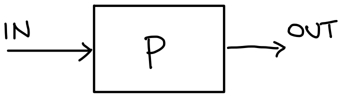
COMPUTABILITY (10/12/2024)

Rice - Shapiro's Theorem



properties of the behaviour

two views

1) properties of the function computed by a program P

$$\mathcal{A} \subseteq \mathcal{C}$$

$\mathcal{T} = \{ f \in \mathcal{C} \mid f \text{ is total} \} = \{ f \in \mathcal{C} \mid dom(f) = \mathbb{N} \}$

$ONE = \{ \mathbb{1} \}$

$\mathcal{B}_m = \{ f \in \mathcal{C} \mid m \in cod(f) \}$ $\quad m \in \mathbb{N}$ fixed

$\vdots$

2) extensional / saturated property of programs $A \subseteq \mathbb{N}$

$T = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{T} \}$

$P_{ONE} = \{ x \in \mathbb{N} \mid \varphi_x = \mathbb{1} \}$

$B_m = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{B}_m \}$

<u>Rice's Theorem</u> : no extensional property, apart from the trivial (true / false) is decidable

Rice - shapiro's Theorem

an extensional property <u>can be</u> semi-decidable only if it is <u>finitary</u>

depends only on a finite amount of input

Rice-Shapiro's Theorem

Let $\mathcal{A} \subseteq \mathcal{C}$ be a set of computable functions

and let $A = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{A} \}$

If $\boxed{A \text{ is r.e.}}$ $(*)$ then

$$\boxed{\forall f \quad (f \in \mathcal{A} \iff \exists \vartheta \subseteq f, \vartheta \text{ finite}, \vartheta \in \mathcal{A})} \quad (**)$$

proof

In order to show $(*) \Rightarrow (**)$

we prove $\neg (**) \Rightarrow \neg (*)$

This splits in two

1. $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f, \vartheta$ finite s.t. $\vartheta \in \mathcal{A} \quad \Rightarrow \quad A$ not r.e.
2. $\exists f \quad f \in \mathcal{A}$ and $\forall \vartheta \subseteq f, \vartheta$ finite $\vartheta \notin \mathcal{A} \quad \Rightarrow \quad A$ not r.e.

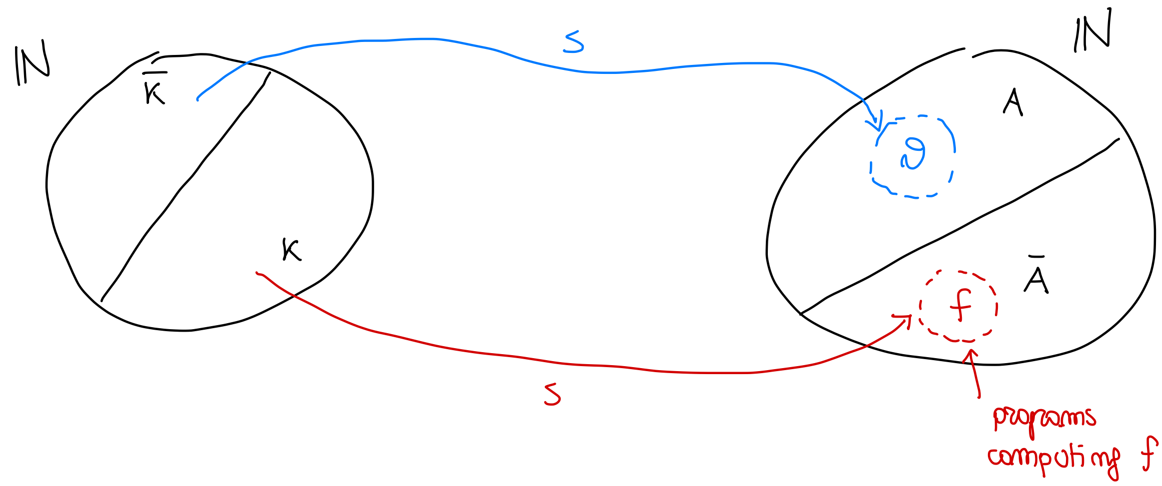
1. $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f, \vartheta$ finite s.t. $\vartheta \in \mathcal{A} \quad \Rightarrow \quad A$ not r.e.

