$BV(U) = \{ f \in L^1(U),\ \exists C > 0 \ \int_U f \operatorname{div} \underline{\Phi}\, dx \leq C$ for all $\underline{\Phi} \in C^1_c(U; \mathbb{R}^n)$ $\|\underline{\Phi}\|_\infty \leq 1 \}$

$= \{ f \in L^1(U)$ s. that $\forall i$, $\forall \varphi \in C^1_c(U; \mathbb{R}^n)$

$\left(-\int_U \frac{\partial \phi}{\partial x_i} f \right) = \frac{\partial}{\partial x_i} T_f(\phi) = \int_U \phi \, d\mu_i$ where μ_i is a finite signed Radon measure on $U \}$.

for $f \in BV(U)$

$V(f, U) :=$ total variation of f in U

$$= \sup \left\{ \int_U f \operatorname{div}(\underline{\Phi}) dx,\ \underline{\Phi} \in C^1_c(U; \mathbb{R}^n), \|\underline{\Phi}\|_\infty \leq 1 \right\}$$

- $f \in W^{1,1}(U)$ $V(f, U) = \sup \{ -\int Df \cdot \underline{\Phi}\, dx,\ \underline{\Phi} \in C^\infty_c(U, \mathbb{R}^n), \|\underline{\Phi}\|_\infty \leq 1 \} = \| |Df| \|_{L^1}$

- $\int_U f \operatorname{div} \underline{\Phi}\, dx = -\int_U \sum_i \varphi_i \underbrace{d\mu_i}_{\mu_i \ll |\mu_i|} = -\sum_i \int_U \phi_i \underbrace{\xi_i \, d|\mu_i|}_{|\xi_i| \leq 1} \leq + \sum_i |\mu_i|(U)$

Proposition Let $f_n \in BV(U)$ such that $f_n \to f$ in $L^1(U)$

① Assume $\exists C > 0$ $V(f_n, U) \leq C$ $\forall n$.
Then $f \in BV(U)$ and $V(f, U) \leq \liminf_n V(f_n, U)$.

② Assume $f \in BV(U)$ then $V(f, U) \leq \liminf_n V(f_n, U)$.

$\boxed{\text{So } f \to V(f, U) \text{ is LSC with respect to } L^1 \text{ CONVERG}}$

Proof. $V(f_n, U) = \sup\left\{ \int_U \operatorname{div} \Phi \, f_n \, , \ \|\Phi\|_\infty \leq 1 \ \ \Phi \in C^1_c(U; \mathbb{R}^n) \right\}$

$\forall \Phi$.
$$\geq \int_U (\operatorname{div} \Phi) f_n \xrightarrow[L^1 \text{ conv.}]{} \int_U (\operatorname{div} \Phi) f$$ SINCE $\operatorname{div} \Phi \in C_c(U)$

$$\liminf_n V(f_n, U) \geq \int_U (\operatorname{div} \Phi) f \, dx \qquad \forall \Phi \in C^1_c(U, \mathbb{R}^n), \ \|\Phi\|_\infty \leq 1.$$

1) If $\liminf_n V(f_n, U) \leq C \Rightarrow \int_U f \operatorname{div}\phi \leq C \quad \forall \phi \in C^1_c(U, \mathbb{R}^n)$
$\|\phi\|_\infty \leq 1$

$\Rightarrow f \in BV(U)$

2) If $f \in BV(U) \Rightarrow \liminf_n V(f_n, U) \geq \int_U f \operatorname{div}\phi \quad \forall \phi \in C^1_c(U, \mathbb{R}^n)$
$\|\phi\|_\infty \leq 1$

taking the supremum on all ϕ

$\Rightarrow \liminf_n V(f_n, U) \geq V(f, U)$.

