POINCARÉ INEQUALITY

$U \subseteq \mathbb{R}^n$ open bdd of class C^1 and CONNECTED

there $\forall p \in [1, +\infty]$. $\exists\, C = C(n, p, U) > 0$ such that.

$$\left\| f - \frac{1}{|U|}\int_U f(y)dy \right\|_{L^p(U)} \leq C(n,p,U) \, \| |Df| \|_{L^p(U)}$$

If $U = B(x,r)$ $\quad C(n,p,B(x,r)) = r\, C(n,p,B(0,1))$

Observation: $C = \{ f \in W^{1,p}(U) \quad \int_U f(y)dy = 0 \} \subseteq W^{1,p}(U)$ (closed)

$(C, \|\cdot\|_{1,p})$ Banach. In C, $\| |Df| \|_{L^p}$ is an equivalent norm to $\|f\|_{W^{1,p}}$.

proof by contradiction $\forall k \; \exists f_k \quad \|f_k - \frac{1}{|U|}\int_U f_k \|_p \geq k \| |Df_k| \|_{L^p}$

$$v_k(y) = \frac{f_k(y) - \frac{1}{|U|}\int_U f_k\, dx}{\| f_k - \frac{1}{|U|}\int_U f_k \|_{L^p}} \qquad \|v_k\|_{L^p} = 1 \qquad \int_U v_k\, dx = 0$$

$$\|Dv_k\|_{L^p} = \frac{\|Df_k\|_{L^p}}{\|f_k - \frac{1}{|U|}\int_U f_a\|_{L^p}} \leq \frac{1}{k} \leq 1$$

v_k is bdd in $W^{1,p}(U)$ $\rightarrow$ $\exists\, v_{k_j} \rightarrow v$ strongly in L^p

$\Rightarrow$ since $\|v_{k_j}\|_{L^p} = 1 \Rightarrow \|v\|_{L^p} = 1$

$\int_U v_{k_j}\, dx = 0 \Rightarrow \int_U v\, dx = 0$

$$\left[\int_U v = \int_U v_{k_j} + \int_U v - v_{k_j} = \int_U v - v_{k_j} \longrightarrow 0 \quad \left(\begin{array}{l}\text{since} \\ v - v_{k_j} \xrightarrow{L^p} 0\end{array}\right)\right]$$

$\| |Dv_k| \|_{L^p} \to 0$ as $k \to +\infty$ $\Rightarrow$ v has weak derivatives and $\frac{\partial v}{\partial x_i} \equiv 0$.

$\forall \phi \in C_c^\infty(U)$ $\quad \left| \int_U \phi \cdot \frac{\partial}{\partial x_i} v_k \right| \leq \|\phi\|_{L^{p'}} \|\frac{\partial}{\partial x_i} v_k\|_{L^p} \longrightarrow 0$

$$0 \leftarrow \int_U \phi \frac{\partial}{\partial x_i} v_k = (-1)\int_U \frac{\partial}{\partial x_i}\phi\, v_k \longrightarrow (-1)\int_U \left(\frac{\partial}{\partial x_i}\phi\right) v \Rightarrow \frac{\partial}{\partial x_i} v = 0 \quad \forall i$$

$$\left[\begin{array}{l} U \text{ connected} \\ \frac{\partial v}{\partial x_i} = 0 \quad \forall i \end{array}\right] \Rightarrow v \text{ constant} \Rightarrow \text{but } \left.\begin{array}{l} \|v\|_{L^p} = 1 \\ \int_U v = 0 \end{array}\right] \underline{\text{ABSURD}}$$

$$\sim \cdot \sim$$

If $U = B(x,r)$ $\quad \left\| f - ⨍_{B(x,r)} f(y) \right\|_{L^p} \le C(n,p)\, r \, \| |Df| \|_{L^p}$.

$\hookrightarrow$ POINCARE' CONSTANT of $B(0,1)$

$f \in W^{1,p}(B(x,r))$ $\quad v(y) := f(x + r y) \quad y \in B(0,1)$

$\to$ then $v \in W^{1,p}(B(0,1))$ $\quad$ Poincaré per v:

$$\left\| v - ⨍_{B(0,1)} v(y)\,dy \right\|_{L^p} \le C(n,p) \|Dv\|_{L^p}$$

$$\|Dv\|_{L^p} = \left[\int_{B(0,1)} r^p |Df(x+ry)|^p dy \right]^{1/p} = \left[r^p r^{-n} \int_{B(x,r)} |Df|^p dy \right]^{1/p} = r^{1-\frac{n}{p}} \| |Df| \|_{L^p}$$

$$\frac{1}{\omega_n} \int_{B(0,1)} v(y)\,dy = \frac{1}{\omega_n} \int_{B(0,1)} f(x+ry)\,dy = \frac{r^{-n}}{\omega_n} \int_{B(x,r)} f(y)\,dy = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy .$$

$$\|v - c\|_{L^p} = \left[\int_{B(0,1)} [f(x+ry) - c]^p dy \right]^{1/p} = \left[r^{-n} \int_{B(x,r)} [f(y) - c]^p dy \right]^{1/p} = r^{-n/p} \|f - c\|_{L^p}$$

$$\cancel{r^{-n/p}} \left\| f - ⨍_{B(x,r)} f(y) \right\|_{L^p} \le C(n,p)\, r^{1 \cancel{-n/p}} \|Df\|_{L^p}$$