let f be such that

$$f \notin \mathcal{A}$$

and let $\vartheta \subseteq f$, ϑ finite $\vartheta \in \mathcal{A}$

We show $\quad \bar{K} = \underbrace{\{ x \in \mathbb{N} \mid \varphi_x(x) \uparrow \}}_{\text{not r.e.}} \leq_m A$, hence A not r.e.



Define

$$g(x,y) = \begin{cases} \vartheta(y) & \text{if } x \in \bar{K} \\ f(y) & \text{if } x \in K \end{cases}$$

$$= \begin{cases} \vartheta(y) = f(y) & \text{if } x \in \bar{K} \text{ and } y \in dom(\vartheta) \\ \uparrow & \text{if } x \in \bar{K} \text{ and } y \notin dom(\vartheta) \\ f(y) & \text{if } x \in K \end{cases}$$

$$= \begin{cases} f(y) & \text{if } x \in K \text{ or } y \in dom(\vartheta) \\ \uparrow & (\text{if } x \in \bar{K} \text{ and } y \notin dom(\vartheta)) \text{ otherwise} \end{cases}$$

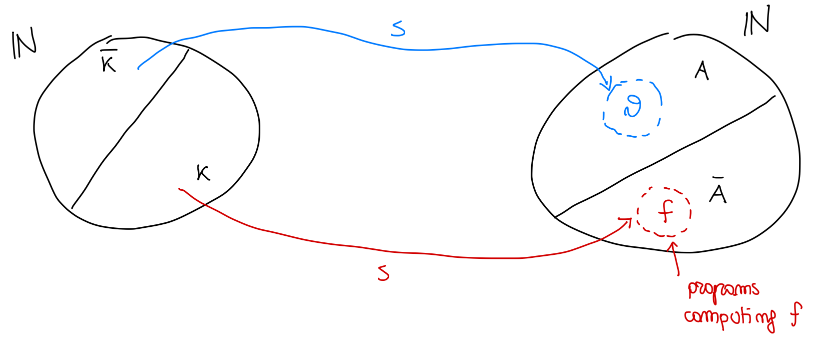
$$Q(x,y) \equiv \underbrace{\underbrace{x \in K}_{\text{semi-decidable}} \text{ or } \underbrace{y \in \underbrace{dom(\vartheta)}_{\text{finite}}}_{\text{decidable}}}_{\text{semidecidable} \Rightarrow sc_Q \text{ computable}}$$

$$= \begin{cases} f(y) & \text{if } Q(x,y) \\ \uparrow & \text{otherwise} \end{cases} = \underbrace{f(y) \cdot \underbrace{sc_Q(x,y)}_{\substack{1 \text{ if } Q(x,y) \\ \uparrow \text{ otherwise}}}}_{\text{computable}}$$

By smn theoreme there is $s: \mathbb{N} \to \mathbb{N}$ total computable such that for all x, y

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} \vartheta(y) & \text{if } x \in \bar{K} \\ f(y) & \text{if } x \in K \end{cases}$$

We show that s is the reduction function for $\bar{K} \leq A$



* if $x \in \bar{K}$ then $s(x) \in A$

if $x \in \bar{K}$ then $\forall y \quad \varphi_{s(x)}(y) = \vartheta(y)$. Hence $\varphi_{s(x)} = \vartheta$ and thus $s(x) \in A$

* if $x \in K$ then $s(x) \in \bar{A}$

if $x \in K$ then $\forall y \quad \varphi_{s(x)}(y) = f(y)$. Hence $\varphi_{s(x)} = f$ and thus $s(x) \in \bar{A}$

