ Y \in M^2 \text{ such that } \forall B \in \mathcal{B}(\mathbb{R}) \quad Y^{-1}(B) \in \mathcal{G} \}$$

$E(X \mid \mathcal{G})$ = Conditional expectation of X with respect to $\mathcal{G}$ = orthogonal projection of X in $M^2_{\mathcal{G}}$

1) $E(X \mid \mathcal{G}) \in M^2_{\mathcal{G}}$ $\qquad$ 2) $E\left[\left(X - E(X \mid \mathcal{G})\right)^2\right] = \min\limits_{Y \in M^2_{\mathcal{G}}} E\left[(X - Y)^2\right]$

3) $\forall Y \in M^2_{\mathcal{G}}$ $\qquad X - E(X \mid \mathcal{G}) \perp Y$

which means $E\left(\left(X - E(X \mid \mathcal{G})\right) Y\right) = 0$

Conditional expectation with respect to a random variable

$\mathbb{E}(X|Y) = \mathbb{E}(X \mid \sigma(Y))$ where $\sigma(Y)$ is the smallest σ-algebra which contains all elements $Y^{-1}(B)$, for $B \in \mathcal{B}(\mathbb{R})$.

$M^2_{\sigma(Y)} = \{ g(Y)$ for some $g: \mathbb{R} \to \mathbb{R}$ such that $g(Y) \in M^2 \}$
so g measurable and $\mathbb{E}(g(Y))^2 < +\infty \}$

$\mathbb{E}(X|Y)$ is the orthogonal projection on $M^2_{\sigma(Y)}$

1) $\mathbb{E}(X|Y) = h(Y) \quad \exists h: \mathbb{R} \to \mathbb{R}$

2) $\displaystyle \min_{\substack{g:\mathbb{R}\to\mathbb{R} \\ \text{meas.}}} \mathbb{E}(X - g(Y))^2 = \mathbb{E}(X - \mathbb{E}(X|Y))^2$

3) $\mathbb{E}((X - \mathbb{E}(X|Y))\, g(Y)) = 0 \quad \forall g: \mathbb{R} \to \mathbb{R}$ measurable

In particular if $g(r) = c$

$$E((X - E(X|Y))c) = 0 \Rightarrow E(X) = E(E(X|Y))$$

if $g(r) = r$

$$E((X - E(X|Y))Y) = 0 \Rightarrow E(XY) = E(E(X|Y) \cdot Y)$$

so if X and Y are independent $E(X|Y) = \underline{E(X)}$ constant

$E(X|Y)$ is the best predictor of X given Y.

Since it is difficult in general to find $E(X|Y)$ $(= h(Y))$

we consider an easier problem:

Let $\underbrace{V = \{aY + b\ ,\ a, b \in \mathbb{R}\}} \subseteq M^2_{\sigma(Y)}$

V is a finite dimensional subspace of $M^2_{\sigma(Y)}$, given by the linear functions of Y.

$$V \xrightarrow{\sim} \mathbb{R}^2 \qquad\qquad \mathbb{R}^2 \longrightarrow V$$
$$aY+b \longmapsto (a,b) \qquad\qquad (a,b) \longrightarrow aY+b$$

V is 2-dimensional since it is isomorphic as a vectorial space to $\mathbb{R}^2$.

$\bar{a}Y+\bar{b}$ is the best LINEAR MEAN SQUARE ESTIMATOR of X given Y if

$\bar{a}Y+\bar{b}$ is the orthogonal projection of X on V.

1) $\bar{a}Y+\bar{b} \in V$

2) $E(X-\bar{a}Y-\bar{b})^2 = \min\limits_{(a,b)\in\mathbb{R}^2} E(X-aY-b)^2$

3) $E\left[(X-\bar{a}Y-\bar{b})(aY+b)\right]=0 \quad \forall (a,b)\in\mathbb{R}^2$

How to compute the orthogonal projection on V?

FIRST METHOD

1) we minimize $\min_{(a,b)\in\mathbb{R}^2} E(X-aY-b)^2$ and find $\bar{a}, \bar{b}$ MINIMA

SECOND METHOD

2) general method based on $\boxed{\text{orthonormal basis}}$

Def: $(e_i)_{i\in I}$ is a ORTHONORMAL BASIS of a Hilbert space if it is a basis (of the vectorial space) and
$\|e_i\|=1 \ \forall i$ $\quad (e_i, e_j)=0 \ \forall i\neq j$

Let H be a Hilbert space and V be a subspace of H of FINITE DIMENSION

so V has a finite basis $\langle v_1 \dots v_n\rangle$.

