lu FRATTI SEMPLICI
mer CAMBIO VARIABILE
gio ESERCIZI misti

$\int \arctg x$ $\qquad \int \frac{x}{1+x^2}$

$$\int tg x = \int \frac{\operatorname{sen} x}{\cos x} dx =$$

$$= -\int \frac{1}{\cos x} \cdot (\cos x)' dx$$

$$= -\lg |\cos x|$$

$$\int \frac{3x+1}{2x^2+3} dx = \int \frac{3x}{2x^2+3} dx + \int \frac{1}{2x^2+3} dx$$

$g(x) = 2x^2+3 \qquad g' = 4x$

$$\frac{3}{4} \int \frac{4x}{2x^2+3} + \int \frac{1}{2x^2+3} dx$$

$$\frac{3}{4} \lg |2x^2+3| + \frac{1}{\sqrt{2} \cdot \sqrt{3}} \arctg \left(\sqrt{\frac{2}{3}}\, x \right) + C$$

Giovedì 12 dicembre 9–11.40

Venerdì 13 dicembre non c'è lezione di Analisi.

$$\int f(x)g(x)\,dx = \underbrace{F(x)\,g(x)} - \underbrace{\int F(x)\,g'(x)\,dx} + c$$

$\downarrow$

F è primitiva di f $\quad (F'(x) = f(x))$

$$\int \operatorname{arctg} x\,dx = \int \underset{f}{1}\cdot \underset{g}{\operatorname{arctg} x}\,dx = x\cdot\operatorname{arctg} x - \underbrace{\int x\,\frac{1}{1+x^2}\,dx}_{+c}$$

$f(x) = 1 \longrightarrow F(x) = x$

$g(x) = \operatorname{arctg} x \rightarrow g'(x) = \dfrac{1}{1+x^2}$

quindi devo calcolare $\int x \cdot \frac{1}{1+x^2}\,dx = \boxed{\int \frac{x}{1+x^2}\,dx}$

$(1+x^2)' = 2x$ $\qquad h(x) = 1+x^2 > 0 \qquad x = \frac{1}{2} \cdot 2x = \frac{1}{2} \cdot h'(x)$

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{2}\frac{h'(x)}{h(x)}\,dx = \frac{1}{2} \int \frac{h'(x)}{h(x)}\,dx$$

$$= \frac{1}{2} \lg(h(x)) + c = \frac{1}{2} \lg(1+x^2) + c$$

$$(\lg(h(x)))' = \frac{1}{h(x)} \cdot h'(x)$$

$$\int \operatorname{arctg} x \, dx = x \operatorname{arctg} x - \frac{1}{2} \lg(1+x^2) + C$$

OSSERVAZIONE se $h(x) > 0$

$$\int \frac{h'(x)}{h(x)} dx = \lg(h(x)) + C$$

perché $\left(\lg(h(x))\right)' = \frac{h'(x)}{h(x)}$

$$\int \frac{h'(x)}{h(x)} dx = \lg|h(x)| + C$$

(perché $h(x) \neq 0$)

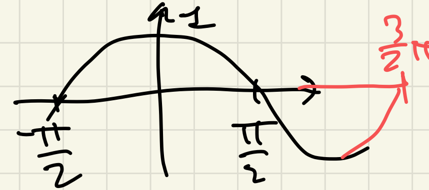
$\forall$ intervallo dove $h(x) > 0 \quad \forall x \in I$

oppure $h(x) < 0 \quad \forall x \in I$

$\int \operatorname{tg} x \, dx$

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\operatorname{tg} x = \frac{\operatorname{sen} x}{\cos x}$



$\int \frac{\operatorname{sen} x}{\cos x} \, dx$

$f(x) = \cos x > 0$

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f'(x) = (\cos x)' = -\operatorname{sen} x$

$= -\int \frac{-\operatorname{sen} x}{\cos x} \, dx = -\int \frac{f'(x)}{f(x)} \, dx = -\log(\cos x) + c$

