Back to the example of calculus of variations

Let $g \in L^p(U)$ given. U bdd of class C^1.

$\exists\ \bar{f}$ minimizer of $E(f) = \int_U |Df|^p + |f-g|^p \, dx$

$\bar{f}$ is obtained by the direct method in the calculus of variation.

<u>Obs</u> $\bar{f}$ is UNIQUE $E(f) = \int_U F(x, f, Df) \, dx$

with $F(x, \cdot, \cdot)$ STRICTLY CONVEX

assume $\exists\ \bar{f}_1 \neq \bar{f}_2$ minimizers $E(\bar{f}_1) = E(\bar{f}_2) = c = \min E(f)$

$\lambda \in (0,1)$

$c \geq E(\lambda \bar{f}_1 + (1-\lambda)\bar{f}_2) > \lambda E(\bar{f}_1) + (1-\lambda) E(\bar{f}_2) = c$

impossible

CHARACTERIZATION of the MINIMIZER (AS SOLUTION of a DIFFERENTIAL EQUATION)

$$p = 2 \qquad E(\bar{f}) = \min_{f \in W^{1,2}(U)} \int_U |Df|^2 + |f-g|^2 \, dx$$

$$\forall \phi \in C_c^\infty(U) \qquad E(\bar{f}+\varepsilon\phi) \geq E(\bar{f}) \qquad \forall \varepsilon \neq 0$$

$$E(\bar{f}+\varepsilon\phi) - E(\bar{f}) = \int_U 2\varepsilon \cdot D\bar{f} \cdot D\phi + \varepsilon^2 |D\phi|^2 + 2\varepsilon(\bar{f}-g)\cdot\phi + \varepsilon^2|\phi|^2$$

$$\lim_{\varepsilon \to 0} \frac{E(\bar{f}+\varepsilon\phi) - E(\bar{f})}{\varepsilon} = \int_U 2 D\bar{f} \cdot D\phi + 2(\bar{f}-g)\cdot\phi = 0$$

$\Rightarrow$ $\bar{f}$ solves in the sense of DISTRIBUTIONS $-\Delta\bar{f} + (\bar{f}-g) = 0$ in U

HILBERT XIX problem: from regularity of g deduce regularity of $\bar{f}$, distributional sol of $-\Delta\bar{f} = g - \bar{f}$

(SOLVED by DE GIORGI NASH $\to$ $g \in L^2(U) \Rightarrow f \in W^{2,2}(U)$)

Note that $\forall \phi \in C_c^\infty(\mathbb{R}^n)$ (so $\phi \in C^\infty(\bar{U})$)

$\bar{f} + \varepsilon\phi \in W^{1,p}(U) \implies$

$$\frac{E(\bar{f}+\varepsilon\phi) - E(\bar{f})}{\varepsilon} = \int_U \{ D\bar{f} \cdot D\phi + 2(\bar{f}-g)\phi \, dx =$$

$$= \underbrace{\int_U -2\bar{f}\Delta\phi + 2(\bar{f}-g)\phi}_{=0} + \int_{\partial U} D\phi \cdot \nu_U \, Tr\bar{f} \, d\mathcal{H}^{n-1}(y) = 0$$

$$\rightarrow \int_{\partial U} \overbrace{D\phi \cdot \nu_U}^{\text{NORMAL DERIVATIVE of } \phi} \, Tr\bar{f} \, d\mathcal{H}^{n-1}(y) = 0 \qquad \forall \phi \in C^\infty(\mathbb{R}^n)$$

(this is a sort of very weak boundary condition)

$\downarrow$ in very weak sense

"normal derivative of $\bar{f} = 0$"

THE SPACE of FUNCTIONS of BOUNDED VARIATION

IMPORTANT : CONSERVATION LAWS (dim 1), CALCULUS of VARIATIONS problems with WITH LINEAR GROWTH, ISOPERIMETRIC PROBLEMS/GEOMETRIC PROBLEMS.

Ref : Ambrosio-Fusco-Pallara

Obs $f \in L^p(U)$ $U \subseteq \mathbb{R}^n$ open. $p \in [1,+\infty]$

$$f \in W^{1,p}(U) \Longrightarrow \exists C>0 \quad \forall \Phi \in C_c^\infty(U,\mathbb{R}^n)$$

$$\left| \int_U (\operatorname{div} \Phi) f \right| = \left| \sum_i \int_U \frac{\partial \phi_i}{\partial x_i} f \right| \leq \|\Phi\|_{L^{p'}} C.$$

$$C = \sum_i \left\| \frac{\partial f}{\partial x_i} \right\|_{L^p} \quad (\text{Hölder inequality}).$$

where the viceversa is true? $\Longleftarrow$ $p \in [1,+\infty]$ $\forall i$ $\underset{C_c^\infty(U)}{\phi} \longmapsto \int_U f \frac{\partial \phi}{\partial x_i}$ is cont. with respect to $L^{p'}$ norm

$T_i(\phi) = \int_U f \frac{\partial \phi}{\partial x_i} dx$ can be extended to a linear continuous functional

$T_i : L^{p'} \longrightarrow \mathbb{R}$ $\Rightarrow$ by (Riesz) $(L^{p'})' = L^p$ $p \in (1,+\infty)$

$T_i \in (L^{p'})'$ (dual) $\qquad (L^1)' = L^\infty$ $p = +\infty$.