($\forall v\in V \quad v=\sum_{i=1}^{n} a_i v_i$ every element of V is written as a linear combination of elements of the basis).

Starting from $\langle v_1 \dots v_n \rangle$ we may construct an ORTHONORMAL BASIS of V (by the GRAM-SCHMIDT ORTHONORMALIZATION PROCEDURE)

$$v_1 \longrightarrow \quad e_1 = \frac{v_1}{\|v_1\|} \quad \text{so} \quad \|e_1\| = 1 = (e_1, e_1)$$

$$e_2 := \frac{v_2 - (v_2, e_1) e_1}{\|v_2 - (v_2, e_1) e_1\|}$$

Note that $\|e_2\| = 1 = (e_2, e_2)$

$$(e_2, e_1) = \frac{(v_2 e_1) - (v_2 e_1)(e_1, e_1)}{\|v_2 - (v_2 e_1) e_1\|} = 0$$

$$e_3 := \frac{v_3 - (v_3, e_2) e_2 - (v_3, e_1) e_1}{\|v_3 - (v_3, e_2) e_2 - (v_3, e_1) e_1\|}$$

$\|e_3\| = 1$

$(e_3, e_2) = 0 = (e_3, e_1)$

$e_4 := \dots$

$\langle v_1 \dots v_n \rangle \longrightarrow \langle e_1 \dots e_n \rangle$

where $\|e_i\| = 1 \ \forall i$

$(e_i, e_j) = 0 \ \forall i \neq j$

$\forall\, Q \in H$ the orthogonal projection of Q in V is given by $\sum_{i=1}^{n} (Q, e_i)\, e_i$ where e_i is an orthonormal basis of V.

So let $V = \{aY + b,\ a, b \in \mathbb{R}\} \subseteq M^2_{\sigma(Y)} \subseteq M^2$

a basis of V is $\langle 1, Y \rangle$ (every element of V is written as $b \cdot 1 + a \cdot Y$

$v_1 = 1 \qquad v_2 = Y$

$$e_1 = \frac{1}{\|1\|} = \frac{1}{(E(1))^{1/2}} = 1$$

$$e_2 = \frac{v_2 - (v_2, e_1)\, e_1}{\| v_2 - (v_2, e_1)\, e_1 \|}$$

$$e_2 := \frac{Y - E(Y \cdot 1) \cdot 1}{E((Y - E(Y))^2)^{1/2}}$$

the orthonormal basis of $\bar{V}$ is

$$\left\langle 1, \frac{Y - \mathbb{E}(Y)}{\sqrt{Var(Y)}} \right\rangle$$

$$Var(Y) = \mathbb{E}((Y - \mathbb{E}(Y))^2) = \\ = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2.$$

$\downarrow$

the orthogonal projection of X in $\bar{V}$ is given by

$$(X, e_1) e_1 + (X, e_2) e_2 =$$

$$= \mathbb{E}(X \cdot 1) \cdot 1 + \mathbb{E}\left(X \cdot \left(\frac{Y - \mathbb{E}(Y)}{\sqrt{Var Y}}\right)\right) \frac{(Y - \mathbb{E}(Y))}{\sqrt{Var Y}} =$$

$$= \mathbb{E}(X) + \left[\frac{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}{Var(Y)}\right] (Y - \mathbb{E}(Y)) =$$

$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

$$= \left[\mathbb{E}(X) - \frac{Cov(X,Y)}{Var(Y)} \mathbb{E}(Y)\right] + \left[\frac{Cov(X,Y)}{Var(Y)}\right] \cdot Y$$

PRODUCT OF CONVOLUTION

$L^1(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \quad \int_{\mathbb{R}} |f(x)|\, dx < +\infty \}$ not Hilbert (just Banach)

$L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \quad \int_{\mathbb{R}} |f(x)|^2\, dx < +\infty \}$ Hilbert

By Hölder inequality we have that $f, g \in L^2(\mathbb{R})$

$f \cdot g \in L^1(\mathbb{R})$ $\qquad \underbrace{\int_{\mathbb{R}} |f(x) g(x)|\, dx} \le \underbrace{\left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{1/2} \left(\int_{\mathbb{R}} |g(x)|^2 dx\right)^{1/2}}$

we define another notion of product of functions

PRODUCT of CONVOLUTION:

