Es determinare il carattere della serie

$$\sum_{n=0}^{+\infty} \frac{3^{n-2}}{n!\, 2^{n+2}}$$

$$a_n = \frac{3^{n-2}}{n!\, 2^{n+2}} = \frac{3^n \cdot 3^{-2}}{n!\, 2^n \cdot 2^2}$$

criterio del rapporto $\frac{a_{n+1}}{a_n}$

$$a_{n+1} = \frac{3^{n+1-2}}{(n+1)!\, 2^{n+1+2}} = \frac{3^{n-1}}{(n+1)\, n!\, 2^{n+3}} = \frac{3^n \cdot 3^{-1}}{(n+1)\, n!\, 2^n \cdot 2^3}$$

$$\frac{a_{n+1}}{a_n} = a_{n+1} \cdot \frac{1}{a_n} = \frac{\cancel{3^n} \cdot 3^{-1}}{(n+1)\, \cancel{n!}\, \cancel{2^n} \cdot 2^3} \cdot \frac{\cancel{n!}\, \cancel{2^n} \cdot 2^2}{\cancel{3^n}\, 3^{-2}} = \frac{3^{-1}\, 2^2}{(n+1)\, 2^3 \cdot 3^{-2}} \xrightarrow{n \to +\infty} 0$$

$L = 0 < 1$ LA SERIE CONVERGE

Es

1) Determinare al variare di $\alpha \in \mathbb{R}$ il limite della successione

$$a_n = \underbrace{n - n\cos\left(\frac{1}{n}\right)} - \sin\left(\frac{\alpha}{n}\right)$$

2) Determinare al variare di α il carattere della serie $\sum_{n=1}^{+\infty} |a_n|$

$n \to +\infty \qquad \frac{1}{n} \to 0 \qquad \frac{\alpha}{n} \to 0$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4) \qquad \sin x = x - \frac{1}{6}x^3 + o(x^3)$$

$$\cos\frac{1}{n} = 1 - \frac{1}{2}\frac{1}{n^2} + \frac{1}{24}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) \qquad \sin\left(\frac{\alpha}{n}\right) = \frac{\alpha}{n} - \frac{1}{6}\frac{\alpha^3}{n^3} + o\left(\frac{1}{n^3}\right)$$

$$a_n = n - n\cos\frac{1}{n} - \sin\frac{\alpha}{n} =$$

$$= n - n\left[1 - \frac{1}{2}\,\frac{1}{n^2} + \frac{1}{24}\,\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)\right] - \left[\frac{\alpha}{n} - \frac{1}{6}\,\frac{\alpha^3}{n^3} + o\left(\frac{1}{n^3}\right)\right]$$

$$= \cancel{n} - \cancel{n} + \frac{1}{2}\,\frac{1}{n} - \underbrace{\frac{1}{24}\,\frac{1}{n^3}} + o\left(\frac{1}{n^3}\right) - \frac{\alpha}{n} + \underbrace{\frac{1}{6}\,\frac{\alpha^3}{n^3}} + o\left(\frac{1}{n^3}\right)$$

$$= \underbrace{\left[\frac{1}{2} - \alpha\right]\cdot\frac{1}{n}} + \underbrace{\left[-\frac{1}{24} + \frac{\alpha^3}{6}\right]\frac{1}{n^3}} + o\left(\frac{1}{n^3}\right)$$

$$\lim_{n} a_n = 0$$

$$\sum_{n=1}^{+\infty} |a_n|$$

$$|a_n| = \left| \underbrace{\left(\frac{1}{2}-\alpha\right)}\cdot\frac{1}{n} + \left[-\frac{1}{24}+\frac{\alpha^3}{6}\right]\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right|$$

Se $\left(\frac{1}{2}-\alpha\right) \neq 0$

$\alpha \neq \frac{1}{2}$

$$|a_n| = \frac{1}{n}\left| \frac{1}{2}-\alpha + \left(-\frac{1}{4}+\frac{\alpha^3}{6}\right)\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right|$$

$$= \frac{1}{n}\left| \frac{1}{2}-\alpha + o(1)\right|$$

$$|a_n| \sim \frac{1}{n}$$

$\sum_{n=1}^{+\infty} \frac{1}{n}$ DIVERGE $\Longrightarrow$ $\sum_{n=1}^{\infty} |a_n|$ DIVERGE

