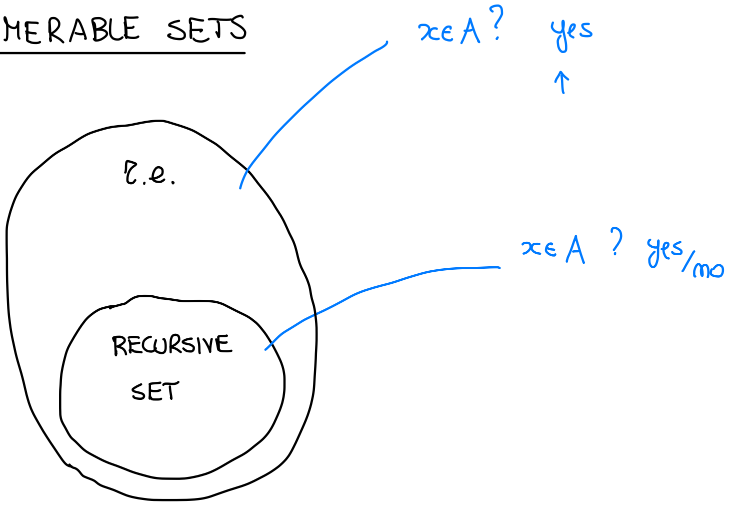
COMPUTABILITY (03/12/2024)

RECURSIVELY ENUMERABLE SETS



Def. (r.e. set): A set $A \subseteq \mathbb{N}$ is recursively enumerable (r.e.) if the semi-characteristic function $sc_A : \mathbb{N} \to \mathbb{N}$

$$sc_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$ is computable

A property $Q(\vec{x}) \subseteq \mathbb{N}^k$ is <u>semi-decidable</u> if

$$sc_Q(\vec{x}) = \begin{cases} 1 & \text{if } Q(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases}$$ is computable

OBSERVATION: if $Q(x) \subseteq \mathbb{N}$

$Q(x)$ is semi-decidable iff $\{x \mid Q(x)\} \subseteq \mathbb{N}$ is r.e.

(we could define recursive / r.e. sets $A \subseteq \mathbb{N}^k$)

OBSERVATION: Let $A \subseteq \mathbb{N}$ be a set

A recursive $\iff$ $A, \bar{A}$ r.e.

proof

($\Rightarrow$) Let $A \subseteq \mathbb{N}$ be recursive, i.e.

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$ computable

we want to show that

$$SC_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

Intuitively, I have $P_{\chi_A} \begin{array}{l} \nearrow 1 \\ \searrow 0 \end{array}$ for $x \in A$?

then def $P_{SC_A}(x)$:

if $P_{\chi_A}(x) = 1$
return 1
else loop

formally:

$$SC_A(x) = \mathbb{1}\Big(\underbrace{\mu w. \underbrace{|\chi_A(x) - 1|}_{\begin{cases} 1 & \text{if } x \notin A \\ 0 & \text{if } x \in A \end{cases}}}_{\begin{cases} 0 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}}\Big)$$

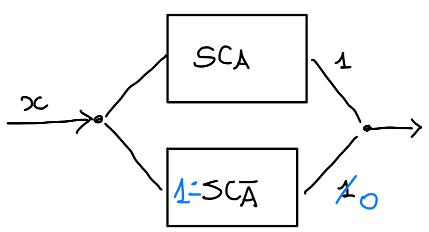
$$\underbrace{\qquad\qquad}_{\begin{cases} 1 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}}$$

computable (composition + minimisation of computable functions)

Concerning $\bar{A}$, just observe that since A is recursive $\Rightarrow$ $\bar{A}$ is recursive
hence by the argument above $\bar{A}$ r.e.

($\Leftarrow$) Let $A, \bar{A}$ r.e., i.e. the functions below are computable

$$SC_A(x) = \begin{cases} 1 & x \in A \\ \uparrow & \text{otherwise} \end{cases} \qquad 1 \dot{-} SC_{\bar{A}}(x) = \begin{cases} \cancel{1}\ 0 & x \notin A \\ \uparrow & \text{otherwise} \end{cases}$$