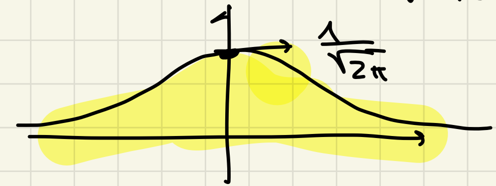
$$(f * g)(x) = \underbrace{\int_{\mathbb{R}} f(x-y)\, g(y)\, dy} = \underbrace{\int_{\mathbb{R}} f(y)\, g(x-y)\, dy}$$

$$\ne$$

$$\cancel{f(x)\, g(x)}$$

TO UNDERSTAND HOW CONVOLUTION WORKS:

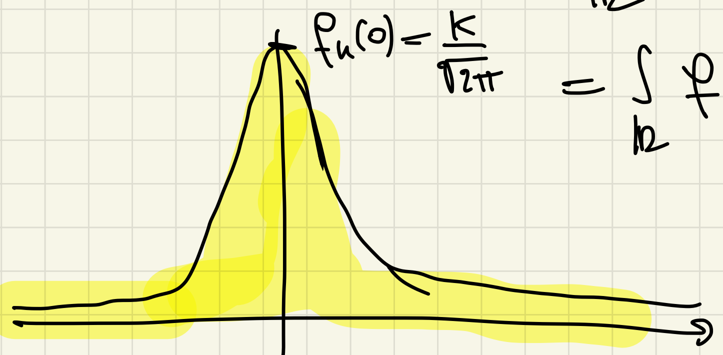
Let $f \geq 0$, f smooth, $\int_{\mathbb{R}} f(x)\,dx = 1$ $\quad \left(f(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}} \right)$



for $k \in \mathbb{N}$ $\quad f_k(x) := k f(kx)$

$$\left(= \frac{k}{\sqrt{2\pi}} e^{-k x^2/2} \right)$$

then $f_k \geq 0$ $\quad \int_{\mathbb{R}} f_k(x)\,dx = \int_{\mathbb{R}} k f(kx)\,dx = \begin{pmatrix} y = kx \\ dy = k\,dx \end{pmatrix}$



$$= \int_{\mathbb{R}} f(y)\,dy = 1$$

$g \in L^1(\mathbb{R})$ $\quad g * f_k(x) = \int_{\mathbb{R}} g(x-y) f_k(y)\, dy =$

$= \int_{\mathbb{R}} g(x-y)\, k f(ky)\, dy =$ change variable $=$
$z = ky$
$dz = k\,dy$

$= \int_{\mathbb{R}} g\left(x - \frac{z}{k}\right) f(z)\, dz \xrightarrow[k \to +\infty]{} g(x) \int_{\mathbb{R}} f(z)\, dz = g(x)$

$g * f_k(x) \xrightarrow{k \to +\infty} g(x)$

$g * f_k$ is an "approximation" of g by taking at every x the average of the values of g around x.

$g * f_k$ is as regular as f_k

since $\frac{d}{dx}(g * f_k)(x) = \int_{\mathbb{R}} g(y) \frac{d}{dx} f_k(x-y)\, dy$

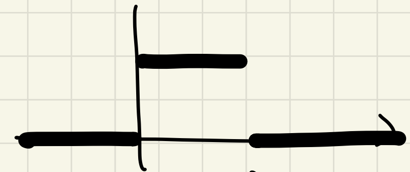
In particular if f is differentiable n-times

$\Rightarrow g * f$ is differentiable n times.

Proposition: If $f, g \in L^2(\mathbb{R})$ then $f * g \in C(\mathbb{R})$ and $\lim_{|x| \to +\infty} f * g(x) = 0$

(NO PROOF / convolution of 2 L^2 functions is CONTINUOUS and bounded – even if the starting functions were not continuous).

Example $f(x) = g(x) = \begin{cases} 0 & x \le 0 \\ 1 & 0 < x < 1 \\ 0 & x \ge 1 \end{cases}$



(f is the density of a uniform random variable).

then $f * f$ is CONTINUOUS and $\lim_{|x| \to +\infty} f*f(x) = 0$.

