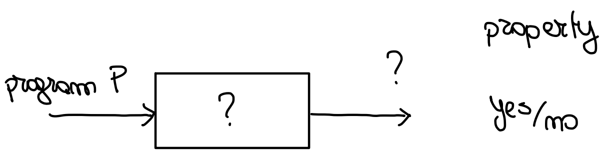
COMPUTABILITY (02/12/2024)

RICE'S THEOREM



every property of the I/O behaviour of programs is undecidable

" P is terminating for every input "

" P has some fixed $m \in \mathbb{N}$ as an output "

" P computes a function f "

⋮

undecidable

" the program P has size $\leq m$ "
(number of instructions)

decidable

What is a behavioural property of programs ?

$$A \subseteq \mathbb{N}$$

$\uparrow$ set of programs $\equiv$ program property

$T = \{ m \mid P_m$ is halting on every input $\}$

$\quad = \{ m \mid \varphi_m$ is total $\}$

ONE $= \{ m \mid P_m$ is a sound implementation of function $\mathbb{1} \}$

$\quad = \{ m \mid \varphi_m = \mathbb{1} \}$

$A \subseteq \mathbb{N}$ (program property) is a behavioural property :

given $m \in \mathbb{N}$ (program) the $m \in A$ or $m \notin A$

<u>only depends on</u> φ_m

Def (saturated / extensional set) : $A \subseteq \mathbb{N}$ is <u>saturated</u> (<u>extensional</u>)

if for all $m, n \in \mathbb{N}$

if $m \in A$ and $\varphi_m = \varphi_n$ then $n \in A$

$\Updownarrow$

$A = \{ n \in \mathbb{N} \mid \varphi_n$ satisfies some property of functions $\}$

$= \{ n \in \mathbb{N} \mid \varphi_n \in \mathcal{A} \}$

where $\mathcal{A} \subseteq \mathcal{F}$ ← set of all functions

property of functions

<u>Examples</u> :

* $T = \{ n \mid P_n$ halts on every input $\}$ $\boxed{\text{SATURATED}}$

$= \{ n \mid \varphi_n$ is total $\}$

$= \{ n \mid \varphi_n \in \mathcal{T} \}$ where $\mathcal{T} = \{ f \mid f$ is total $\}$

* ONE $= \{ n \mid P_n$ is sound implementation of $\mathbb{1} \}$ $\boxed{\text{SATURATED}}$

$= \{ n \mid \varphi_n = \mathbb{1} \} = \{ n \mid \varphi_n \in \{ \mathbb{1} \} \}$

* LEN 10 $= \{ n \mid P_n$ has length $\leq 10 \}$ $\boxed{\text{NOT SATURATED}}$

$m \in$ LEN 10

$\varphi_m = \varphi_n$

$n \notin$ LEN 10

e.g.

$m = \gamma(S(1)) \in$ LEN 10

$\varphi_m = \varphi_n =$ successor

$$n = \gamma\left(\left.\begin{matrix} S(1) \\ S(2) \\ \vdots \\ S(2) \end{matrix}\right\} \text{10 times} \right) \notin \text{LEN } 10$$

$\ast \quad K = \{ n \in \mathbb{N} \mid \varphi_n(n)\downarrow \}$

$\quad\;\; = \{ n \in \mathbb{N} \mid \varphi_n \in \mathcal{K} \}$

$\mathcal{K} = \{ f \mid f(?)\downarrow \}$

apparently not saturated.

formally we must find $m, n \in \mathbb{N}$

$m \in K$ i.e. $\varphi_m(m)\downarrow$

$n \notin K$ i.e. $\varphi_n(n)\uparrow$

and $\varphi_m = \varphi_n$

$$\boxed{\text{if there exists a program } m \in \mathbb{N} \text{ s.t. } \quad \varphi_m(x) = \begin{cases} 1 & x = m \\ \uparrow & \text{otherwise} \end{cases}}$$

then we conclude easily that K is not saturated

(1) $\boxed{m \in K}$ since $\varphi_m(m)\downarrow$

(2) for a computable functions there are infinitely many programs

hence there $n \neq m$ s.t. $\boxed{\varphi_n = \varphi_m}$

(3) $\boxed{n \notin K}$

$$\varphi_n(n) \underset{\varphi_m = \varphi_n}{=} \varphi_m(n) \underset{m \neq n}{\uparrow}$$

Is $(\ast)$ true?

