END of PROOF of R–K.

$f_k^\varepsilon := Ef_k * \eta_\varepsilon$

$\| Ef_k - f_k^\varepsilon \|_{L^1(\mathbb{R}^n)} \leq C\varepsilon \quad \forall k$ (1)

$\forall \varepsilon \in (0,1)$ $\exists\, k_i^\varepsilon$ such that $f_{k_i^\varepsilon}^\varepsilon \to \tilde{f}^\varepsilon$ in $L^1(\mathbb{R}^n) \Rightarrow f_{k_i^\varepsilon}^\varepsilon$ is CAUCHY in $L^1(\mathbb{R}^n)$ (2)

NOW CLAIM.

Fix $\delta > 0 \Rightarrow \exists$ a subsequence $\boxed{k_i \text{ (depending on } \delta)}$ such that

$$\lim_{k_i, k_j \to +\infty} \| Ef_{k_i} - Ef_{k_j} \|_{L^1} \leq \delta$$

proof

by (1) $\exists\, \varepsilon_0$ such that $\| Ef_k - f_k^\varepsilon \|_{L^1} \leq \frac{\delta}{2} \quad \forall\, \varepsilon \leq \varepsilon_0$

Fix $\bar{\varepsilon} = \frac{\varepsilon_0}{2} \to$ by (2) there exists a subsequence k_i depending on $\bar{\varepsilon}$ (and so actually on δ) such that

$$\lim_{k_i, k_j \to +\infty} \| f_{k_i}^{\bar{\varepsilon}} - f_{k_j}^{\bar{\varepsilon}} \| = 0$$

$$\lim_{k_i,k_j\to 0} \| Ef_{k_i} - Ef_{k_j} \|_{L^1} \leq \lim_{k_i k_j \to 0} \Big[\| Ef_{k_i} - f^{\varepsilon}_{k_i} \|_{L^1} + \| Ef_{k_j} - f^{\varepsilon}_{k_j} \|_{L^1} + \| f^{\varepsilon}_{k_i} - f^{\varepsilon}_{k_j} \|_{L^1} \Big] \leq \frac{\delta}{2} + \frac{\delta}{2} + 0 = \delta$$

So for $\delta = 1$ we have a subsequence $k^1_{i_1}$

From the subsequence k^1_i, extract a subsequence $k^2_{i_2}$ such that $\lim_{k^2_i, k^2_j \to +\infty} \| Ef_{k^2_{i_2}} - Ef_{k^2_{j_2}} \|_{L^1} \leq \frac{1}{2}$

$\dots \to k^n_i$ subsequence of k^{n-1}_i such that

$$\lim_{k^n_{i_n}, k^n_{j_n} \to +\infty} \| Ef_{k^n_{i_n}} - Ef_{k^n_{j_n}} \|_{L^1} \leq \frac{1}{n}$$

$\to$ extract a diagonal subsequence $Ef_{k^n_{n_n}}$ $\to$ this is Cauchy in L^1.

Ex [HARMONIC EXTENSION
U open, bdd, class C^1.

fix $u_0 \in W^{1,2}(U)$ $\quad I(f) =$ Dirichlet integral $= \int_U |Df|^2 dx$

$I(f)$ is well defined $\forall f \in W^{1,2}(U)$.

Q: $\exists \bar{f}$ such that $\bar{f} - u_0 \in W_0^{1,2}(U)$, $I(\bar{f}) = \min_{f - u_0 \in W_0^{1,2}(U)} I(f)$?

(I'm asking $Tr(f) = Tr(u_0)$, so "$f = u_0$" on ∂U).

(IF IN PLACE OF 2 I put $p>1$, EXACTLY SAME ARGUMENT

$I(f) \geq 0$. Let $c = \inf\{I(f),\ f \in W^{1,2}(U)\ f - u_0 \in W_0^{1,2}(U)\} \leq I(u_0)$.

Let (f_k) be a MINIMIZING SEQUENCE

$f_k \in W^{1,2}(U)$ $\quad f_k - u_0 \in W_0^{1,2}(U)$ $\quad c \leq I(f_k) \leq c + \frac{1}{k}$.