let e_0, e_1 be indexes for $1 \dot{-} sc_{\bar{A}}$ and sc_A

$$1 \dot{-} sc_{\bar{A}} = \varphi_{e_0} \qquad sc_A = \varphi_{e_1}$$

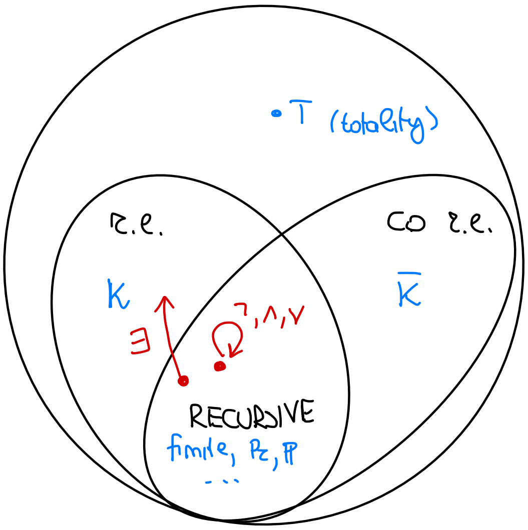
$$\chi_A(x) = \text{"}\left(\mu(y,t) \, . \, S(e_0, x, y, t) \vee S(e_1, x, y, t) \right)_y \text{"}$$

$$= \left(\mu \omega \, . \, S(e_0, x, (\omega)_1, (\omega)_2) \vee S(e_1, x, (\omega)_1, (\omega)_2) \right)_1$$

$$\omega = \begin{array}{l} (\omega)_1 = y \\ (\omega)_2 = t \end{array}$$

computable, hence A is recursive

co r.e. $\Leftrightarrow$ complement is r.e.



* **K not recursive but r.e.**

$$sc_K(x) = \begin{cases} 1 & \text{if } \varphi_x(x) \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mathbb{1}\left(\varphi_x(x)\right)$$

$$= \mathbb{1}\left(\varphi_U(x, x)\right)$$

* **$\bar{K}$ is not r.e.**

otherwise $K, \bar{K}$ r.e. $\leadsto$ $K, \bar{K}$ recursive

* **Existential quantification**

$Q(t, \vec{x}) \subseteq \mathbb{N}^{k+1}$ decidable

$P(\vec{x}) \equiv \exists t. \; Q(t, \vec{x})$ semi-decidable

STRUCTURE THEOREM :

Let $P(\vec{x}) \subseteq \mathbb{N}^k$ be a predicate

$P(\vec{x})$ semi-decidable $\Leftrightarrow$ there is $Q(t, \vec{x}) \subseteq \mathbb{N}^{k+1}$ decidable such that $P(\vec{x}) \equiv \exists t. \; Q(t, \vec{x})$

proof

($\Rightarrow$) Let $P(\vec{x}) \subseteq \mathbb{N}^k$ semi-decidable

$$sc_P(\vec{x}) = \begin{cases} 1 & \text{if } P(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases} \quad \text{is computable}$$

i.e. there is $e \in \mathbb{N}$ s.t. $\varphi_e^{(k)} = sc_P$

Observe $P(\vec{x})$ iff $sc_P(\vec{x}) = 1$ iff $\exists t.\ S^{(k)}(e, \vec{x}, 1, t)$

If we define

$$Q(t, \vec{x}) \equiv S^{(k)}(e, \vec{x}, 1, t) \quad \text{decidable}$$

and

$$P(\vec{x}) \equiv \exists t.\ Q(t, \vec{x})$$

($\Leftarrow$) We assume $P(\vec{x}) \equiv \exists t.\ Q(t, \vec{x})$ with $Q(t, \vec{x})$ decidable

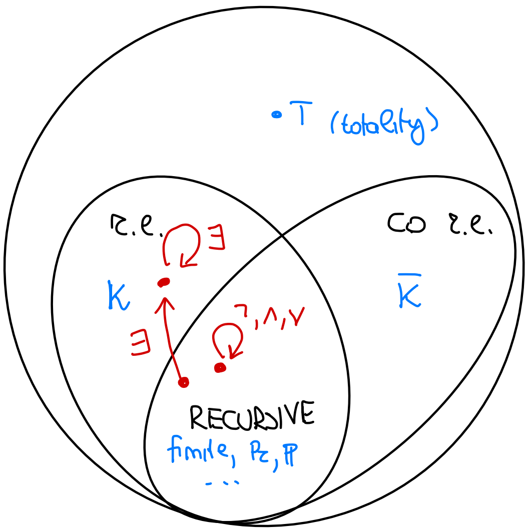
$$sc_P(\vec{x}) = \begin{cases} 1 & \text{if } P(\vec{x}) \Leftrightarrow \exists t.\ Q(t,\vec{x}) \Leftrightarrow \exists t.\ \chi_Q(t,\vec{x}) = 1 \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mathbb{1}(\mu t.\ |\chi_Q(t, \vec{x}) - 1|) \quad \text{computable}$$

hence $P(\vec{x})$ r.e.