$\alpha = \frac{1}{2}$ $\quad |a_n| = \left| \left(\frac{1}{2} - \alpha\right)\frac{1}{n} + \left(-\frac{1}{24} + \frac{\alpha^3}{6}\right)\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right| =$

$$= \left| 0 \cdot \cancel{\frac{1}{n}} + \left(-\frac{1}{24} + \frac{1}{8}\cdot\frac{1}{6}\right)\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right| =$$

$$= \frac{1}{n^3}\left| -\frac{1}{48} + o(1)\right|$$

$\alpha^3 = \frac{1}{8}$

$\frac{\alpha^3}{6} = \frac{1}{8}\cdot\frac{1}{6}$

$|a_n| \sim \frac{1}{n^3}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^3} < +\infty$ CONVERGE

$$\Downarrow$$

$\sum_{n=1}^{+\infty} |a_n| < +\infty$ CONVERGE

Es

Determinare al variare di α il limite di

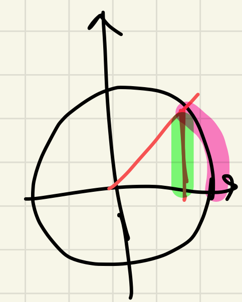
$$a_n = n^{\alpha}\left(\frac{1}{n^2} - \sin\frac{1}{n^2}\right) \geq 0 \quad \forall n \geq 1$$

e il carattere della serie $\sum_{n=1}^{\infty} a_n$.

$n \geq 1$

$$\boxed{0 < \frac{1}{n^2} \leq 1}$$

$x \geq \sin x \qquad x \in \left(0, \frac{\pi}{2}\right)$



$$\frac{1}{n^2} \geq \sin\frac{1}{n^2}$$

$$\operatorname{sen} x = x - \frac{1}{6}x^3 + o(x^3)$$

$$\operatorname{sen}\frac{1}{n^2} = \frac{1}{n^2} - \frac{1}{6}\left(\frac{1}{n^2}\right)^3 + o\left(\frac{1}{n^2}\right)^3 = \frac{1}{n^2} - \frac{1}{6}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right)$$

$$a_n = n^\alpha \left[\frac{1}{n^2} - \left(\frac{1}{n^2} - \frac{1}{6}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right)\right)\right] =$$

$$= n^\alpha\left[\cancel{\frac{1}{n^2}} - \cancel{\frac{1}{n^2}} + \frac{1}{6}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right)\right] =$$

$$= n^\alpha \cdot \frac{1}{n^6}\underbrace{\left[\frac{1}{6} + o(1)\right]} \xrightarrow{n \to +\infty} \begin{cases} \boxed{\alpha = 6} & = \frac{1}{6} \\ \alpha > 6 & = +\infty \\ \alpha < 6 & = 0 \end{cases}$$

$\alpha \geq 6$ la serie sicuramente diverge perché $\lim_{n} a_n \neq 0$.

$\boxed{\alpha < 6}$ la serie potrebbe convergere oppure no.

$$a_n = n^{\alpha} \cdot \frac{1}{n^6}\left[\frac{1}{6} + o(1)\right] \sim \frac{1}{n^{6-\alpha}} = \frac{n^{\alpha}}{n^6}$$

per il criterio del confronto asintotico

$$\sum_{n=1}^{+\infty} a_n \text{ converge} \iff \sum_{n=1}^{\infty} \frac{1}{n^{6-\alpha}} \text{ converge}$$

SERIE ARMONICA GENERALIZZATA

Converge se $6-\alpha>1$ $\quad \alpha<5$

La serie diverge per $\alpha \geq 5$

converge per $\alpha < 5$

~ . ~

Studiare la convergenza al variare di $\alpha \in \mathbb{R}$

della serie

$$\sum_{n=1}^{+\infty} \frac{\alpha^{2n}}{3^n \cdot \sqrt{n}}$$

$$\alpha^{2n} = (\alpha^2)^n \geq 0$$

RADICE n-esima $\quad a_n = \dfrac{\alpha^{2n}}{3^n \cdot \sqrt{n}}$

$\urcorner$

$$\sqrt[n]{a_n} = (a_n)^{\frac{1}{n}} = \left(\frac{\alpha^{2n}}{3^n \sqrt{n}} \right)^{\frac{1}{n}} = \frac{\left(\alpha^{2n}\right)^{\frac{1}{n}}}{\left(3^n\right)^{\frac{1}{n}} \cdot \left(\sqrt{n}\right)^{\frac{1}{n}}} =$$