Obs If $f_n \to f$ in $L^1(U)$ and $f_n, f \in BV(U)$, it may happen that $\liminf_n V(f_n, U) > V(f, U)$

ex $f_n(x) = \frac{1}{n}\sin(nx) \in W^{1,1}(-\pi, \pi)$ $\qquad f_n'(x) = \cos(nx)$

$f_n(x) \to 0$ in $L^1(-\pi, \pi)$ $\qquad V(f_n, (-\pi, \pi)) = \|\cos(nx)\|_{L^1} = 4 > V(0, (-\pi, \pi))$

Note that by Riemann Lebesgue $\lim_n \int_{-\pi}^{\pi} f_n'(x)\phi(x) = 0 \quad \forall \phi \in C_0(-\pi, \pi)$

$\|f\|_{BV} = \|f\|_{L^1(U)} + V(f,U)$. If $f \in W^{1,1}(U)$ $\|f\|_{BV}$ is equivalent to $\|f\|_{W^{1,1}}$

Thm $(BV(U), \|\cdot\|_{BV})$ is Banach. $(W^{1,1}(U), \|\cdot\|_{1,1}) \xhookrightarrow{\text{CONT.}} (BV(U), \|\cdot\|_{BV})$.

f_n Cauchy in $BV(U) \Rightarrow f_n$ is Cauchy in $L^1(U) \Rightarrow f_n \to f$ in $L^1(U)$

f_n is bdd in $BV(U) \Rightarrow V(f_n, U) \le C \Rightarrow f \in BV(U)$.

$f_n - f_m \to f - f_m$ $n \to +\infty$ $\Rightarrow 0 \le V(f - f_m, U) \le \liminf_n V(f_n - f_m, U) \to 0$.

CONVERGENCE IN BV(U).

1) STRONG CONVERGENCE $f_n \to f$ in BV $\|f_n - f\|_{BV} \to 0$ i.e

$\not\uparrow\downarrow$ $f_n \to f$ in $L^1(U)$ and $V(f_n - f, U) \to 0$

2) STRICT CONVERGENCE $f_n \to f$ in strict sense if

$\not\uparrow\downarrow$ $f_n \to f$ in $L^1(U)$ and $V(f_n, U) \to V(f, U)$

(METRIZABLE $d(f,g) = \int_U |f-g| + |V(f,U) - V(g,U)|$

3) "WEAK CONVERGENCE" (should be weak* convergence) $f_n \xrightarrow{*} f$ in BV if

$\mu_i = \frac{\partial}{\partial x_i} T_f$

$\mu_i^n = \frac{\partial}{\partial x_i} T_{f_n}$

$f_n \to f$ in $L^1(U)$ and $\forall \phi \in C_0(U)$ $\forall i$ $\int_U \phi \, d\mu_i^n \to \int_U \phi \, d\mu_i$

ex $f_n = \begin{cases} 0 & 0<x<\frac{1}{2} \\ n(x-\frac{1}{2}) & x\in(\frac{1}{2},\frac{1}{2}+\frac{1}{n}) \\ 1 & x\in(\frac{1}{2}+\frac{1}{n},1) \end{cases}$

$f_n \to \chi_{(\frac{1}{2},1)}$ in $L^1(0,1)$

$\|f_n'\|_{L^1} = 1 \to V(\chi_{(\frac{1}{2},1)},(0,1))$

] $f_n \to \chi_{(\frac{1}{2},1)}$ strictly non strongly

STRICT $\not\Rightarrow$ STRONG

ex $f_n = \frac{1}{n}\sin(nx) \xrightarrow{*} 0$ weakly non strictly

$\int_{-\pi}^{\pi} \phi \cdot \cos(nx)\,dx \to 0$ (by Riemann-Lebesgue lemma)

WEAK $\not\Rightarrow$ STRICT

weak convergence is NOT the convergence in $BV(U)$ with respect to dual space (hard to characterize).

(is strong convergence in L^1 + weak* convergence of the Radon measures associated to the distributional gradients)

it is actually a weak* convergence since BV can be seen as the dual of a space

$BV \subseteq \mathcal{M}^{n+1} = \{(\mu_0 \dots \mu_n)$ μ_i signed Radon measures

$T: f \longmapsto \mu_0 = f\,dx \quad \mu_i = \frac{\partial}{\partial x_i} T_f$.