(even if f is NOT CONTINUOUS).

proof $f*f(x) = \int_{\mathbb{R}} f(x-y) f(y)\,dy = \int_{-\infty}^{0} f(x-y)\underbrace{\cancel{f(y)}}_{=0} dy +$

$$+ \int_0^1 f(x-y) \underbrace{f(y)}_{=1} dy + \int_1^{+\infty} \cancel{f(x-y)} \underbrace{f(y)}_{=0} dy =$$

$$= \int_0^1 f(x-y)\,dy = \begin{pmatrix} t = x-y \\ dt = -dy \end{pmatrix} = \int_x^{x-1} f(t)(-1)\,dt =$$

$$= \int_{x-1}^{x} f(t)\,dt$$

$$f * f(x) = \int_{x-1}^{x} f(t)\,dt .$$

1) if $x < 0$ $\quad (x-1, x) \subseteq (-\infty, 0) \Rightarrow \int_{x-1}^{x} f(t)\,dt = 0$

2) if $0 < x < 1$ $\quad x-1 < 0$ and $x > 0$

$$\int_{x-1}^{x} f(t)\,dt = \cancel{\int_{x-1}^{0} \underbrace{f(t)}_{=0}\,dt} + \int_{0}^{x} \underbrace{f(t)}_{=1}\,dt = \int_{0}^{x} 1\,dt = x$$

3) if $1 < x < 2$ $\quad x-1 > 0$ $\quad x > 1$

$$\int_{x-1}^{x} f(t)\,dt = \int_{x-1}^{1} \underbrace{f(t)}_{=1}\,dt + \cancel{\int_{1}^{x} \underbrace{f(t)}_{=0}\,dt} = \int_{x-1}^{1} 1\,dt = 1-(x-1)$$
$$= 2 - x$$

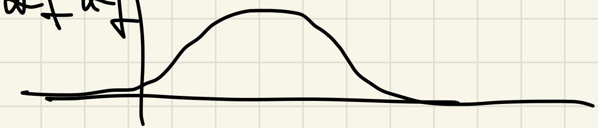
4) $x > 2$ $\quad x-1 > 1$ $\quad (x-1, x) \subseteq (1, +\infty) \rightarrow \int_{x-1}^{x} f(t)\,dt = 0$

$$f*f(x) = \begin{cases} 0 & x<0 \\ x & 0\le x<1 \\ 2-x & 1\le x<2 \\ 0 & x\ge 2 \end{cases}$$

$f*f$ IS CONTINUOUS !

$f*f*f$ .. becomes more and more regular

$f*f*f*f$



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Prop If X, Y are absolutely continuous independent random variables, with density f and g then $X+Y$ is an absolutely cont. random variable with density $f * g$

proof

f is the density of X (density = derivative of the cumulative dist. f)

$$F_X(a) = \int_{-\infty}^{a} f(x)\,dx = \mathbb{P}(\omega \in \Omega \mid X(\omega) \leq a\}$$

$$\mathbb{P}(\omega \mid Y(\omega) \leq b) = \int_{-\infty}^{b} g(x)\,dx = F_Y(b)$$

I want to compute the density of $X+Y$

$t \in \mathbb{R}$ $\quad \mathbb{P}\{\omega \mid (X+Y)(\omega) \le t\} =$

$$= \mathbb{P}\{\omega \mid \underbrace{X(\omega) \le x}, \underbrace{Y(\omega) \le y} \quad x+y \le t\}$$

$$= \int_{\{(x,y)\in\mathbb{R}^2,\ x+y\le t\}} f(x)\,g(y)\,dx\,dy = \int_{\mathbb{R}} \int_{-\infty}^{t-x} f(x)\,g(y)\,dy\,dx$$

$x \in \mathbb{R} \qquad y \le t-x \longrightarrow z = y+x \le t$

$$= \int_{\mathbb{R}} \int_{-\infty}^{t} f(x)\,g(z-x)\,dz\,dx = \int_{-\infty}^{t} \underbrace{\int_{\mathbb{R}} f(x)\,g(z-x)\,dx}\,dz$$

$\boxed{z = x+y}$

$x = x$

$$= \int_{-\infty}^{t} (f*g)(z)\,dz$$

(if X, Y are discrete random variables and independent

$$\mathbb{P}\left(\left(X+Y\right)(\omega)=t\right) = \sum_{k \in \mathbb{Z}} \underbrace{\mathbb{P}(X(\omega)=k)\,\mathbb{P}(Y(\omega)=t-k)}\ \Bigg)$$

FOURIER TRANSFORM

$f : \mathbb{R} \to \mathbb{C}$ $\quad f(x) = f_1(x) + i f_2(x)$ $\quad i =$ imaginary unit.