$$= \frac{\alpha^{2\cancel{n} \cdot \frac{1}{\cancel{n}}}}{3^{\cancel{n} \frac{1}{\cancel{n}}} \left(\sqrt{n}\right)^{\frac{1}{n}}} = \frac{\alpha^2}{3 \left(n^{\frac{1}{2}}\right)^{\cdot \frac{1}{n}}} \longrightarrow \frac{\alpha^2}{3} < 1$$

$$n^{\frac{1}{2} \cdot \frac{1}{n}} = e^{\frac{1}{2} \cdot \frac{1}{n} \lg n} = e^{\frac{1}{2} \frac{\lg n}{n}} \longrightarrow e^{\frac{1}{2} \cdot 0} = e^0 = 1$$

$\forall k$

$$\left(n^k\right)^{\frac{1}{n}} = e^{\frac{k}{n} \lg n} \longrightarrow e^0$$

$\frac{\alpha^2}{3} = L > 1$ $\qquad \alpha^2 > 3$ $\qquad \boxed{\alpha > \sqrt{3} \quad \alpha < -\sqrt{3}}$

$\alpha^2 - 3 > 0$

la serie DIVERGE

$\frac{\alpha^2}{3} = L < 1$ $\qquad \alpha^2 < 3 \qquad \alpha^2 - 3 < 0 \quad \boxed{-\sqrt{3} < \alpha < \sqrt{3}}$

la serie CONVERGE

$\alpha = +\sqrt{3} \quad \alpha = -\sqrt{3}$

se $\alpha^2 = 3$ $\quad L = 1$ NON HO INFORMAZIONI

devo tornare alla serie di partenza e sostituire ad α il valore $+\sqrt{3}$ e poi $-\sqrt{3}$.

$$\sum_{n=1}^{\infty} \frac{\alpha^{2n}}{3^n \cdot \sqrt{n}}$$

$\alpha = \sqrt{3}$

$$(\sqrt{3})^{2n} = [(\sqrt{3})^2]^n = 3^n$$

$$\sum_{n=1}^{\infty} \frac{\cancel{(\sqrt{3})^{2n}}}{\cancel{3^n}\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

DIVERGE

$\alpha = -\sqrt{3}$ ottengo la stessa serie

$$\sum_{n=1}^{\infty} \frac{(-\sqrt{3})^{2n}}{3^n \cdot \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$(-\sqrt{3})^{2n} = ((-\sqrt{3})^2)^n = 3^n$

DIVERGE

$-\sqrt{3} < \alpha < \sqrt{3}$ la serie converge, $\alpha \geq \sqrt{3}$ $\alpha \leq -\sqrt{3}$ diverge

Es $\quad f(x) = x \arctan\left(\frac{1}{x}\right)$

D: $(-\infty, 0) \cup (0, +\infty)$ $\quad x \neq 0$

$\arctan\left(-\frac{1}{x}\right) = -\arctan\frac{1}{x}$

$$f(-x) = (-x) \cdot \arctan\left(\frac{1}{-x}\right) = -x \cdot \arctan\left(-\frac{1}{x}\right) =$$

$$= -x \cdot \left(-\arctan\frac{1}{x}\right) = x \arctan\frac{1}{x}$$

f è pari

segno $\quad \arctan\frac{1}{x} > 0$ se $\frac{1}{x} > 0$ $\quad \arctan\frac{1}{x} < 0$ se $\frac{1}{x} < 0$

$x > 0$

$x > 0$

$\underset{+}{x} \cdot \underset{+}{\arctan\frac{1}{x}} > 0$ $\quad x < 0$ $\quad \underset{<0}{x} \cdot \underbrace{\arctan\frac{1}{x}}_{<0} > 0$

f è sempre positiva

$$\lim_{x\to 0} x \cdot \arctan\left(\frac{1}{x}\right) = 0$$

$x \to 0$

$\arctan\left(\frac{1}{x}\right)$ LIMITATA

$$-\frac{\pi}{2} < \arctan\frac{1}{x} < +\frac{\pi}{2}$$

$x=0$ è una SINGOLARITÀ ELIMINABILE

aggiungo $x=0$ al DOMINIO

$f(0) = 0$

altrimenti: (va bene lo stesso se si procede così:)

$$\lim_{x\to 0^+} x \cdot \arctan\left(\frac{1}{x}\right) = 0 \cdot \frac{\pi}{2} = 0$$