$x \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$

$\int \operatorname{tg} x \, dx = -\log|\cos x| + c$

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Es

$$\int_0^1 \frac{e^x}{e^x+1}\,dx = \lg(e^1+1) - \lg(e^0+1)$$

$$= \lg(e+1) - \lg(2) = \lg\left(\frac{e+1}{2}\right)$$

$$\int \frac{e^x}{e^x+1}\,dx =$$

$$h(x) = e^x + 1 > 0$$

$$h'(x) = e^x + 0 = e^x$$

$$= \int \frac{h'(x)}{h(x)}\,dx = \lg(e^x+1) + c$$

Es $\int_1^2 \frac{2x+3}{x^2+5}\,dx = \left(\lg(2^2+5) + \frac{3}{\sqrt5}\,\text{arctg}\,\frac{1}{\sqrt5}\cdot 2\right) + $

$- \left(\lg(1^2+5) + \frac{3}{\sqrt5}\,\text{arctg}\,\frac{1}{\sqrt5}\cdot 1\right)$

$$\int \frac{2x+3}{x^2+5}\,dx = \int \frac{2x}{x^2+5}\,dx + \int \frac{3}{x^2+5}\,dx$$

$$\int \frac{a'(x)}{a(x)}\,dx$$

$a(x) = x^2+5 > 0$

$a' = 2x$

$$\int \frac{1}{ax^2+b}\,dx = \frac{1}{\sqrt{a}\cdot\sqrt{b}}\cdot\text{arctg}\left(\sqrt{\frac{a}{b}}\,x\right)+c \qquad a,b>0$$

$$= \lg(x^2+5) + 3\,\frac{1}{\sqrt5}\,\text{arctg}\left(\frac{1}{\sqrt5}\,x\right) + c$$

Es

$$\int \frac{x-2}{x^2+2x-3}\,dx$$

$$x^2+2x-3=0 \iff x_{1,2}=\frac{-2\pm\sqrt{4+12}}{2}=$$

$$=\frac{-2\pm4}{2}=\begin{cases}-3\\1\end{cases}$$

METODO DEI FRATTI SEMPLICI

$$\int \frac{dx+e}{ax^2+bx+c}\,dx$$

$d, e \in \mathbb{R}$

$a, b, c \in \mathbb{R}$

ax^2+bx+c polinomio di 2° grado con 2 soluzioni (radici) reali x_1, x_2

$$ax^2+bx+c = a(x-x_1)(x-x_2)$$

$$1x^2+2x-3 = 1\cdot(x-(-3))(x-1) = (x+3)(x-1)$$

$a=1$ $\quad x_1=-3$ $\quad x_2=1$

$$\frac{dx+e}{ax^2+bx+c} = \frac{dx+e}{a\cdot(x-x_1)(x-x_2)} = \frac{1}{a}\cdot\frac{dx+e}{(x-x_1)(x-x_2)}$$

$$\frac{dx+e}{(x-x_1)(x-x_2)} = \frac{A}{(x-x_1)} + \frac{B}{(x-x_2)} = \frac{A(x-x_2)+B(x-x_1)}{(x-x_1)(x-x_2)}$$

$$dx+e = Ax - Ax_2 + Bx - Bx_1$$

$$x(A+B) + (-Ax_2 - Bx_1)$$

$$\begin{cases} A+B = d \\ -Ax_2 - Bx_1 = e \end{cases}$$

le incognite sono A, B

$d = 1 \quad e = -2$

$$\frac{x-2}{x^2+2x-3} = \frac{x-2}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

$$x_1 = -3 \qquad x_2 = 1$$

$$= \frac{A(x-1) + B(x+3)}{(x+3)(x-1)}$$

$$= \frac{Ax - A + Bx + 3B}{(x+3)(x-1)}$$

$$= \frac{(A+B)x - A + 3B}{(x+3)(x-1)}$$

$$\begin{cases} A+B = d \\ -Ax_1 - Bx_2 = e \end{cases}$$

$$\begin{cases} A+B = 1 \\ -A+3B = -2 \end{cases}$$

$$\begin{cases} A = 1-B \\ -(1-B)+3B = -2 \end{cases} \quad \begin{cases} A = 1-B \\ -1+B+3B = -2 \end{cases} \quad \begin{cases} A = \frac{5}{4} \\ B = -\frac{1}{4} \end{cases}$$

$$\frac{x-2}{x^2+3x-2} = \frac{5/4}{x+3} + \frac{-1/4}{x-1} = \frac{5}{4}\frac{1}{(x+3)} - \frac{1}{4}\frac{1}{(x-1)}$$

