$$\nabla^2 \phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\nabla^2 \phi(\vec{r}) = f(\vec{r}) \qquad \vec{r} \in \Omega$$

condizione per $\vec{r} \in \partial\Omega$ $\quad f(\vec{r}) = g(\vec{r})$

$\Omega = [a_x, b_x] \times [a_y, b_y] \times [a_z, b_z] \qquad \vec{r} = (x, y, z)$

griglie equispaziate

$$\begin{cases} x_1 = a_x \\ x_{N_x} = b_y \end{cases} \qquad \begin{cases} y_1 = a_y \\ y_{N_y} = b_y \end{cases} \qquad \begin{cases} z_1 = a_z \\ z_{N_z} = b_z \end{cases}$$

$$h_x = \frac{b_x - a_x}{N_x - 1} \qquad h_y = \frac{b_y - a_y}{N_y - 1} \qquad h_z = \frac{b_z - a_z}{N_z - 1}$$

$$\vec{r}_{ijk} \Leftrightarrow (x_i, y_j, z_k)$$

$$\nabla^2 \phi = f \qquad \text{dove} \quad \phi_{i,j,k} = \phi(r_{i,j,k})$$

$$\frac{\partial^2 \phi(\vec{r})}{\partial x^2} + \frac{\partial^2 \phi(\vec{r})}{\partial y^2} + \frac{\partial^2 \phi(\vec{r})}{\partial z^2} = f(\vec{r}) \qquad \text{con} \quad x_i = a_x + (i-1) h_x$$

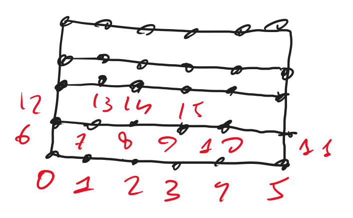
$$\frac{\phi_{i+1,j,k} - 2\phi_{i,j,k} + \phi_{i-1,j,k}}{h_x^2} + \frac{\phi_{i,j+1,k} - 2\phi_{i,j,k} + \phi_{i,j-1,k}}{h_y^2} + \frac{\phi_{i,j,k+1} - 2\phi_{i,j,k} + \phi_{i,j,k-1}}{h_z^2} = f_{i,j,k}$$

Con le condizione al contorno

$$\begin{cases} \phi_{1jk} = g_{1jk} & \forall j,k \\ \phi_{N_x jk} = g_{N_x jk} & \forall j,k \\ \phi_{i1k} = g_{i,1,k} & \forall i,k \\ \phi_{iN_y k} = g_{i,N_y,k} & \forall i,k \\ \phi_{ij1} = g_{ij1} & \forall i,j \\ \phi_{ijN_k} = g_{i,j,N_k} & \forall i,j \end{cases}$$

Voglio trovare una corrispondenza $(i,j,k) \Leftrightarrow l$

$l = 0, \dots, N_x N_y N_z - 1$



$$(i,j,k) \to l = (k-1) N_x N_y + (y-1) N_x + (i-1)$$

trasformazione inversa

dove floor(x) è l'intero uguale o minore di x più grande possibile

$$l \to (i,j,k) = \begin{cases} k = \text{floor}(l / N_x N_y) + 1 \\ j = \text{floor}((l - (k-1) N_x N_y)) / N_x) + 1 \\ i = l - (k-1)(N_x N_y) - (j-1) N_x + 1 \end{cases}$$

Sia $l \Rightarrow (i,j,k) \quad m = (i', j', k')$

$$M_{lm} = \begin{cases} \frac{1}{h_x^2} & \text{se } (i',j',k') = (i+1,j,k) \\ \frac{1}{h_x^2} & \text{se } (i',j',k') = (i-1,j,k) \\ \frac{1}{h_y^2} & \text{se } (i',j',k') = (i,j+1,k) \\ \frac{1}{h_y^2} & \text{se } (i',j',k') = (i,j-1,k) \\ \frac{1}{h_z^2} & \text{se } (i',j',k') = (i,j,k+1) \\ \frac{1}{h_z^2} & \text{se } (i',j',k') = (i,j,k-1) \\ -2\left(\frac{1}{h_x^2} + \frac{1}{h_y^2} + \frac{1}{h_z^2}\right) & \text{se } (i',j',k') = (i,j,k) \end{cases}$$

$$\boxed{\sum_m M_{lm} \phi_m = f_l + b_l}$$

$$\text{con} \quad b_l = \begin{cases} 0 & \text{se } l \notin \partial\Omega \\ g_l & \text{se } l \in \partial\Omega \end{cases}$$