$\frac{1}{0^+} = +\infty \qquad \arctan_{+\infty} = \frac{\pi}{2}$

$$\lim_{x\to 0^-} x \cdot \arctan\left(\frac{1}{x}\right) = 0 \cdot \left(-\frac{\pi}{2}\right) = 0$$

$\frac{1}{x} \to -\infty$

Deduco anche che $x=0$ è pto di MINIMO ASSOLUTO

$$f(0)=0 \qquad f(x)\geq 0 \quad \forall x \in D$$

$$\lim_{x\to+\infty} x \, \boxed{\operatorname{arctg}\left(\frac{1}{x}\right)}$$

$x \to +\infty$

$\operatorname{arctg}\left(\frac{1}{x}\right) \to \operatorname{arctg} 0 = 0$

se $x \to +\infty$ $\quad \frac{1}{x} \to 0$

la funzione arcotangente vicino a 0 può essere approssimata con un polinomio (di TAYLOR)

$x \to 0 \qquad \arctan(x) = x + o(x)$

$x \to +\infty$

$x \to -\infty$

$$\arctan\left(\frac{1}{x}\right) = \frac{1}{x} + o\left(\frac{1}{x}\right) = \underbrace{\frac{1}{x}\left(1+o(1)\right)}$$

$$\lim_{x \to +\infty} x \cdot \arctan\left(\frac{1}{x}\right) = \lim_{x\to+\infty} \cancel{x} \cdot \frac{1}{\cancel{x}} \cdot \left(1+o(1)\right) = 1$$

$$\lim_{x \to -\infty} x \arctan\left(\frac{1}{x}\right) = \lim_{x\to-\infty} \cancel{x} \cdot \frac{1}{\cancel{x}}\left(1+o(1)\right) = 1$$

$$\boxed{y = 1 \text{ È ASINTOTO ORIZZONTALE a } +\infty \text{ e a } -\infty.}$$

$$f(x) = \underbrace{x}\cdot \underbrace{\operatorname{arctg}\left(\frac{1}{x}\right)}$$

$$(\operatorname{arctg} x)' = \frac{1}{1+x^2}$$

$$\left(\frac{1}{x}\right)' = \left(x^{-1}\right)' = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

$$(x^{\alpha})' = \alpha \cdot x^{\alpha-1}$$

$$f'(x) = 1 \cdot \operatorname{arctg}\left(\frac{1}{x}\right) + \cancel{x}\,\frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(-\frac{1}{x^{\cancel{2}}}\right)$$

$$= \operatorname{arctg}\frac{1}{x} \cdot - \frac{1}{x}\,\frac{1}{1+\frac{1}{x^2}} =$$

$$= \operatorname{arctg}\left(\frac{1}{x}\right) - \frac{1}{x} \cdot \frac{1}{\frac{x^2+1}{x^2}} =$$

$$= \operatorname{arctg}\frac{1}{x} - \frac{1}{\cancel{x}}\,\frac{x^{\cancel{2}}}{x^2+1} = \underline{\operatorname{arctg}\left(\frac{1}{x}\right) - \frac{x}{x^2+1}}\,.$$

$$\boxed{f'(x) = \operatorname{arctg}\left(\frac{1}{x}\right) - \frac{x}{x^2+1} \quad \forall x \in D} \quad x \neq 0$$