$\overline{E} = \{(\phi_0 \dots \phi_n),\ \operatorname{div}\phi_0 = (\phi_1 \dots \phi_n)\} \subseteq (C_0(U), \|\ \|_\infty)^{n+1}$

$(Tf, \phi) = 0 \quad \forall \phi \in \overline{E}$ $\qquad BV(\mathbb{R}^n) \cong (C_0(U), \|\cdot\|_\infty)^{n+1} / \overline{E}$

Theorems for BV functions

MEYERS-SERRIN for BV (density of smooth functions in STRICT SENSE)

Let $U \subseteq \mathbb{R}^n$. $f \in BV(U) \iff \exists f_k \in C^\infty(U) \cap W^{1,1}(U)$ such that
$f_k \to f$ in $L^1(U)$ & $V(f_k, U) \to V(f, U)$ (i.e. $f_k \to f$ in strict sense)
(NOT $\|f_k - f\|_{BV} \to 0$! (otherwise $f \in W^{1,1}(U)$..))

NO PROOF JUST IDEA with the same argument as in Meyers-Serrin

we reduce to $f \in BV(U)$ with compact support $\operatorname{supp} f \subset W \subset\subset V$

$\varepsilon < \frac{\operatorname{dist}(W, \mathbb{R}^n \setminus U)}{2}$ $f_\varepsilon = f * \eta_\varepsilon \to f$ in L^1 $\Rightarrow$ $V(f, U) \leq \liminf_{\varepsilon \to 0} V(f_\varepsilon, U)$.

Moreover $f_\varepsilon \in C^\infty_c(U) \Rightarrow f_\varepsilon \in W^{1,1}(U)$

$$V(f_\varepsilon, U) = \sup \Big[\int_U (f * \eta_\varepsilon) \underbrace{\operatorname{div} \Phi}_{\|\Phi\|_\infty \leq 1} = \int_U f \, (\eta_\varepsilon * \operatorname{div} \Phi) = \int_U f \operatorname{div}(\eta_\varepsilon * \Phi)$$

component by component

sufficient to consider Φ such that $\operatorname{supp} \Phi \subseteq W + B(0, \varepsilon)$

$\leq V(f, U)$.

EXTENSION: U bdd set of class C^1 $(n \geq 1)$
For any V bdd $V \supset\supset U$ $E: BV(U) \to BV(\mathbb{R}^n)$ continuous and app $E(f) \subseteq V$ $\forall f \in BV(U)$.

TRACE : U bdd set of class C^1, $n > 1$.
$Tr: BV(U) \longrightarrow L^1(\partial U)$

GAGLIARDO-NIRENBERG-SOBOLEV INEQ: $n > 1$, $\forall f \in BV(\mathbb{R}^n)$

$$\|f\|_{L^{n/n-1}} \leq V(f, \mathbb{R}^n).$$

proof By MEYERS-SERRIN $\exists g_k \in W^{1,1}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, $g_k \to f$ in L^1

and $V(f, \mathbb{R}^n) = \lim_k V(g_k, \mathbb{R}^n) = \lim_k \| |Dg_k| \|_{L^1(\mathbb{R}^n)}$

by (GNS) applied to $g_k \Rightarrow \|g_k\|_{L^{n/n-1}} \leq \| |Dg_k| \|_{L^1}$

by FATOU lemma since $g_k \to f$ in L^1 (and a.e. up to passing to a subsequence)

$$\liminf_k \|g_k\|_{L^{n/n-1}} \geq \|f\|_{L^{n/n-1}}$$

$$V(f, \mathbb{R}^n) = \lim_k \| |Dg_k| \|_{L^1} \geq \lim_k \|g_k\|_{L^{n/n-1}} \geq \|f\|_{L^{n/n-1}}$$

HELLY'S THEOREM (Compactness theorem for weak convergence)

U open bdd of class C^1 ($n = 1$ U open interval)

$f_k \in BV(U)$ $\exists C > 0$ $\|f_k\|_{BV(U)} \le C$ $\forall k$.