$$x \longrightarrow \boxed{P} \longrightarrow \begin{array}{l} \text{if } x = P \text{ then } 1 \\ \text{if } x \neq P \quad \uparrow \end{array}$$

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def P(x):
    if x == " def P(x):
                 if ...     "
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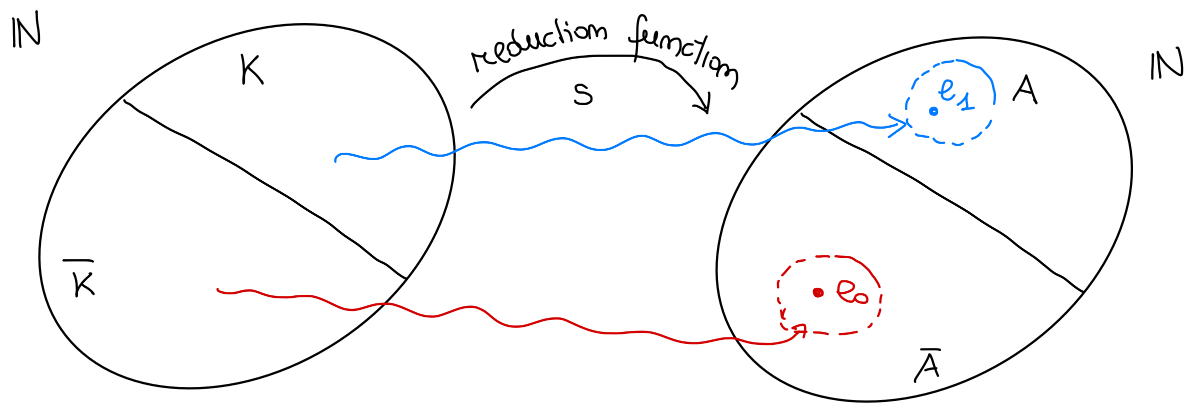
Rice's Theorem :

Let $A \subseteq \mathbb{N}$ if A is saturated, $A \neq \emptyset$, $A \neq \mathbb{N}$

then A is not recursive

proof

we show that $K \leq_m A$ (since K is not recursive this will imply that A not recursive)



Let $e_0 \in \mathbb{N}$ s.t. $\varphi_{e_0}(x)\uparrow \ \forall x$ (program for the function always undefined)

(1) Assume $e_0 \in \bar{A}$

let $e_1 \in A$ (it exists since $A \neq \emptyset$)

Define

$$g(x,y) = \begin{cases} \varphi_{e_1}(y) & \text{if } x \in K \\ \varphi_{e_0}(y) & \text{if } x \notin K \end{cases}$$

$$= \begin{cases} \varphi_{e_1}(y) & \text{if } x \in K & [\varphi_x(x)\downarrow] \\ \uparrow & \text{if } x \notin K & [\varphi_x(x)\uparrow] \end{cases}$$

$$= \varphi_{e_1}(y) \cdot \underbrace{\mathbb{1}(\varphi_x(x))}_{\begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{if } \varphi_x(x)\uparrow \end{cases}}$$

$$= \varphi_{e_1}(y) \cdot \mathbb{1}(\Psi_U(x,x))$$

computable !

By the smn theorem there $s: \mathbb{N} \to \mathbb{N}$ total and computable s.t.

$$\forall x,y \qquad \varphi_{s(x)}(y) = g(x,y) = \begin{cases} \varphi_{e_1}(y) & \text{if } x \in K \\ \varphi_{e_0}(y) & \text{if } x \notin K \end{cases}$$

s is the reduction function : $\forall x \quad x \in K$ iff $s(x) \in A$

* if $x \in K \rightsquigarrow s(x) \in A$

if $x \in K$ then $\varphi_{s(x)}(y) = g(x,y) = \varphi_{e_1}(y) \quad \forall y$

hence $\varphi_{s(x)} = \varphi_{e_1}$. Since $e_1 \in A$ and A saturated then $s(x) \in A$.

* if $x \notin K \rightsquigarrow s(x) \notin A$

if $x \notin K$ then $\varphi_{s(x)}(y) = g(x,y) = \varphi_{e_0}(y) \quad \forall y$

hence $\varphi_{s(x)} = \varphi_{e_0}$. Since $e_0 \notin A$ and A saturated, $s(x) \notin A$

Hence s is the reduction function for $K \leq_m A$, since K is not recursive we deduce that A is not recursive.