From Poincaré we deduce the following:

$W^{1,n}(\mathbb{R}^n)$ is embedded in $BMO(\mathbb{R}^n)$

Recall $W^{1,n}(\mathbb{R}^n) \not\subset L^\infty(\mathbb{R}^n)$ but $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ $\forall q<+\infty$.

$f \in BMO$ (BOUNDED MEAN OSCILLATION) space introduced in the '60 by John - Nirenberg

$$BMO(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n),\ \sup_{B \subseteq \mathbb{R}^n} \frac{1}{|B|}\int_B \left| f(y) - \frac{1}{|B|}\int_B f(z)\,dz \right| dy < +\infty \right\}$$

(obviously $L^\infty(\mathbb{R}^n) \subseteq BMO(\mathbb{R}^n)$ but also other functions → typically eg $(\log(1+|x|))$)

$f \sim g$ in BMO if $\exists C \in \mathbb{R}$ $f(x) = g(x) + C$ a.e. $x \in \mathbb{R}^n$

$BMO/\sim$ is a Banach space with norm

$$\|f\|_{BMO} = \sup_{B \subseteq \mathbb{R}^n} \frac{1}{|B|}\int_B \left| f(y) - \frac{1}{|B|}\int_B f(z)\,dz \right| dy$$

Let $f \in W^{1,n}(\mathbb{R}^n)$ → $f \in BMO(\mathbb{R}^n)$

proof: $f \in W^{1,n}(\mathbb{R}^n) \to f \in L^1(\mathbb{R}^n) \to$ take $B(x,r) \subseteq \mathbb{R}^n$ and apply Poincaré

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \left| f - \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y) \right| \overset{\text{POINCARE' for } p=1}{\leq} \frac{1}{|B(x,r)|} C(n,1)\, r \int_{B(x,r)} |Df|\, dx \leq (\text{Hölder } L^n, L^{\frac{n}{n-1}})$$

$$\leq \frac{1}{|B(x,r)|} C(n,1)\, r\, \| Df \|_{L^n(B(x,r))} \cdot |B(x,r)|^{1-\frac{1}{n}} =$$

$$\leq \frac{1}{\omega_n \cancel{r^n}} C(n,1) \cancel{r}\, \| |Df| \|_{L^n(\mathbb{R}^n)}\, \omega_n^{1-\frac{1}{n}} \cancel{(r^n)^{1-\frac{1}{n}}} =$$

$$= \frac{C(n,1)}{\omega_n^{1/n}} \| |Df| \|_{L^n(\mathbb{R}^n)}$$

$$\boxed{\| f \|_{BMO} \leq \frac{C(n,1)}{\omega_n^{1/n}} \| |Df| \|_{L^n(\mathbb{R}^n)}}$$

$C(n,1)$ Poincaré constant for $B(0,1)$

Typical problem in calculus of variations

Let $U \subseteq \mathbb{R}^n$ open bdd of class C^1.

$p \in (1, +\infty)$ $\quad g \in L^p(U)$ fixed.

Define $E(f) := \int_U |Df|^p + |f-g|^p \, dx \qquad f \in W^{1,p}(U)$

Pb: $\exists \bar{f} \in W^{1,p}(U)$ s. that $E(\bar{f}) = \min_{f \in W^{1,p}(U)} E(f)$?

1) Let $c = \inf_{f \in W^{1,p}(U)} E(f) \qquad c \in [0, +\infty)$ obvious.

2) Let f_k be a MINIMIZING SEQUENCE: $\quad f_k \in W^{1,p}(U)$

$c \leq E(f_k) \leq c + \frac{1}{k} \quad \forall k.$

We observe that $\exists C > 0 \quad \|f_k\|_{W^{1,p}(U)} \leq C$

$$\bar{C} + 1 \geq C + \frac{1}{k} \geq \int_U |Df_k|^p + |f_k - g|^p \rightarrow \|f_k\|_{W^{1,p}} \leq C(\bar{C}, p, \|g\|_p)$$

[since $\|f_k - g\|_{L^p} \geq \|f_k\|_{L^p} - \|g\|_{L^p}$]