$\| |Df_k| \|_{L^2} \leq c + 1$ $\forall k$. Moreover $f_k - u_0 \in W_0^{1,2}(U)$ and so

$$\underset{\substack{\vee\!\mid \\ \|f_k\|_{W^{1,2}} - \|u_0\|_{W^{1,2}}}}{\|f_k - u_0\|_{W^{1,2}(U)}} \leq C \, \| \, |D(f_k - u_0)| \, \|_{L^2(U)} \leq \bar{C} \| \, |Df_k| \, \|_{L^2} + C \| Du_0 \|_{L^2}$$

by (GNS) $\|f_k - u_0\|_{L^2} \leq C \| D(f_k - u) \|_{L^2}$

$$\Rightarrow \quad \|f_k\|_{W^{1,2}} \leq C \|Df_k\|_{L^2} + \tilde{C} \|u_0\|_{W^{1,2}} \leq \bar{C}$$

$\exists \, k_j \quad f_{k_j} \to f$ strongly in L^2 $\qquad f \in W^{1,2}(U)$.

$Df_{k_j} \rightharpoonup Df$ weakly in L^2.

$$\begin{bmatrix} f_{k_j} - u_0 \to f - u_0 \\ Df_{k_j} - Du_0 \rightharpoonup Df - Du_0 \end{bmatrix} \Rightarrow f - u_0 \in W_0^{1,2}(U)$$

(since $W_0^{1,2}(U)$ is weakly closed)

$$C + \frac{1}{k} \geq \int_U |Df_k|^2 \, dx \overset{\text{by convexity}}{\geq} \int_U |Df|^2 dx + \underbrace{\int_U Df \cdot (Df_k - Df) \, dx}_{\to 0 \text{ by weak convergence}}$$

$$C \leq I(f) \leq \liminf_k I(f_k) = C \quad \Rightarrow \quad f \text{ is a MINIMIZER}$$

- f is UNIQUE (assume $\exists f_1 \neq f_2$ minimizer)

$\lambda \in (0,1)$ $\quad \underset{\substack{\vee| \\ c}}{I(\lambda f_1 + (1-\lambda) f_2)} < \lambda I(f_1) + (1-\lambda) I(f_2) = c$

IMPOSSIBLE

- PDE characterization. Compute Gateaux derivative of $I(\cdot)$

$\phi \in C_c^\infty(U)$ $\quad f + \varepsilon\phi \in W^{1,2}(U)$ $\quad f + \varepsilon\phi - u_0 \in W_0^{1,2}(U)$.

$I(f+\varepsilon\phi) \geq I(f) \quad \forall \varepsilon$

$$\frac{I(f+\varepsilon\phi) - I(f)}{\varepsilon} = \frac{1}{\varepsilon}\int_U Df \cdot \varepsilon D\phi + \varepsilon^2 |D\phi|^2 \, dx = \int_U Df \cdot D\phi + \varepsilon |D\phi|^2$$

for $\varepsilon > 0$ $\quad \frac{I(f+\varepsilon\phi) - I(f)}{\varepsilon} \geq 0$ $\qquad$ for $\varepsilon < 0$ $\quad \frac{I(f+\varepsilon\phi) - I(f)}{\varepsilon} \leq 0$

$$\lim_{\varepsilon \to 0} \frac{I(f+\varepsilon\phi) - I(f)}{\varepsilon} = 0 = \int_U Df \cdot D\phi = -\int_U f \cdot \Delta\phi$$

$\rightarrow \forall \phi \in C_c^\infty(U)$ $\quad \int_U f \Delta\phi = 0 \Rightarrow \Delta f = 0$ in the sense of DISTRIBUTIONS.

So f solves $\begin{cases} -\Delta f = 0 & \text{in } U \text{ in the sense of distributions} \\ f = u_0 & \text{on } \partial U \text{ in the sense of the trace} \end{cases}$

$\Delta f = 0 \Rightarrow f \in C^\infty(U)$ (by Weyl lemma).

$\underline{f \text{ is the harmonic extension of } u_0}$ in U

(harmonic function in U which coincide with u_0 on ∂U).

POINCARÉ INEQUALITY

Let $p \in [1, +\infty]$, U bdd open of class C^1 and connected
$\exists\, C = C(n, p, U)$ such that $\forall f \in W^{1,p}(U)$.

$$\left\| f - \frac{1}{|U|} \int_U f(y)\, dy \right\|_{L^p(U)} \leq C(n,p,U) \, \| |Df| \|_{L^p(U)}$$

Observation: $C = \{ f \in W^{1,p}(U) \ \int_U f(y)\, dy = 0 \}$ is closed in $W^{1,p}(U)$

In C $\|f\|_{W^{1,p}}$ is equivalent to $\| |Df| \|_{L^p}$.