Then up to a subsequence $f_k \to f$ in $L^p(U)$ $\forall p \in [1, \frac{n}{n-1})$ (1^*)

$f \in BV(U) \cap L^{\frac{n}{n-1}}(U)$ and $V(f, U) \le \liminf_k V(f_k, U)$.

Proof by density in strict sense of $W^{1,1}$ functions we have $\forall k$ $\exists g_k \in W^{1,1}(U)$ such that

$$\|f_k - g_k\|_{L^1(U)} \le \frac{1}{k} \qquad |V(f_k, U) - V(g_k, U)| \le \frac{1}{k}.$$

$$\Rightarrow \|g_k\|_{BV} \le C \Rightarrow \|g_k\|_{W^{1,1}} \le \bar{C}$$

by Rellich-Kondrachov $\exists$ subsequence g_k and $f \in L^{n/n-1}(U)$ such that $\|g_k - f\|_{L^q(U)} \to 0$ $\forall q \in [1, \frac{n}{n-1})$

$$\|f_k - f\|_{L^1} \leq \|f_k - g_k\|_{L^1} + \|g_k - f\| \leq \frac{1}{k} + \|g_k - f\|_{L^1}$$

$\Rightarrow f_k \to f$ in $L^1(U)$ $\Rightarrow$ by Proposition - since f_k is bounded in BV $(V(f_k, U) \leq C)$. $f \in BV(U)$ and liminf $V(f_k, U) \geq V(f, U)$.

By (GNS) applied to the extension of f, f_k

$$\|f_k - f\|_{L^{n/n-1}} \leq \tilde{C}\|f_k - f\|_{BV(U)} \leq \tilde{C}\|f_k\|_{BV} + \tilde{C}\|f\|_{BV} \leq \overline{C}.$$

$\Rightarrow$ by interpolation $\forall\, q \in [1, \frac{n}{n-1})$

$$\|f_k - f\|_{L^q} \leq \|f_k - f\|_{L^1}^{\theta} \|f_k - f\|_{L^{n/n-1}}^{1-\theta} \leq \|f_k - f\|_{L^1}^{\theta} \cdot \overline{C}^{1-\theta} \to 0.$$

Poincaré inequality

U bdd of class C^1 and CONNECTED

$\exists\, C>0$ s.that $\forall f \in BV(U)$

$$\left\| f - \frac{1}{|U|}\int_U f(y)\,dy \right\|_{L^{\frac{n}{n-1}}} \le C\, V(f,U).$$

proof Assume not true (proof by contradiction)

$\forall k \; \exists f_k \in BV(U) \quad \left\| f - \frac{1}{|U|}\int_U f(y)\,dy \right\|_{L^{\frac{n}{n-1}}} \ge k\, V(f_k, U)$

$$\Rightarrow v_k = \frac{f_k - \frac{1}{|U|}\int_U f_k}{\left\| f_k - \frac{1}{|U|}\int_U f_k \right\|_{L^{\frac{n}{n-1}}}} \qquad V(v_k,U) = \frac{V(f_k,U)}{\left\| f_k - \frac{1}{|U|} f_k \right\|_{L^{\frac{n}{n-1}}}}.$$

$\underline{\|v_k\|_{L^{\frac{n}{n-1}}} = 1} \qquad \boxed{V(v_k,U) \le \tfrac{1}{k} < 1} \quad \xRightarrow{\text{by HELLY}} \exists f \quad v_{k_i} \to f \text{ in } L^1$

$\Downarrow \|v_k\|_{L^1} \le C$ (Hölder)

$$0 = \liminf_{k_j} V(v_k, U) \ge V(f,U)$$

$V(f,U) = 0 \Longrightarrow \frac{\partial}{\partial x_i} T_f \equiv 0 \;\; \forall i \Rightarrow f$ is constant