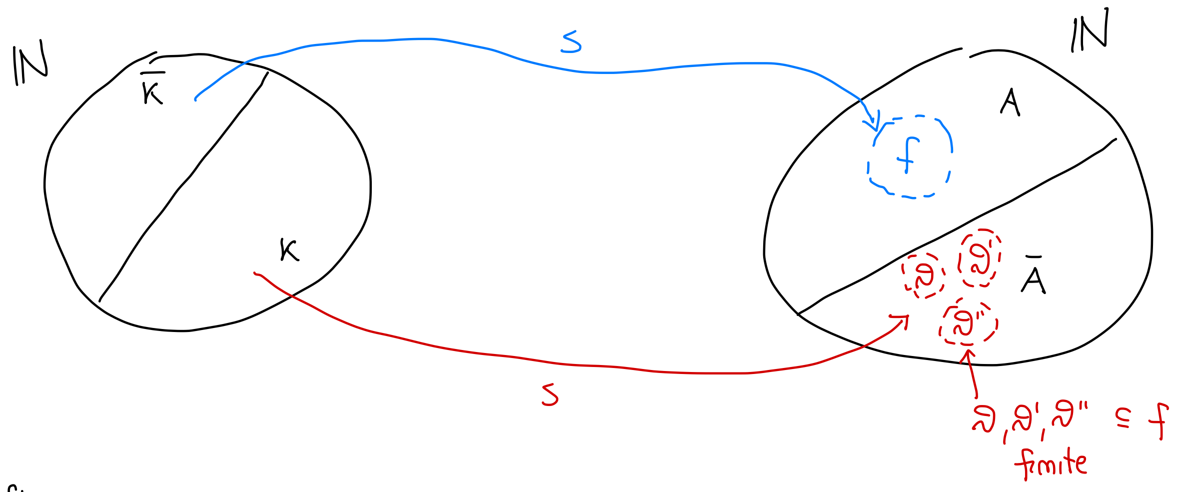
Hence s reduces $\bar{K} \leq_m A$ and we conclude A not r.e.

(2) $\exists f \quad f \in \mathcal{A}$ and $\forall \vartheta \subseteq f$, ϑ finite $\vartheta \notin \mathcal{A} \quad \Rightarrow A$ not r.e.

Let f be such that

$$f \in \mathcal{A} \quad \text{and} \quad \forall \vartheta \subseteq f, \ \vartheta \text{ finite} \quad \vartheta \notin \mathcal{A}$$

and let us show that $\bar{K} \leq_m A$ (hence A not r.e.)



Define

$$g(x,y) = \text{“}\begin{cases} f(y) & \text{if } x \in \bar{K} \\ \vartheta(y) & \text{if } x \in K \end{cases}\text{”}$$

ϑ: any subfunction of f

$\varphi_x(x)\uparrow \iff P_x(x)\uparrow$

$\varphi_x(x)\downarrow \iff P_x(x)\downarrow$

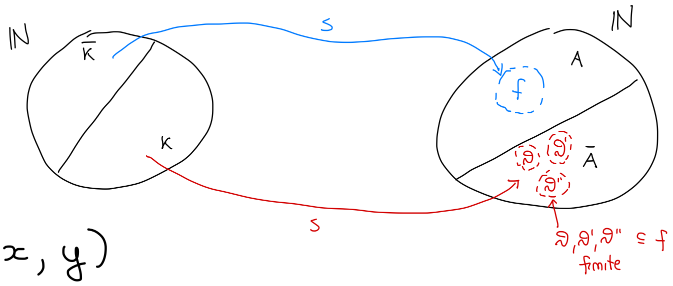
$$= \begin{cases} f(y) & \text{if } \neg H(x,x,y) \\ \uparrow & \text{if } H(x,x,y) \end{cases}$$

$$= f(y) + \underbrace{\mu w\, \underbrace{\chi_H(x,x,y)}_{\begin{cases}1 & \text{if } H(x,x,y)\\ 0 & \text{if } \neg H(x,x,y)\end{cases}}}_{\begin{cases}\uparrow & \text{if } H(x,x,y)\\ 0 & \text{if } \neg H(x,x,y)\end{cases}} \qquad \text{computable}$$

By smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ total computable s.t. $\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} f(y) & \text{if } \neg H(x,x,y) \\ \uparrow & \text{otherwise} \end{cases}$$

This is the reduction function for $\bar{K} \leq_m A$



* if $x \in \bar{K}$ then $s(x) \in A$

If $x \in \bar{K}$ then $\varphi_x(x) \uparrow$

i.e. $P_x(x) \uparrow$ and thus $\forall y \; \neg H(x,x,y)$

and therefore $\forall y \qquad \varphi_{s(x)}(y) = g(x,y) = f(y)$ i.e. $\varphi_{s(x)} = f$

hence $f(x) \in A$

* if $x \in K$ then $s(x) \notin A$

if $x \in K$ then $\varphi_x(x) \downarrow$ i.e. $P_x(x) \downarrow$. Let y_0 the number steps needed for P_x to halt on x i.e.

$$\forall y < y_0 \qquad \neg H(x,x,y)$$
$$\forall y \geq y_0 \qquad H(x,x,y)$$

Thus we have

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} f(y) & y < y_0 \\ \uparrow & \text{otherwise} \end{cases}$$

Therefore $\varphi_{s(x)} \subseteq f$ and $\mathrm{dom}(\varphi_{s(x)}) \subseteq [0, y_0)$ finite

hence $s(x) \in \bar{A}$

Therefore s reduces $\bar{K}$ to A and thus A not r.e. $\square$

Typical use of Rice-Shapiro : show that $A \subseteq \mathbb{N}$ not r.e.

by arguing that

- A is extensional / saturated $\qquad A = \{x \mid \varphi_x \in \mathcal{A}\}$

$\mathcal{A} \subseteq \mathcal{C}$

- $\mathcal{A}$ not finitary ((1) or (2))

(1) $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f$, ϑ finite s.t. $\vartheta \in \mathcal{A}$

(2) $\exists f \quad f \in \mathcal{A}$ and $\forall \vartheta \subseteq f$, ϑ finite $\vartheta \notin \mathcal{A}$

Exercise :

* T is not r.e. $\qquad$ ($T = \{x \mid \varphi_x$ total $\}$

$= \{x \mid \varphi_x \in \mathcal{T}\}$ where

$\mathcal{T} = \{f \mid f$ is total $\}$)

$id \in \mathcal{T} \qquad dom(id) = \mathbb{N}$

$\forall \vartheta \subseteq id$, ϑ finite $\quad \underbrace{dom(\vartheta)}_{\text{finite}} \subsetneq \mathbb{N} \quad$ i.e. $\vartheta \notin \mathcal{T}$

$\Rightarrow$ by Rice-Shapiro T is not r.e.

* $\bar{T}$ is not r.e.

$id \notin \bar{\mathcal{T}}$ $\qquad$ and if we let $\underbrace{\vartheta = \emptyset}_{\text{finite}} \subseteq id \qquad \vartheta \in \bar{\mathcal{T}}$

$\Rightarrow$ by Rice-Shapiro $\bar{T}$ is not r.e.

<u>EXERCISE</u> : $ONE = \{x \mid \varphi_x = \mathbb{1}\}$

$$\varphi_x \in \{\mathbb{1}\}$$

* <u>ONE is not r.e.</u>

  $\mathbb{1} \in \{\mathbb{1}\}$ and $\forall \vartheta \subseteq \mathbb{1}$, ϑ finite $\vartheta \notin \{\mathbb{1}\}$

  hence by Rice - Shapiro ONE is not r.e.

* <u>$\overline{ONE}$ is not r.e.</u>

  $\mathbb{1} \notin \overline{\{\mathbb{1}\}}$ and $\underset{\text{finite}}{\vartheta = \emptyset} \subseteq \mathbb{1}$ and $\vartheta \in \overline{\{\mathbb{1}\}}$

  hence by Rice - Shapiro $\overline{ONE}$ is not r.e.