$$\frac{dx+e}{ax^2+bx+c} = \frac{1}{a}\left(\frac{A}{x-x_1} + \frac{B}{x-x_2}\right)$$

$$\int \frac{dx+e}{ax^2+bx+c}\,dx = \frac{1}{a}\left[\int \frac{A}{x-x_1}\,dx + \int \frac{B}{x-x_2}\,dx\right] =$$

$$= \frac{1}{a}\left[A \lg|x-x_1| + B \lg|x-x_2|\right] + c$$

$$= \frac{A}{a}\lg|x-x_1| + \frac{B}{a}\lg|x-x_2| + c$$

$$\begin{cases} A+B=d \\ -Ax_2 - Bx_1 = e \end{cases}$$

$x_1\ x_2$ soluz

$ax^2+bx+c=0$

$$\int \frac{x-2}{x^2+3x-2}\,dx = \frac{5}{4}\lg|x+3| - \frac{1}{4}\lg|x-1| + C$$

$$\int \frac{x-2}{x^2+3x-2}\,dx = \int \frac{5}{4}\,\frac{1}{x+3}\,dx - \int \frac{1}{4}\,\frac{1}{x-1}\,dx$$

$A = \frac{5}{4}$ $\quad x_1 = -3$

$B = -\frac{1}{4}$ $\quad x_2 = 1$

$$\int_2^4 \frac{x-2}{x^2+3x-2}\,dx = \left[\frac{5}{4}\lg|4+3| - \frac{1}{4}\lg|4-1|\right] + $$

$$-\left[\frac{5}{4}\lg|2+3| - \frac{1}{4}\lg|2-1|\right] =$$

$$= \frac{5}{4}\lg 7 - \frac{1}{4}\lg 3 - \frac{5}{4}\lg 5 .$$

es

$$\int \frac{x-1}{x^2+3x-2}\,dx = \int \frac{1}{x+3}\,dx = \lg|x+3| + c$$

$x_1 = -3 \quad x_2 = 1$

A, B

$$\begin{cases} A+B = 1 \\ -A\cdot 1 - B(-3) = -1 \end{cases}$$

$$\begin{cases} A+B = d \\ -A x_2 - B x_1 = e \end{cases}$$

$$\begin{cases} A+B = 1 \\ -A + 3B = -1 \end{cases} \qquad \begin{cases} A = 1 \\ B = 0 \end{cases}$$

$$\frac{x-1}{x^2+3x-2} = \frac{A}{x+3} + \frac{B}{x-1} = \frac{1}{x+3} + \frac{0}{x-1}$$

$$\frac{x-1}{(x+3)(x-1)}$$

Es $\int_0^1 \frac{3}{2x^2+6x+4}\,dx$

$2x^2+6x+4=0 \rightarrow x_{1,2} = \frac{-6 \pm \sqrt{36-32}}{4}$

$= \frac{-6 \pm 2}{4} = \begin{cases} -1 = x_1 \\ -2 = x_2 \end{cases}$

$2x^2+6x+4 = 2\,(x-(-1))\,(x-(-2)) =$

$= 2\,(x+1)\,(x+2)$

$ax^2+bx+c = a\,(x-x_1)\,(x-x_2)$

$$\frac{3}{2x^2+6x+4} = \frac{3}{2(x+1)(x+2)} = \frac{3}{2}\left[\frac{1}{(x+1)(x+2)}\right] =$$

$$= \frac{3}{2}\left[\frac{A}{x+1} + \frac{B}{x+2}\right] = \frac{3}{2}\left[\frac{Ax+2A+Bx+B}{(x+1)(x+2)}\right]$$

$$= \frac{3}{2}\left[\frac{x(A+B)+2A+B}{(x+1)(x+2)}\right] = \frac{3}{2}\left[\frac{1}{x+1} - \frac{1}{x+2}\right]$$

$$\begin{cases} A+B = 0 = d \\ 2A+B = 1 = e \end{cases} \quad \begin{cases} A = -B \\ -2B+B = 1 \end{cases} \quad \begin{cases} A = -B \\ -B = 1 \end{cases}$$

$$\begin{cases} A = 1 \\ B = -1 \end{cases}$$

$$\int \frac{3}{2x^2+6x+4}\,dx = \frac{3}{2}\left[\int \frac{1}{x+1}\,dx - \int \frac{1}{x+2}\,dx\right] =$$