$\square$

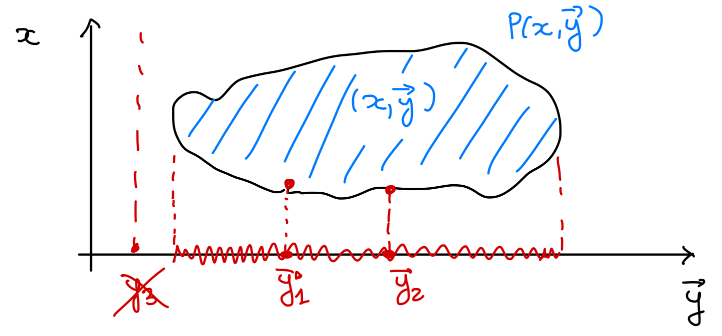
What about further existential quantification ?



Projection Theorem

Let $P(x,\vec{y}) \subseteq \mathbb{N}^{k+1}$ semi-decidable.

Then also $R(\vec{y}) \equiv \exists x.\ P(x,\vec{y})$ is semi-decidable



proof

let $P(x,\vec{y})$ semi-decidable. By the structure theorem

$$P(x,\vec{y}) \equiv \exists t.\ Q(t,x,\vec{y}) \quad \text{with} \quad Q(t,x,\vec{y}) \subseteq \mathbb{N}^{k+2} \text{ decidable}$$

Then

$$\boxed{R(\vec{y})} \equiv \exists x.\ P(x,\vec{y}) \equiv \exists x.\ \exists t.\ Q(t,x,\vec{y})$$

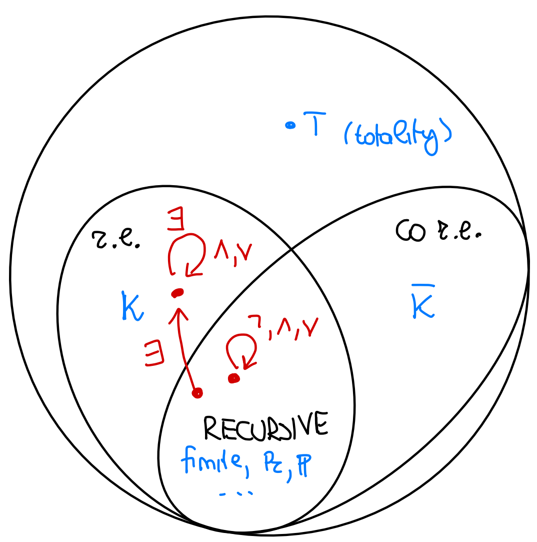
$$\equiv \exists \omega.\ \underbrace{Q((\omega)_1, (\omega)_2, \vec{y})}$$

$Q'(\omega,\vec{y}) \equiv Q((\omega)_1,(\omega)_2,\vec{y})$ decidable

$$\boxed{\equiv \exists \omega.\ Q'(\omega,\vec{y})}$$

By the structure theorem, $R(\vec{y})$ is semi-decidable.

$\square$



Conjunction / Disjunction

OBSERVATION : Let $P(\vec{x}), Q(\vec{x}) \subseteq \mathbb{N}^k$ be semi-decidable

Then ① $P(\vec{x}) \wedge Q(\vec{x})$
② $P(\vec{x}) \vee Q(\vec{x})$
are semi-decidable

proof

By the structure theorem

$$P(\vec{x}) \equiv \exists t.\ P'(t, \vec{x})$$
$$Q(\vec{x}) \equiv \exists t.\ Q'(t, \vec{x})$$

with $P'(t,\vec{x})$, $Q'(t,\vec{x})$ decidable

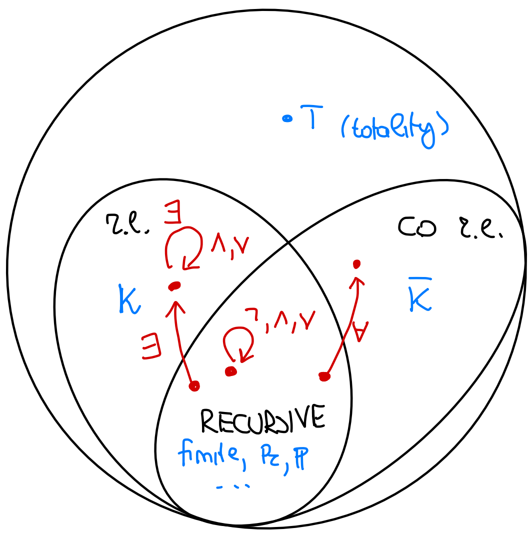
Then

① $$P(\vec{x}) \wedge Q(\vec{x}) \equiv \exists t_1.\ P'(t_1, \vec{x}) \wedge \exists t_2.\ Q'(t_2, \vec{x})$$
$$\equiv \exists \omega.\ (\underbrace{P'((\omega)_1, \vec{x}) \wedge Q'((\omega)_2, \vec{x})}_{\text{decidable}})$$

hence $P(\vec{x}) \wedge Q(\vec{x})$ is semi-decidable by the structure theorem.

