Es Abbiamo visto che $\lim_{n\to+\infty} \sqrt[n]{n^k} = 1 \quad \forall k>0$

$$\sqrt[n]{n^k} = \left(n^k\right)^{\frac{1}{n}} = n^{\frac{k}{n}} = e^{\frac{k}{n}\log n} = e^{k\frac{\log n}{n}}$$

$\frac{\log n}{n} \to 0$

$\to e^{k\cdot 0} = e^0 = 1$

$$\boxed{\text{Ultimo criterio per serie a termini positivi è il criterio del CONFRONTO ASINTOTICO}}$$

$a_n \geq 0$ $\qquad$ $b_n \geq 0$

$\downarrow$

$\sum_{n=0}^{+\infty} a_n \qquad \sum_{n=0}^{+\infty} b_n$

se $\lim_{n \to +\infty} \frac{a_n}{b_n} = L \neq 0$
$\neq +\infty$

$a_n \approx L b_n$
n grande

$a_n \sim b_n$

$\sum_{n=0}^{+\infty} a_n$ CONVERGE $\Longleftrightarrow$ $\sum_{n=0}^{+\infty} b_n$ CONVERGE

$\sum_{n=0}^{+\infty} a_n$ DIVERGE $\Longleftrightarrow$ $\sum_{n=0}^{+\infty} b_n$ DIVERGE

Es Determinare al variare di $\alpha > 0$ il carattere della serie

$$\sum_{n=0}^{+\infty} \frac{\alpha^n + 5^n}{7^n + n^7}$$

$$a_n = \frac{\alpha^n + 5^n}{7^n + n^7}$$

$$a_n = \frac{\alpha^n + 5^n}{7^n + n^7} = \frac{?}{7^n \left[1 + \underbrace{\frac{n^7}{7^n}}_{\to 0}\right]}$$

per confronto infiniti

$\boxed{\text{SE } \alpha < 5}$

$$a_n = \frac{\alpha^n + 5^n}{7^n\left(1+\frac{n^7}{7^n}\right)} = \frac{5^n\left(\frac{\alpha^n}{5^n}+1\right)}{7^n\left(1+\frac{n^7}{7^n}\right)} =$$

$\frac{\alpha^n}{5^n} \to 0$

$$a_n = \left(\frac{5}{7}\right)^n \cdot (1+o(1))$$

$$= \left(\frac{5}{7}\right)^n \frac{\left(\overset{0}{\cancel{\frac{\alpha^n}{5^n}}}+1\right)}{\left(1+\underset{0}{\cancel{\frac{n^7}{7^n}}}\right)} \to \frac{1}{1} = 1$$

$$a_n \sim \left(\frac{5}{7}\right)^n$$

$$\lim_n \frac{a_n}{\left(\frac{5}{7}\right)^n} = \lim_n \frac{\cancel{\left(\frac{5}{7}\right)^n}\frac{\left(\frac{\alpha^n}{5^n}+1\right)}{\left(1+\frac{n^7}{7^n}\right)}}{\cancel{\left(\frac{5}{7}\right)^n}} = 1 \neq 0$$

$$\sum_{n=0}^{\infty} a_n < +\infty \iff \sum_{n=0}^{+\infty} \left(\frac{5}{7}\right)^n \text{ converge}$$

CONVERGE $q = 5/7 < 1$

Se $\alpha = 5$

$$a_n = \frac{5^n + 5^n}{7^n + n^7} = \frac{2 \cdot 5^n}{7^n\left(1 + \frac{n^7}{7^n}\right)} =$$

$$= \left(\frac{5}{7}\right)^n \left(\frac{2}{1 + n^7/7^n}\right) \to \frac{2}{1} = 2$$

$$a_n \sim \left(\frac{5}{7}\right)^n \qquad \sum_{n=0}^{+\infty} \left(\frac{5}{7}\right)^n \text{ converge}$$

$$\Downarrow$$

$$\sum_{n=0}^{+\infty} a_n \text{ converge}$$

Se $\alpha > 5$

$$a_n = \frac{\alpha^n \left[1 + \frac{5^n}{\alpha^n}\right]}{7^n \left[1 + \frac{n^7}{7^n}\right]} = \left(\frac{\alpha}{7}\right)^n \frac{\left[1 + \frac{5^n}{\alpha^n}\right]}{\left[1 + \frac{n^7}{7^n}\right]}$$

$\frac{5^n}{\alpha^n} \to 0$, $\frac{n^7}{7^n} \to 0$

$\to 1$

$$a_n \sim \left(\frac{\alpha}{7}\right)^n$$

$\alpha < 7$

$$\frac{1}{1-\left(\frac{\alpha}{7}\right)}$$

$$\sum_{n=0}^{+\infty} a_n < +\infty \iff \sum_{n=0}^{+\infty} \left(\frac{\alpha}{7}\right)^n \text{ converge}$$