$e^{ix} = \cos x + i \sin x$ for $x \in \mathbb{R}$ $\quad |e^{ix}| = 1$

$z \in \mathbb{C} \iff z = x + iy$ $\quad |z| = \sqrt{x^2+y^2}$

Take $f \in L^1(\mathbb{R})$

$$\hat{f}(\lambda) = \text{Fourier transform of } f := \int_{\mathbb{R}} f(y) e^{ixy} dy =$$

$$= \int_{\mathbb{R}} \cos(xy) f(y) dy + i \int_{\mathbb{R}} \sin(xy) f(y) dy$$

$\hat{f} : \mathbb{R} \to \mathbb{C}$.

$\hat{f}$ is bounded : $\forall x \in \mathbb{R}$ $\quad |\hat{f}(\lambda)| = \left| \int_{\mathbb{R}} f(y) e^{ixy} dy \right| \leq$

$$\leq \int_{\mathbb{R}} |f(y)| |e^{ixy}| dy = \int_{\mathbb{R}} |f(y)| dy = \|f\|_1$$

2) $\hat{f}$ is continuous: $\hat{f}(x+h) =$

$$= \int_{\mathbb{R}} f(y)\cos((x+h)y)\,dy + i\int_{\mathbb{R}} f(y)\sin((x+h)y)\,dy$$

$$\xrightarrow{h\to 0} \int_{\mathbb{R}} f(y)\cos(xy)\,dy + i\int_{\mathbb{R}} f(y)\sin(xy)\,dy = \hat{f}(x)$$

$f\in L^1(\mathbb{R}) \longrightarrow \boxed{\hat{f}:\mathbb{R}\to\mathbb{C} \text{ is a continuous bounded function}}$

$$\sup_{x\in\mathbb{R}} |\hat{f}(x)| \le \|f\|_{L^1}$$

Other important properties of FOURIER TRANSFORM:

(1) $f, g \in L^1(\mathbb{R})$ $\qquad \widehat{(f * g)}(x) = \hat{f}(x) \cdot \hat{g}(x)$

(Fourier transform of a convolution of 2 function is the product of the Fourier transform).

proof:

$$e^{ixy} = e^{ix(y-z)} \cdot e^{ixz}$$

$$\widehat{(f*g)}(x) = \int_{\mathbb{R}} \underbrace{(f*g)(y)} \cdot e^{ixy}\, dy =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y-z)\, g(z)\, dz\; e^{ix(y-z)}\, e^{ixz}\, dy =$$

$z \to z$

$y \to w = y - z$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y-z)\, e^{ix(y-z)}\, g(z)\, e^{ixz}\, dz\, dy =$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \underbrace{f(w)\, e^{ixw}}\ \underbrace{g(z)\, e^{izx}}\, dz\, dw =$$

$$= \underbrace{\int_{\mathbb{R}} f(w)\, e^{ixw}\, dw}\ \underbrace{\int_{\mathbb{R}} g(z)\, e^{ixz}\, dz} = \hat{f}(x)\, \hat{g}(x)$$

(2) Let $f \in L^1(\mathbb{R})$ such that $x \cdot f(x) \in L^1(\mathbb{R})$

$$\Rightarrow (\hat{f})'(x) = \frac{d}{dx}\hat{f}(x) = \lim_{h \to 0} \frac{\hat{f}(x+h) - \hat{f}(x)}{h} =$$

$$= [\underline{\underline{i\, y\, f(y)}}]^{\wedge}(x)$$

proof

$$\frac{d}{dx}\hat{f}(x) = \frac{d}{dx} \int_{\mathbb{R}} f(y)\, e^{ixy}\, dy = \int_{\mathbb{R}} f(y) \frac{d}{dx} e^{ixy}\, dy =$$

$$= \int_{\mathbb{R}} f(y)\, iy\, e^{ixy}\, dy = \int_{\mathbb{R}} \underline{iy f(y)} \cdot e^{ixy}\, dy$$

$$= (\underbrace{iy f(y)})^{\wedge}(x)$$

2') $$\frac{d^2}{dx^2} \hat{f}(x) = \int_{\mathbb{R}} f(y) \frac{d^2}{dx^2} e^{ixy}\, dy = \int_{\mathbb{R}} f(y) \cdot (iy)(iy)\, e^{ixy}\, dy$$

$$= \int_{\mathbb{R}} (-y^2) f(y)\, e^{ixy} = \left(-y^2 f(y)\right)^{\wedge}(x)$$

$$\frac{d^k}{dx^k} \hat{f}(x) = \left(\underbrace{(iy)^k f(y)}\right)^{\wedge}(x)$$

