Characterization of BV functions in dimension 1

I interval in $\mathbb{R}$.

BASIC OBSERVATION: f MONOTONE NON DECREASING (NON INCREASING same) $\Rightarrow f \in L^1_{loc}(\mathbb{R})$

moreover $(T_f)'$ is a positive zero order distribution

$\rightarrow T_f'(\phi) = \int \phi \, d\mu$ where μ is a (positive) Radon measure on $\mathbb{R}$. (that is $f \in BV_{loc}(\mathbb{R})$, $f \in BV(a,b)$ $\forall (a,b) \subseteq \mathbb{R}$).

Fix (a,b) $\quad t \in (a,b) \xrightarrow{g} \mu(a,t) \Rightarrow g$ is monotone $\Rightarrow (T_g)' = (T_f)' = \mu$

since $\int_a^b \phi'(t) \int_a^t d\mu(y) = $ (Fubini Tonelli) $= \int_a^b \int_y^b \phi'(t) dt \, d\mu(y) = -\int_a^b \phi(y) d\mu(y)$

$\Rightarrow \exists c \in \mathbb{R}$ such that $f(t) = c + \mu(a,t)$ a.e. $t \in (a,b)$

Recall: f monotone $\Rightarrow \exists$ $f(t^+) = \lim_{x \to t^+} f(x)$ $\quad f(t^-) = \lim_{x \to t^-} f(x)$

$A(f) =$ atoms of f $= \{ t \in \mathbb{R} \mid f(t^+) \neq f(t^-) \}$

$A(f)$ are at MOST COUNTABLE

$f(t) \underset{a.e.}{=} f(a^+) + \mu(a,t) \qquad V(f,(a,b)) = \mu(a,b) = f(b^-) - f(a^+)$

Definition (POINTWISE VARIATION).

Let $f \in L^1(a,b)$ $\quad pV(f,(a,b)) = \sup \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i+1})| \right.$

among all possible finite subdivisions

$\left. a < x_1 < x_2 < \dots < x_{n+1} < b \right\}$

Observation : $pV(f,(a,b)) \geq V(f_h,(a,b))$ (f_h is the piecewise constant "approx" of f)

pV(f,(a,b)) depends on the representative!

$epV(f,(a,b))$ = ESSENTIAL pointwise variation =

$= \inf \{ pV(\tilde{f},(a,b)) \, , \; \tilde{f} = f \text{ a.e in } (a,b) \}$

f monotone $\Rightarrow$ $pV f(a,b) = |f(b^-) - f(a^+)| = V(f,(a,b))$

$\quad \| \quad$

$epV(f,(a,b))$

Prop If $pV(f,(a,b)) < +\infty$ then $f \in L^\infty(a,b)$ and $f = g_1 - g_2$ where g_i are monotone nondecreasing functions

proof f bdd is immediate by $pV(f,(a,b)) < +\infty$.

$$g_1(x) := \sup \left\{ \sum_{i=1}^{m} |f(x_i) - f(x_{i+1})| \quad a < x_1 < x_2 < \dots < x_{m+1} = x \right\}$$

g_1 is increasing. $g_2(x) := g_1(x) - f(x)$.

Take $y > x$. $g_2(y) - g_2(x) = g_1(y) - [g_1(x) + f(y) - f(x)] \geq 0$

since $g_1(x) + f(y) - f(x) \leq g_1(x) + |f(y) - f(x)| \leq$

$$= \sup \left\{ \sum_{i=1}^{n-1} |f(t_{i+1}) - f(t_i)| + |f(y) - f(x)| \,,\quad a < t_1 < t_2 < t_n = x \right\}$$

$$= \sup \left\{ \sum_{i=1}^{n} |f(t_{i+1}) - f(t_i)| \quad a < t_1 < \dots < t_n = x < t_{n+1} = y \right\}$$

$\leq g_1(y)$.

Theorem (Characterization of BV) Let $f \in BV(a,b)$, then $\exists\, \tilde{f}$ such that $\tilde{f} = f$ a.e. such that

$$ep\,V(f,(a,b)) = pV(\tilde{f},(a,b)) = V(\tilde{f},(a,b)) = V(f,(a,b))$$

$\to$ $\tilde{f}$ is the difference of 2 monotone nondecreasing functions.

Every f in $BV(a,b)$ has a right continuous $\tilde{f}_r$ and a left continuous representative $\tilde{f}_l$

$$\tilde{f}_r(t) = c + \mu(a,t] \qquad \tilde{f}_l(t) = c + \mu(a,t) \qquad (\mu = T_f)$$

these representative are continuous up to a set which is at most countable

set of ATOMS of f $\quad A(f) = \{ t \mid \mu\{t\} \neq 0 \}$

idea of proof

$f \in BV(a,b) \Rightarrow (T_f)' = \mu \Rightarrow \mu = \mu^+ - \mu^-$ μ^+, μ^- Radon measures.

$g : t \to \mu^+(a,t) - \mu^-(a,t)$

$(T_g)' = (T_f)' \Rightarrow \exists\, c$ $f \overset{\cdot}{=} g + c = c + \mu(a,t) =$ a.e.

$= c + \mu^+(a,t) - \mu^-(a,t)$

f is the difference of 2 monotone nondecreasing functions (up to a set of measure 0).

$V(f,(a,b)) = \mu^+(a,b) + \mu^-(a,b) = pV(g,(a,b)) = V(g,(a,b))$

$= epV(f,(a,b))$ $\square$

Obs Let f monotone nondecreasing function in (a,b)

$\Rightarrow f \in BV(a,b) \quad (T_f)' = \mu \quad \mu$ Radon measure

$$V(f,(a,b)) = f(b^-) - f(a^+) = \mu(a,b) =$$

$$= \int_a^b \underbrace{f'(t)}_{\text{almost everywhere derivative}}\,dt + \sum_{t \in A(f)} f(t^+) - f(t^-) + \mu^c(a,b).$$

μ has an absolutely continuous part w.r. to Leb. with density given by the a.e derivative $f'(t)$ and a singular part with respect to Lebesgue

① ATOMIC PART: $\sum_{t \in A(f)} (f(t^+) - f(t^-))\,\delta_{\{t\}}$ →

② CANTORIAN PART

μ^c is the derivative in the sense of distribution of a continuous function which has a.e. derivative equal to 0

(main example: μ^c is the derivative in the sense of distribution of the Cantor function).

$$\boxed{\mu = f' dt + \sum_{t \in A} \left(f(t^+) - f(t^-)\right) \delta_{\{t\}} + \mu^c}$$

$f \in SBV(a,b) \iff$ Cantorian part is 0.
(special BV).