$$= \frac{3}{2}\left[\lg|x+1| - \lg|x+2|\right] + c =$$

$$= \frac{3}{2}\lg\left(\frac{|x+1|}{|x+2|}\right) + c$$

$$\lg a - \lg b = \lg \frac{a}{b}$$
$$a, b > 0$$

$$\int_0^1 \frac{3}{2x^2+6x+4}\,dx = \frac{3}{2}\lg\left(\frac{1+1}{1+2}\right) - \frac{3}{2}\lg\left(\frac{0+1}{0+2}\right)$$

$$= \frac{3}{2}\lg\left(\frac{2}{3}\right) - \frac{3}{2}\lg\left(\frac{1}{2}\right) = \frac{3}{2}\lg\left(\frac{4}{3}\right)$$

Es

$$\int_2^e \frac{1}{x \lg x}\, dx = \lg(\lg e) - \lg(\lg 2) =$$

($\lg e = 1$)

$$= \lg(1) - \lg(\lg 2) = -\lg(\lg 2)$$

$$h(x) = \lg x > 0 \qquad h'(x) = \frac{1}{x}$$

$x \in [2, e]$

$$\int \frac{h'(x)}{h(x)}\, dx = \lg|h(x)| + c$$

$$\frac{h'(x)}{h(x)} = h'(x) \cdot \frac{1}{h(x)} = \frac{1}{x}\, \frac{1}{\lg x}$$

$$\int \frac{1}{x \lg x}\, dx = \lg(\lg x) + c$$

es

$$\int_0^1 \frac{3x}{x^2-2}\,dx$$

$$\left[\int_{-1}^{1} \frac{3x}{x^2-2}\,dx = 0\right]$$

1ª strada

$$\int \frac{3x}{x^2-2}\,dx$$

$$h(x) = x^2-2$$

$$h'(x) = 2x$$

$$\int \frac{3x}{x^2-2} = \frac{3}{2}\int \frac{2x}{x^2-2}\,dx = \frac{3}{2}\int \frac{h'(x)}{h(x)}\,dx =$$

$$= \frac{3}{2}\log|h(x)| + c$$

$$= \frac{3}{2}\log|x^2-2| + c$$

$$\int_0^1 \frac{3x}{x^2-2}\,dx = \frac{3}{2}\log|1-2| - \frac{3}{2}\log|0-2|$$

$$= \frac{3}{2}\log 1 - \frac{3}{2}\log 2$$

2ª strada applico fratti semplici

$$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$$

$$\frac{3x + 0}{x^2 - 2} = \frac{A}{x - \sqrt{2}} + \frac{B}{x + \sqrt{2}} = \frac{Ax + \sqrt{2}A + Bx - \sqrt{2}B}{(x - \sqrt{2})(x + \sqrt{2})}$$

$$\begin{cases} A + B = 3 = d \\ \sqrt{2}A - \sqrt{2}B = 0 = e \end{cases} \qquad \begin{cases} 2A = 3 \\ A = B \end{cases} \qquad \begin{cases} A = \frac{3}{2} \\ B = \frac{3}{2} \end{cases}$$

$$\frac{3x}{x^2 - 2} = \frac{3}{2} \frac{1}{x - \sqrt{2}} + \frac{3}{2} \cdot \frac{1}{x + \sqrt{2}}$$

$$\int \frac{3x}{x^2-2}\,dx = \frac{3}{2}\int \frac{1}{x-\sqrt{2}}\,dx + \frac{3}{2}\int \frac{1}{x+\sqrt{2}}\,dx =$$

$$= \frac{3}{2}\lg|x-\sqrt{2}| + \frac{3}{2}\lg|x+\sqrt{2}| + c$$

$$= \frac{3}{2}\lg\left(|x-\sqrt{2}|\,|x+\sqrt{2}|\right) + c \qquad \lg a + \lg b = \lg(a\cdot b)$$

$$= \frac{3}{2}\lg|x^2-2| + c$$

$$\int_0^1 \frac{3x}{x^2-2}\,dx = \cancel{\frac{3}{2}\lg|1-2|} - \frac{3}{2}\lg|0-2| = -\frac{3}{2}\lg 2.$$