$\alpha < 7$ CONVERGE

$\alpha \geq 7$ DIVERGE

$q = \frac{\alpha}{7}$

CONVERGE se $q = \frac{\alpha}{7} < 1$

DIVERGE se $q = \frac{\alpha}{7} \geq 1$

Per applicare criterio del confronto asintotico scrivo

$$\underbrace{a_n}_{\substack{\vee\mid \\ 0}} = \underbrace{b_n}_{\substack{\vee\mid \\ 0}} \left(\dots \right)$$

$\dots \to L \neq 0 \qquad L \neq +\infty$

$$\sum_{n=0}^{\infty} a_n < +\infty \iff \sum_{n=0}^{+\infty} b_n < +\infty$$

$= +\infty \qquad\qquad +\infty$

SERIE ARMONICA GENERALIZZATA

$\alpha > 0$ $\qquad \displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$

TEOREMA (NO DIMOSTR.)

se $\alpha > 1$ LA SERIE $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ CONVERGE

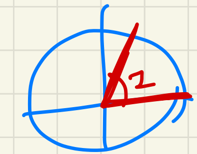
se $\alpha \leq 1$ LA SERIE $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ DIVERGE $= +\infty$

$\alpha = 1$ serie armonica $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$

$\alpha = \frac{1}{2}$ $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{+\infty} \frac{1}{n^{1/2}} = +\infty$

es $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \dots$

$< +\infty$



Es Determinare per quali $\alpha > 0$ è convergente la serie

$$\sum_{n=1}^{+\infty} \underbrace{n^{\alpha}}_{>0} \cdot \underbrace{\sin\left(\frac{1}{n}\right)}_{>0}$$

$0 < \dfrac{1}{n} \leq 1 < \dfrac{\pi}{2}$

$\sin 0 \leq \sin \dfrac{1}{n} < \sin 1 < 1$

$\sin 0 = 0$

$$a_n = n^\alpha \cdot \sin\left(\frac{1}{n}\right)$$

$n \to +\infty$

$\frac{1}{n} \to 0$

$$\sin\left(\frac{1}{n}\right) = \frac{1}{n} + o\left(\frac{1}{n}\right) = \frac{1}{n}\left[1 + \underset{\substack{\downarrow \\ 0}}{o(1)}\right]$$

se $n \to +\infty$

$x \to 0$

$$\sin x = x + o(x)$$

$1 + 0 = 1$

$$a_n = n^\alpha \cdot \frac{1}{n}\left[1 + o(1)\right] = \frac{1}{n^{1-\alpha}}\overbrace{\left[1 + o(1)\right]}$$

$$= n^{\alpha - 1}\left[1 + o(1)\right]$$

$$\boxed{a_n \simeq \frac{1}{n^{1-\alpha}}}$$

$$a_n \simeq n^{\alpha-1}$$

confronto asintotico

$$\sum_{n=1}^{\infty} a_n < +\infty \iff \sum_{n=1}^{+\infty} \frac{1}{n^{1-\alpha}} < +\infty$$

$$= +\infty \qquad\qquad = +\infty$$

serie arm. generalizzata con esponente $1-\alpha$

CONVERGE	se $1-\alpha > 1$	$\rightarrow$	$\alpha < 0$
DIVERGE	se $1-\alpha \leq 1$	$\rightarrow$	$\alpha \geq 0$

Es. 1 Calcolare al variare di α il limite della successione

$$a_n = \frac{n^{\alpha} \cdot \left[\sqrt{1+\frac{1}{4n^2}} - \cos\left(\frac{1}{n}\right) - \frac{1}{n^2} \right]}{\sqrt{n} - \log(n+1)}$$

2 Determinare al variare di α il carattere della serie

$$\sum_{n=1}^{+\infty} |a_n|$$

$$\sqrt{n} - \lg(n+1) = n^{\frac{1}{2}} - \lg(n+1) = n^{\frac{1}{2}}\left[1 - \underbrace{\frac{\lg(n+1)}{n^{1/2}}}_{\to 0}\right]$$

$$= n^{\frac{1}{2}}\left(1 + o(1)\right)$$

per confronto infiniti

$$\underbrace{\sqrt{1+\frac{1}{4n^2}}} - \underbrace{\cos\left(\frac{1}{n}\right)} - \underbrace{\frac{1}{n^2}}$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + o(x^4)$$

$$\cos\frac{1}{n} = \underbrace{1 - \frac{1}{2}\frac{1}{n^2} + \frac{1}{24}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)}$$