$\Rightarrow \exists\, g_i \in L^{p'}$ such that $T_i(\phi) = \int_U \phi g_i = \int_U f \frac{\partial \phi}{\partial x_i}\, dx$

$\forall \phi \in C_c^\infty(U)$

$\to g_i = -\frac{\partial f}{\partial x_i} \in L^p \;.\; \Rightarrow f \in W^{1,p}(U).$

$\boxed{\text{for } p = 1 \text{ NOT TRUE}}$

$BV(U) = \{ f \in L^1(U) \text{ s.th. } \exists\, C \; \int_U f \operatorname{div} \Phi \, dx \le C \|\Phi\|_\infty \;\; \forall \Phi \in C_c^\infty(U, \mathbb{R}^n) \} =$

$= \{ f \in L^1(U) \text{ s.th. } \int_U f \operatorname{div} \Phi \, dx \le C \;\; \forall \Phi \in C_c^\infty(U, \mathbb{R}^n), \|\Phi\|_\infty \le 1 \}$

$= \{ f \in L^1(U) \;\; \forall i \;\; T_i : \phi \longmapsto \int f \frac{\partial \phi}{\partial x_i}$ is a zero order distribution $\}$

$W^{1,1}(U) \subseteq BV(U)$ $\qquad$ $\left(f \in W^{1,1}(U) \to C = \sum_i \left\| \frac{\partial f}{\partial x_i} \right\|_{L^1} \right)$

$W^{1,1}(U) \neq BV(U)$ $\qquad$ [ex $f \in L^1(0,1)$ CANTOR FUNCTION
$(T_f)'$ is a zero order distribution.]

Let $f \in BV(U)$

$T_i : \phi \longrightarrow \int_U \frac{\partial \phi}{\partial x_i} f$ is a distribution of order 0 :

$(T_i(\phi) \leq C\|\phi\|_\infty) \Rightarrow$ T_i can be extended to $(C_0(U), \|\cdot\|_\infty)$ as a linear continuous functional

Obs 1

every linear continuous functional T on $(C_0(U), \|\cdot\|_\infty)$ can be written as the difference between two positive functionals $T = T^+ - T^-$

$\forall \phi \geq 0 \quad \phi \in C_0(U) \qquad T^+(\phi) = \sup \{ T(\psi) \quad 0 \leq \psi \leq \phi \}$

$T^+(\phi) \geq 0 \qquad T^+(\phi) \geq T(\phi)$

$T^+(\phi) = T^+(\phi^+) - T^+(\phi^-) \qquad \phi = \phi^+ - \phi^-$

$\phi^+ = \max(\phi, 0) \geq 0$

$T^-(\phi) := T^+(\phi) - T(\phi \qquad \phi^- = \max(-\phi, 0) \geq 0$

(NOTE THAT $\phi \geq 0 \quad T^+(\phi) \geq T(\phi)$).

Obs 2 By Riesz theorem every T positive linear functional on $\mathcal{C}_c(U)$ is associated to a Radon measure μ (POSITIVE MEASURE)

$$T(\phi) = \int \phi \, d\mu \quad \forall \phi \in \mathcal{C}_c(U)$$

T LINEAR POSITIVE $\Rightarrow$ T satisfies: $\forall K \subset\subset U$ $\exists C_K > 0$ such that

$$|T(\varphi)| \leq C_K \|\varphi\|_\infty \quad \forall \phi \in \mathcal{C}_c(U) \text{ with } \operatorname{supp} \phi \subseteq K.$$

If the previous inequality is satisfied for C independent of K $\left(|T(\varphi)| \leq C \|\phi\|_\infty \ \forall \phi \in \mathcal{C}_c(U) \right)$

T can be extended to a positive linear functional on $C_0(U)$, and $\mu(U) < +\infty$ (the associated Radon measure is finite) by Obs 1 + Obs 2 we get that every linear continuous functional T on $C_0(U)$ is ASSOCIATED TO a SIGNED FINITE RADON MEASURE on U.

$$T = T^+ - T^-$$

$$T^+(\varphi) = \int_U \phi \, d\mu^+ \qquad \mu^+(U) < +\infty$$

$$T^-(\varphi) = \int_U \phi \, d\mu^- \qquad \mu^-(U) < +\infty$$

$$\Rightarrow T(\varphi) = \int_U \phi \, d\mu = \int_U \phi \, d\mu^+ - \int_U \phi \, d\mu^- \qquad \boxed{\mu = \mu^+ - \mu^-}$$

Let $f \in BV(U)$. So $f \in L^1(U)$ and $T_i = \frac{\partial}{\partial x_i} T_f$ is a linear continuous functional on $C_0(U)$ $\forall i = 1, \dots, n$

$\Longrightarrow$

$\exists\ \mu_i \overset{\mu_i^+ - \mu_i^-}{=}$ signed (finite) Radon measure on U such that $T_i(\phi) = \int_U \phi \, d\mu_i(x) = \int_U \phi \, d\mu_i^+ - \int_U \phi \, d\mu_i^-$

$BV(U) = \{ f \in L^1(U)$ such that the distributional derivatives of f are signed Radon measures$\}$

Let $f \in BV(U)$. Then $f \in W^{1,1}(U)$ IF AND ONLY IF $\forall\, i = 1..n$ $\mu_i = \mu_i^+ - \mu_i^-$ (signed finite Radon measure such that $\int_U f \frac{\partial}{\partial x_i}\phi \, dx = -\int_U \phi \, d\mu_i$) is ABSOLUTELY CONTINUOUS with respect to Lebesgue ($\mu_i \ll \mathcal{L}$) and $\forall\, i$ $\frac{\partial f}{\partial x_i}$ (weak derivative of f) is the density of μ_i $\left(d\mu_i = \frac{\partial f}{\partial x_i} dx\right)$