② $$P(\vec{x}) \vee Q(\vec{x}) \equiv \exists t.\ P'(t, \vec{x}) \vee \exists t.\ Q'(t, \vec{x})$$
$$\equiv \exists t.\ (\underbrace{P'(t, \vec{x}) \vee Q'(t, \vec{x})}_{\text{decidable}})$$

hence $P(\vec{x}) \vee Q(\vec{x})$ is semi-decidable by the structure theorem.



* NEGATION

$Q(x) \equiv$ " $x \in K$ " semi-decidable

but

$\neg Q(x) \equiv$ " $x \notin K$ "

not semi-decidable

* <u>Universal quantification</u>

$R(t,x) \equiv \neg H(x,x,t)$ decidable

$$\forall t.\ R(t,x) \equiv \forall t.\ \neg H(x,x,t) \equiv x \notin K$$

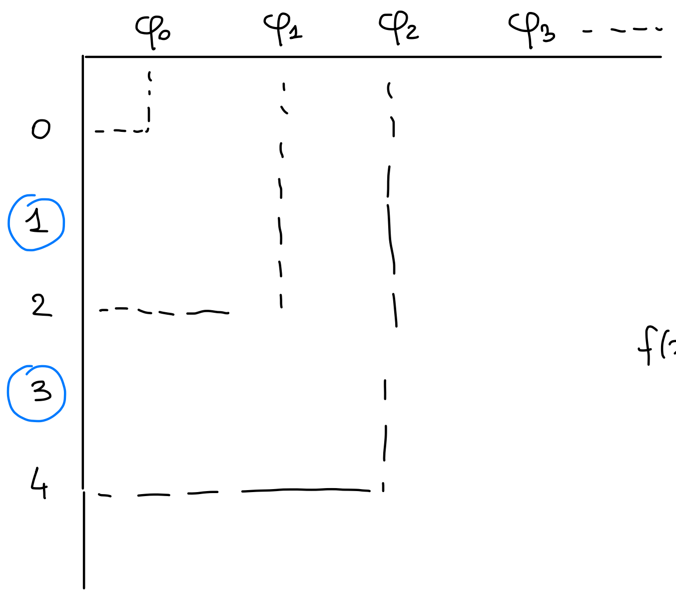
non semi-decidable.

<u>EXERCISE</u> : Define a function total and non computable

$f : \mathbb{N} \to \mathbb{N}$ s.t. $f(x) = x$ on infinitely many $x \in \mathbb{N}$

($\{x \mid f(x) = x\}$ is infinite)

<u>1st idea</u>



$$f(x) = \begin{cases} x & \text{if } x \text{ is odd} \\ \varphi_{x/2}(x) + 1 & \text{if } x \text{ is even, } \varphi_{x/2}(x)\downarrow \\ 0 & \text{if } x \text{ even, } \varphi_{x/2}(x)\uparrow \end{cases}$$

- f total
- $f(x) = x \quad \forall x$ odd $\Rightarrow$ infinite
- f not computable since it differs from all total computable functions ($\forall x$ if φ_x is total $f(2x) = \varphi_x(2x) + 1 \neq \varphi_x(2x)$

<u>2nd idea</u>

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_x(x)\downarrow \\ x & \text{otherwise} \end{cases}$$

- f total
- f not computable since it differs from all total computable functions ($\forall x$ if φ_x total $f(x) = \varphi_x(x) + 1 \neq \varphi_x(x)$)
- $f(x) = x$ $\forall x$ s.t. $\varphi_x(x)\uparrow$ i.e. $\forall x \in \overline{K}$ which is infinite (e.g. because if $\overline{K}$ were finite it would be recursive and it is not)

3rd idea

$$f(x) = \begin{cases} x+1 & \text{if } \varphi_x(x)\downarrow \\ x & \text{otherwise} \end{cases}$$

- f total
- $f(x) = x$ $\forall x \in \overline{K}$ (infinite)
- f not computable since $\chi_K(x) = \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases} = f(x) \doteq x$

EXERCISE : If f is computable and $g(x) = f(x)$ almost everywhere ($\underset{\text{finite}}{\{x \mid f(x) = g(x)\}}$)

$\Rightarrow$ g computable.