By compact embedding $\underline{\exists f \in W^{1,p}(U)}$ (true ONLY for $p > 1$)

and a subsequence f_{k_i} (just continue to call it f_k)

$$f_k \longrightarrow \bar{f} \text{ strongly in } L^p(U) \Rightarrow \int_U |f_k - g|^p dx \rightarrow \int_U |\bar{f} - g|^p$$

$$\underbrace{\frac{\partial}{\partial x_i} f_k \longrightarrow \frac{\partial}{\partial x_i} \bar{f}}_{} \text{ weakly in } L^p(U)$$

by convexity $\displaystyle \int_U |Df_k|^p \geq \int_U |D\bar{f}|^p + \underbrace{\int_U \overbrace{p\, |D\bar{f}|^{p-2} D\bar{f}}^{\in L^{p'}} \cdot (D f_k - D\bar{f})\, dx}_{\rightarrow 0}$

$$\Longrightarrow \liminf_k \int_U |Df_k|^p \geq \int_U |D\bar{f}|^p$$

[also directly: the norm $\|\cdot\|_p$ is weakly LSC)
that is $g_k \rightharpoonup g$ in $L^p \Rightarrow \liminf_k \|g_k\|_p \geq \|g\|_p$
(consequence of Banach–Steinhaus thm.
UNIFORM BOUNDEDNESS PRINCIPLE)]

So $c = \liminf_k E(f_k) \geq E(\bar{f}) \Rightarrow E(\bar{f}) = \min_{f \in W^{1,p}(U)} E(f)$.

$\bar{f}$ is the best "regular" approximation of g.

$\bar{f}$ UNIQUE in this case by convexity

assume $f_1 \neq f_2$ minima. $\Rightarrow E\left(\frac{f_1+f_2}{2}\right) < \frac{E(f_1)}{2} + \frac{E(f_2)}{2}$

$E(f_1) = E(f_2) = c$

IMPOSSIBLE

CHARACTERIZATION of the MINIMIZER.

$p = 2 \qquad E(\bar{f}) = \min_{f \in W^{1,2}(U)} \int_U |Df|^2 + |f-g|^2 \, dx$

$\forall \phi \in C^\infty_c(U) \qquad E(\bar{f} + \varepsilon\phi) \geq E(\bar{f}) \qquad \forall \varepsilon \neq 0$

$E(\bar{f}+\varepsilon\phi) - E(\bar{f}) = \int_U 2\varepsilon \cdot D\bar{f} \cdot D\phi + \varepsilon^2 |D\phi|^2 + 2\varepsilon(\bar{f}-g)\cdot\phi + \varepsilon^2 |\phi|^2$

$\lim_{\varepsilon \to 0} \frac{E(\bar{f}+\varepsilon\phi) - E(\bar{f})}{\varepsilon} = \int_U 2 D\bar{f} \cdot D\phi + 2(\bar{f}-g)\cdot\phi = 0$

$\Rightarrow \bar{f}$ solves in the sense of DISTRIBUTIONS $-\Delta \bar{f} + (\bar{f}-g) = 0$ in U

HILBERT XIX problem: from regularity of g deduce regularity of $\bar{f}$, distributional sol of $-\Delta\bar{f} = g - \bar{f}$

(SOLVED by DE GIORGI NASH $\to g \in L^2(U) \Rightarrow f \in W^{2,2}(U)$

$g \in L^k(U) \to f \in W^{2+k,2}(U)$

"Partial differential equation"

PDE characterization of the minimizer done with the Gateaux derivative of $E(f)$

$\phi \in C_c^\infty(U)$ $\qquad f \in \mathcal{A}$ minimizer

$f + \varepsilon \phi \in \mathcal{A}$ for $\varepsilon \in \mathbb{R}$ (maybe suff. small)

by minimality $\dfrac{E(f+\varepsilon\phi) - E(f)}{\varepsilon} \geq 0$ if $\varepsilon > 0$

≤ 0 if $\varepsilon < 0$

$$\lim_{\varepsilon \to 0^+} \frac{E(f+\varepsilon\phi) - E(f)}{\varepsilon} = 0 \qquad \forall \phi \in C_c^\infty(U)$$

→ from this one deduces an equation satisfied by f in the SENSE of DISTRIBUTIONS.

What we used?

$$E(f) = \int_U F(x, f, Df)\, dx \qquad \boxed{F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}}$$

1) F coercive $\quad F(x, u, q) \geq C(|u|^p + |q|^p) - k$

(to have uniform bound on minimizing sequences)

2) $F(x, u, \cdot)$ convex (to have LSC with respect to weak convergence of the gradient)

3) $F(x, \cdot, \cdot)$ convex to have uniqueness of the minimizer

4) Gateaux derivative of $E(f)$ at the minimizer is 0

characterization of the minimum as a solution in sense of distribution of a differential equation

"DIRECT METHODS in the CALCULUS of VARIATIONS"

(by Tonelli – 1920-1930)

for $p = 1$ NOT WORKING $\quad f_k \to f \qquad f \notin W^{1,1}(U)$!