PROBLEMA

Supponiamo $N_x = 100, \; N_y = 100, \; N_z = 100$

la matrice M ha $(100 \times 100 \times 100)^2$ elementi

10^{12} elementi

in virgola mobile ho bisogno di 7.45 TB

MA: Noi sappiamo calcolare $\hat{M} \cdot \vec{\phi}$

<u>Algoritmi iterativi</u>

Per risolvere $\hat{M} \cdot \vec{\phi} = \vec{s}$

permettono di trovare $\vec{\phi}$ sapendo calcolare

$\hat{M} \cdot \vec{t}$

METODO DI JACOBI E DI GAUSS-SEIDEL

$$\sum_m M_{lm} \phi_m = s_l$$

Definiamo la matrice A

$$\begin{cases} A_{ll} = \frac{1}{M_{ll}} \\ A_{lm} = 0 & \text{per } l \neq m \end{cases}$$

il problema diventa

$$(\hat{A}\hat{M}) \cdot \vec{\phi} = \hat{A} \cdot \vec{s}$$

$$(AM)_{lm} = \begin{cases} 1 & \text{se } l = m \\ \frac{M_{lm}}{M_{ll}} & \text{se } l \neq m \end{cases}$$

$$\tilde{\vec{s}} = \hat{A}\vec{s} \qquad \text{con} \quad \tilde{s}_l = \frac{s_l}{M_{ll}}$$

$$\tilde{M}_{lm} = \begin{cases} 0 & \text{se } l = m \\ -\frac{M_{lm}}{M_{ll}} & \text{se } l \neq m \end{cases}$$

$$(\mathbb{1} - \tilde{\hat{M}}) \cdot \vec{\phi} = \tilde{\vec{s}}$$

Idea $\quad \vec{\phi}^{(0)} = \tilde{\vec{s}}$

(indice di iterazione)

$$\vec{\phi}^{(i+1)} = \tilde{\hat{M}} \cdot \vec{\phi}^{(i)} + \tilde{\vec{s}}$$

se si ha convergenza $\quad \vec{\phi}^{(i+1)} = \vec{\phi}^{(i)}$

$$\vec{\phi}^{(i)} = \tilde{\hat{M}} \vec{\phi}^{(i)} + \tilde{\vec{s}}$$

$$(\mathbb{1} - \tilde{\hat{M}}) \cdot \vec{\phi}^{(i)} = \tilde{\vec{s}}$$

infatti:

$$\phi^{(0)} = \tilde{\vec{s}}$$

$$\phi^{(1)} = \tilde{\hat{M}} \tilde{\vec{s}} + \tilde{\vec{s}}$$

$$\phi^{(2)} = \tilde{\hat{M}}^2 \tilde{\vec{s}} + \tilde{\hat{M}} \tilde{\vec{s}} + \tilde{\vec{s}}$$

però altra maniera

$$(\mathbb{1} - \tilde{\hat{M}}) \vec{\phi} = \tilde{\vec{s}}$$

$$\vec{\phi} = (\mathbb{1} - \tilde{\hat{M}})^{-1} \tilde{\vec{s}}$$

$$= (\mathbb{1} + \tilde{\hat{M}} + \tilde{\hat{M}}^2 + \tilde{\hat{M}}^3 + \dots) \tilde{\vec{s}}$$

Variante Gauss-Seidel

$$\phi_l^{(i+1)} = \sum_{m<l} \tilde{M}_{lm} \phi_m^{(i+1)} + \sum_{m \geq l} \tilde{M}_{lm} \phi^{(i)}$$