$n \to +\infty \quad \frac{1}{n} \to 0$

$$\sqrt{1+\frac{1}{4n^2}} = \left(1+\frac{1}{4n^2}\right)^{\frac{1}{2}} = 1+\frac{1}{2}\left(\frac{1}{4n^2}\right)-\frac{1}{8}\left(\frac{1}{4n^2}\right)^2+o\left(\frac{1}{4n^2}\right)^2$$

$$(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x+\underbrace{\frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{2}-1\right)}\,x^2+o(x^2) = 1+\frac{1}{2}x-\frac{1}{8}x^2+o(x^2)$$

$x \to 0$

$$(1+x)^\alpha = 1+\alpha x+\frac{1}{2}\alpha(\alpha-1)x^2+o(x^2)$$

$$= 1+\frac{1}{2}\cdot\frac{1}{4n^2}-\frac{1}{8}\cdot\frac{1}{16n^4}+o\left(\frac{1}{n^4}\right) =$$

$8 \cdot 16 = 128$

$$= 1+\frac{1}{8}\frac{1}{n^2}-\frac{1}{128}\frac{1}{n^4}+o\left(\frac{1}{n^4}\right) =$$

$$\sqrt{1+\frac{1}{4n^2}} - \cos\frac{1}{n} - \frac{1}{n^2} =$$

$$= 1 + \frac{1}{8}\frac{1}{n^2} - \frac{1}{128}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) - \left[1 - \frac{1}{2}\frac{1}{n^2} + \frac{1}{24}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)\right] - \frac{1}{n^2} =$$

$$= \cancel{1} + \frac{1}{8}\frac{1}{n^2} - \frac{1}{128}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) - \cancel{1} + \frac{1}{2}\frac{1}{n^2} - \frac{1}{24}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) - \frac{1}{n^2} = \frac{1}{n^2}\left[\frac{1}{8} + \frac{1}{2} - 1\right] + \frac{1}{n^4}\left[-\frac{1}{128} - \frac{1}{24}\right] + o\left(\frac{1}{n^4}\right)$$

$$= \underbrace{-\frac{3}{8}\frac{1}{n^2}} + \left[-\frac{1}{128}-\frac{1}{24}\right]\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) =$$

$$= \frac{1}{n^2}\left[-\frac{3}{8} + \boxed{\left(-\frac{1}{128}-\frac{1}{24}\right)\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)}\right]$$

$\to 0 \quad n \to +\infty$

$$= \frac{1}{n^2}\left[-\frac{3}{8} + o(1)\right]$$

$$a_n = \frac{n^{\alpha} \cdot \frac{1}{n^2}\left[-\frac{3}{8}+o(1)\right]}{n^{\frac{1}{2}} \cdot \left[1+o(1)\right]} =$$

$$= \frac{n^{\alpha} \cdot n^{-2}}{n^{1/2}} \quad \frac{\left[-\frac{3}{8}+o(1)\right]}{\left[1+o(1)\right]} =$$

$$= n^{\alpha-2-\frac{1}{2}} \frac{\left[-\frac{3}{8}+o(1)\right]}{\left[1+o(1)\right]} = n^{\alpha-\frac{5}{2}} \frac{\left[-\frac{3}{8}+o(1)\right]}{\left[1+o(1)\right]}$$

$$a_n = n^{\alpha - \frac{5}{2}} \, \frac{\left[-\frac{3}{8} + o(1)\right]}{\left[1 + o(1)\right]}$$

$\alpha > \frac{5}{2}$ $$\frac{+\infty \cdot \left(-\frac{3}{8}\right)}{1} = -\infty$$

$\alpha = \frac{5}{2}$ $$-\frac{3}{8}$$

$\alpha < \frac{5}{2}$ $$\frac{0 \cdot \left(-\frac{3}{8}\right)}{1} = 0$$

2) $\sum_{n=1}^{+\infty} |a_n|$

$$|a_n| = n^{\alpha - 5/2} \frac{\left|-\frac{3}{8} + o(1)\right|}{|1+o(1)|} \sim n^{\alpha - \frac{5}{2}}$$

$\frac{3}{8} \neq 0$

$$\sum_{n=1}^{\infty} |a_n| < +\infty \iff \sum_{n=1}^{\infty} n^{\alpha - \frac{5}{2}} < +\infty$$

$$n^{\alpha - \frac{5}{2}} = \frac{1}{n^{-(\alpha - \frac{5}{2})}} \qquad \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{5}{2} - \alpha}} < +\infty$$

CONVERGE $\frac{5}{2} - \alpha > 1 \rightarrow \alpha < \frac{3}{2}$

DIVERGE $5/2 - \alpha \leq 1 \rightarrow \alpha \geq 3/2$