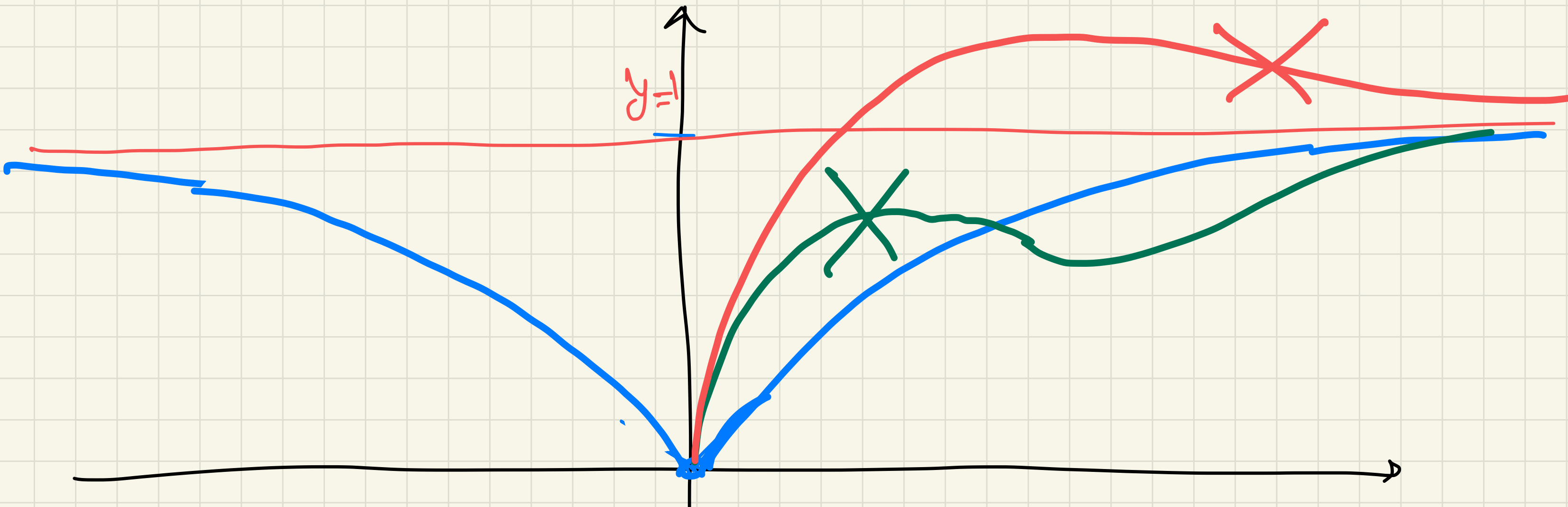
$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \operatorname{arctg}\underbrace{\left(\frac{1}{x}\right)}_{+\infty} - \frac{x}{x^2+1} = \frac{\pi}{2}$$

$$\to \frac{0}{0^2+1} = \frac{0}{1} = 0$$

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} \operatorname{arctg} \underbrace{\frac{1}{x}}_{\to -\infty} - \underbrace{\frac{x}{x^2+1}}_{\to 0} =$$

$$= -\frac{\pi}{2}$$

$$\boxed{x=0 \text{ è pto ANGOLOSO}}$$



$$f'(x) = \underbrace{\operatorname{arctg}\frac{1}{x}} - \underbrace{\frac{x}{x^2+1}}$$

segno della derivata non è facile da studiare

Studio $f''(x)$

$$f'(x) = \boxed{\operatorname{arctg} \frac{1}{x}} - \frac{x}{x^2+1}$$

$$f''(x) = \frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) - \left[\frac{1\cdot(x^2+1) - x\cdot(2x+0)}{(x^2+1)^2}\right]$$

$$= \frac{1}{\frac{x^2+1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) - \left[\frac{x^2+1-2x^2}{(x^2+1)^2}\right] = \frac{\cancel{x^2}}{x^2+1} \cdot \left(-\frac{1}{\cancel{x^2}}\right) - \left[\frac{1-x^2}{(1+x^2)^2}\right]$$

$$= -\frac{1}{x^2+1} - \frac{1-x^2}{(1+x^2)^2} =$$

$$= \frac{-1}{x^2+1} - \frac{1-x^2}{(x^2+1)^2} = \frac{-(x^2+1) - (1-x^2)}{(x^2+1)^2} =$$

$$= \frac{\cancel{-x^2} - 1 - 1 \cancel{+x^2}}{(x^2+1)^2} = \frac{-2}{(x^2+1)^2} = f''(x)$$

$f''(x) < 0 \implies$ $\boxed{f \text{ CONCAVA in tutto il dominio}}$

$x=0$ è pto di MINIMO, ma f non è derivabile né una né 2 volte in $x=0$