* $B_n = \{x \mid n \in E_x\}$

  $= \{x \mid \varphi_x \in \mathcal{B}_n\}$ where $\mathcal{B}_n = \{f \mid n \in \mathrm{cod}(f)\}$

  This is finitary $\forall f \; (f \in \mathcal{B}_n$ iff $\exists \vartheta \subseteq f$, ϑ finite, $\vartheta \in \mathcal{B}_n)$

  EXERCISE

$\Downarrow$

A r.e.

NOOOOOO!

(not for this)

* $\boxed{\text{The converse implication for Rice-Shapiro is } \underline{\text{false}} \quad !}$

$$\mathcal{A} \subseteq \mathcal{C} \qquad A = \{x \mid \varphi_x \in \mathcal{A}\}$$

$$\forall f \ (f \in \mathcal{A} \text{ iff } \exists \vartheta \subseteq f, \ \vartheta \text{ finite}, \ \vartheta \in \mathcal{A})$$

$$\cancel{\Downarrow} \qquad \text{No!}$$

$$A \quad \text{r.e.}$$

<u>counter example</u>

$$\mathcal{A} \subseteq \mathcal{C} \qquad \mathcal{A} = \{ f \in \mathcal{C} \mid dom(f) \cap \bar{K} \neq \emptyset \}$$

observe that

(a) $\mathcal{A}$ is finitary $\quad \forall f \ (f \in \mathcal{A} \iff \exists \vartheta \subseteq f, \vartheta \text{ finite}, \vartheta \in \mathcal{A})$

($\Rightarrow$) let $f \in \mathcal{A}$, i.e.

$$dom(f) \cap \bar{K} \neq \emptyset$$

let $x_0 \in dom(f) \cap \bar{K}$ and we define $\quad \vartheta(x) = \begin{cases} f(x_0) & x = x_0 \\ \uparrow & \text{otherwise} \end{cases}$

then $\vartheta \subseteq f$, finite $\qquad \underbrace{dom(\vartheta)}_{=\{x_0\}} \cap \bar{K} = \{x_0\} \neq \emptyset$

hence $\vartheta \in \mathcal{A}$

($\Leftarrow$) let $\vartheta \subseteq f$ finite and assume $\vartheta \in \mathcal{A}$

$$\underbrace{dom(\vartheta)}_{\subseteq dom(f)} \cap \bar{K} \neq \emptyset$$

hence

$$dom(f) \cap \bar{K} \supseteq dom(\vartheta) \cap \bar{K} \neq \emptyset$$

i.e. $f \in \mathcal{A}$

(b) A not r.e.

$$A = \{x \mid \varphi_x \in \mathcal{A}\} = \{x \mid dom(\varphi_x) \cap \bar{K} \neq \emptyset\}$$

$$= \{x \mid W_x \cap \bar{K} \neq \emptyset\}$$

<u>idea</u>: if we were able to semi-decide $x \in A$ we could semi-decide $x \in \bar{K}$

given $x \in \mathbb{N}$

I build a program

```
def P(y):
    if y = x
        return 0
    else
        loop
```

and check if $\underbrace{\mathrm{dom}(P)}_{=\{x\}} \cap \bar{K} \neq \emptyset$ $\quad (\Leftrightarrow x \in \bar{K})$

More precisely $\bar{K} \leq_m A$

define

$$g(x,y) = \begin{cases} 0 & \text{if } y = x \\ \uparrow & \text{otherwise} \end{cases} = \mu w.\ |y-x| \qquad \text{computable}$$

$$= \varphi_{s(x)}(y) \qquad \text{with } s \text{ total computable by smn}$$

s is the reduction function for $\bar{K} \leq_m A$

$$x \in \bar{K} \Leftrightarrow \underbrace{\mathrm{dom}(\varphi_{s(x)})}_{=\{x\}} \cap \bar{K} \neq \emptyset \Leftrightarrow s(x) \in A$$