3) if f is differentiable and $\lim_{|x|\to+\infty} f(x)=0$

$$\left(\frac{d}{dy}f\right)^{\wedge}(x) = (-ix)\,\hat{f}(x)$$

proof

$$\left(\frac{d}{dy}f\right)^{\wedge}(x) = \int_{\mathbb{R}} \underbrace{\frac{d}{dy}f(y)}\cdot \underbrace{e^{ixy}}\,dy =$$

$$= \text{integration by part formula} = \cancel{\left[f(y)e^{ixy}\right]_{-\infty}^{+\infty}} +$$

$$- \int_{\mathbb{R}} f(y)\cdot\frac{d}{dy}e^{ixy}\,dy = -ix\int_{\mathbb{R}} f(y)\,e^{ixy}\,dy$$

(3') f is differentiable k - times and

$\lim_{|x| \to +\infty} f^{(n)}(x) = 0 \qquad \forall\, n \leq k-1$

$$f^{(n)}(x) = \frac{d^n}{dx^n} f(x)$$

$$\left(\frac{d^k}{dy^k} f(y) \right)^{\wedge} (x) = (-ix)^k \cdot \hat{f}(x)$$

$\underline{\text{Ex}}$ $f(x) = e^{-x^2}$

$$\hat{f}(x) = \int_{\mathbb{R}} e^{-y^2} e^{ixy}\, dy = \int_{\mathbb{R}} \underbrace{e^{-y^2}\cos(xy)\, dy} + i\int_{\mathbb{R}} e^{-y^2}\sin xy\, dy$$

How to compute explicitly $\hat{f}$?

$$\left(\frac{d}{dx}\hat{f}(x) = (iy f(y))^{\wedge}\right)$$

$$\underbrace{\frac{d}{dx}(\hat{f}(x))} = \int_{\mathbb{R}} \underbrace{iy\, e^{-y^2}} e^{ixy}\, dy =$$

$$= -\frac{i}{2}\int_{\mathbb{R}} \underbrace{(-2y)\, e^{-y^2}}_{\frac{d}{dy} e^{-y^2}} e^{ixy}\, dy = -\frac{i}{2}\int_{\mathbb{R}} \frac{d}{dy}\left(e^{-y^2}\right) e^{ixy} dy$$

$$= -\frac{i}{2}\left[\cancel{\left[e^{-y^2} e^{ixy}\right]_{-\infty}^{+\infty}} - \int_{\mathbb{R}} e^{-y^2} \frac{d}{dy} e^{ixy}\, dy \right] =$$

$$= \underbrace{-\frac{i}{2}}\left[\underbrace{-ix} \int_{\mathbb{R}} e^{-y^2} e^{ixy}\, dy\right] =$$

$$= -\frac{x}{2} \underbrace{\int_{\mathbb{R}} e^{-y^2} e^{ixy}\, dy}_{\hat{f}(x)}$$

$$\frac{d}{dx}\hat{f}(x) = -\frac{x}{2}\hat{f}(x)$$

$$f(y) = e^{-y^2}$$

$$\hat{f}(x) = \int_{\mathbb{R}} e^{-y^2} e^{ixy} dy > 0$$

$$\Longrightarrow$$

$$\frac{1}{\hat{f}(x)} \cdot \frac{d}{dx}\hat{f}(x) = \underbrace{-\frac{x}{2}}$$

$$\frac{d}{dx} \log\left(\hat{f}(x)\right) = \frac{d}{dx}\left(-\frac{x^2}{4}\right)$$

$$\log(\hat{f}(x)) = -\frac{x^2}{4} + c$$

$$e^{\log(\hat{f}(x))} = \hat{f}(x) = e^{-\frac{x^2}{4}+c} = e^{-\frac{x^2}{4}} e^{c}$$

$$\hat{f}(x) = e^{-\frac{x^2}{4}} \underbrace{e^{c}}_{\uparrow}$$

$$\hat{f}(0) = e^{c} = \int_{\mathbb{R}} e^{-y^2} \underline{e^{i0\cdot y}}\, dy = \int_{\mathbb{R}} e^{-y^2} dy = \sqrt{\pi}$$

$$\hat{f}(x) = \sqrt{\pi}\, e^{-x^2/4} \qquad f(x) = e^{-x^2}$$