(2) if instead $e_0 \in A$

if we let $B = \bar{A}$

- B is saturated (since A is saturated)
- $e_0 \in \bar{B}$
- $B \neq \emptyset$ (since $A \neq \mathbb{N}$)
- $B \neq \mathbb{N}$ (since $A \neq \emptyset$)

by (1) B is not recursive

Since $B = \bar{A}$ not recursive, neither A is.

$\square$

Example : Output problem

$B_n = \{x \mid P_x \text{ outputs } n \text{ for some input}\}$

$= \{x \mid n \in E_x\}$ (not recursive since $K \leq_m B_n$)

- B_n is saturated, in fact

$B_n = \{x \mid \varphi_x \in \mathcal{B}_n\}$ where $\mathcal{B}_n = \{f \mid n \in cod(f)\}$

- $B_n \neq \emptyset$

e.g. let $e_1 \in \mathbb{N}$ s.t. $\varphi_{e_1}(y) = y \ \forall y \rightsquigarrow n \in E_{e_1} = \mathbb{N}$

$\rightsquigarrow e_1 \in B_n$

- $B_n \neq \mathbb{N}$

e.g. let $e_2 \in \mathbb{N}$ s.t. $\varphi_{e_2}(y) = m \ \forall y \rightsquigarrow n \notin E_{e_2} = \{m\}$
(with $m \neq n$) $\rightsquigarrow B_n \neq \mathbb{N}$

Hence by Rice's theorem, B_n is not recursive.

EXAMPLE :

$I = \{x \in \mathbb{N} \mid P_x \text{ is halting on infinitely many inputs}\}$

$= \{x \in \mathbb{N} \mid W_x \text{ is infinite}\}$

* saturated

$I = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{I}\}$

where $\mathcal{I} = \{f \mid dom(f) \text{ is infinite}\}$

* $I \neq \emptyset$

if e_1 is as above (identity) $\Rightarrow W_{e_1} = \mathbb{N}$ infinite $\Rightarrow e_1 \in I \neq \emptyset$

* $I \neq \mathbb{N}$

if e_0 is s.t. $\varphi_{e_0}(y)\uparrow \ \forall y \Rightarrow W_{e_0} = \emptyset$ finite $\Rightarrow e_0 \notin I$

$\rightsquigarrow$ by Rice's Theorem I is not recursive.

$\square$

Example : $A = \{ x \mid x \in W_x \cap E_x \}$

saturated ?

$$A = \{ x \mid \varphi_x \in \mathcal{A} \}$$

$$\mathcal{A} = \{ f \mid \ ? \in dom(f) \cap cod(f) \}$$

probably not saturated, hence we do not use Rice's theorem.

We show $K \leq_m A$ (hence A not recursive)

s.t.

$x \in K$ iff $s(x) \in A$

$s(x) \in W_{s(x)}$ $\quad \varphi_{s(x)}(s(x))\downarrow$

$s(x) \in E_{s(x)}$ $\quad \varphi_{s(x)}(y) = s(x)$ for some $y \in \mathbb{N}$

we define

$$g(x,y) = \begin{cases} y & \text{if } x \in K \quad [\varphi_x(x)\downarrow] \\ \uparrow & \text{if } x \notin K \quad [\varphi_x(x)\uparrow] \end{cases}$$

$$= y \cdot \mathbb{1}(\varphi_x(x))$$

$$= y \cdot \mathbb{1}(\varphi_U(x,x)) \qquad \text{computable}$$

by smn theorem, there is $s : \mathbb{N} \to \mathbb{N}$ total and computable s.t.

$\forall x, y$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} y & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

s is the reduction function for $K \leq_m A$

* if $x \in K$ then $\varphi_{s(x)}(y) = y \quad \forall y$

hence $s(x) \in \underset{\mathbb{N}}{\underset{\shortparallel}{W_{s(x)}}} \cap \underset{\mathbb{N}}{\underset{\shortparallel}{E_{s(x)}}} = \mathbb{N}$ $\Rightarrow s(x) \in A$

* if $x \notin K$ then

$$\varphi_{s(x)}(y)\uparrow \quad \forall y$$

hence $s(x) \notin$ $\quad \underset{\emptyset}{\underset{\shortparallel}{W_{s(x)}}} \cap \underset{\emptyset}{\underset{\shortparallel}{E_{s(x)}}} = \emptyset$

$\square$