RELLICH - KONDRACHOV theorem

$p<n$, U bdd of class C^1. $W^{1,p}(U) \hookrightarrow L^q(U)$ compact $\forall q \in [1, p^*)$

Precisely we prove: $\forall f_k \in W^{1,p}(U)$ BOUNDED, UP TO PASSING TO A SUBSEQUENCE, $\exists f \in L^{p^*}(U)$ such that $\|f_k - f\|_{L^q(U)} \to 0$ $\forall q < p^*$.

Remarks

(1) $p<n$, U bdd (NO REGULARITY)
$W_0^{1,p}(U) \xhookrightarrow[\text{COMPACT}]{} L^q(U)$ $\forall q \in [1, p^*)$ (same proof)

(2) for p^* the COMPACT EMBEDDING is NOT TRUE

(3) Since U is bounded $W^{1,p}(U) \subseteq W^{1,r}(U)$ $\forall r<p$.

So for $p \geq n$, $\underset{\forall r<n}{W^{1,p}(U) \xhookrightarrow{\text{CONT.IN.}} W^{1,r}(U)} \xhookrightarrow{\text{COMPACT}} \underset{\forall q<r^*}{L^q(U)}$

$r \nearrow n \longrightarrow r^* \to +\infty$.

$W^{1,p}(U) \hookrightarrow L^q(U)$ compactly for any $q<+\infty$.

$W^{1,p}(U) \hookrightarrow C(\bar{U})$ compactly for $p>n$. (Morrey..)
$\hookrightarrow L^\infty(U), \|\cdot\|_\infty$.

VERY IMPORTANT OBSERVATION

$\forall p \in [1, +\infty]$, U bdd of class C^1, $W^{1,p}(U) \hookrightarrow L^p(U)$ COMPACT

- if $p < n \Rightarrow W^{1,p} \underset{\text{comp.}}{\hookrightarrow} L^q$ by R.K $q < p^*$ $(p < p^*!)$
- if $p \geq n \Rightarrow W^{1,p}(U) \subseteq W^{1,q}(U)$ $\forall q \leq p$ since U is bounded

$\Rightarrow$ take $q < n$ such that $p < q^*$ and conclude

(for $p > n$ also with Morrey..)

So if $f_k \in W^{1,p}(U)$ is bdd $\|f_k\|_{W^{1,p}(U)} \leq C \Rightarrow \exists f \in L^p(U)$

and f_{k_j} s.t. $\underline{f_{k_j} \longrightarrow f \text{ in } L^p(U)}$

$f \in L^q(U)$ $\forall q \in [1, p^*]$

(bdd sequence in $W^{1,p}$ has subsequence STRONGLY CONVERGING in L^p)

Q: $f \in W^{1,p}(U)$?

Observe that $\forall i = 1..n$ $\left(\frac{\partial f_k}{\partial x_i}\right)$ is bdd in $L^p(U)$

if $p \in (1, +\infty)$ $\quad \frac{\partial f_{k_j}}{\partial x_i} \longrightarrow g_i$ weakly in $L^p(U)$

$\exists (f_{k_j})$

that means $\forall h \in L^{p'}(U)$ $\quad \int_U \frac{\partial f_{k_j}}{\partial x_i} h \to \int g_j h$

take $h \in C_c^\infty(U) \Rightarrow \int_U \frac{\partial f_{k_j}}{\partial x_i} h \to \int g_j h$

$$\left.\begin{array}{l} \int_U \frac{\partial f_{k_j}}{\partial x_i} h \to \int g_j h \\ \parallel \\ -\int_U f_{k_j} \frac{\partial h}{\partial x_i} \to -\int_U f \frac{\partial h}{\partial x_i} \end{array}\right] \quad g_j = \frac{\partial f}{\partial x_j}$$

So for $p \in (1, +\infty)$ $\quad \|f_k\|_{W^{1,p}} \leq C \Rightarrow \exists f_{k_j}, \ f \in W^{1,p}(U)$

$$\left[\begin{array}{l} f_{k_j} \rightharpoonup f \text{ in } L^p(U) \\ \frac{\partial f_{k_j}}{\partial x_i} \rightharpoonup \frac{\partial}{\partial x_i} f \text{ in } L^p(U) \end{array}\right.$$

(weak convergence in $W^{1,p}$ is strong convergence in L^p and weak convergence of the derivatives).

$p = +\infty$: $\to$ same argument $\frac{\partial f_n}{\partial x_i} \xrightharpoonup{L^\infty}{}^{*} g_i = \frac{\partial f}{\partial x_i}$

$\Rightarrow f \in W^{1,\infty}(U)$. (also direct proof with Lipschitz) and Ascoli - Arzelà then.

$\exists f_{k_j} \to f$ strongly in L^∞, $\frac{\partial f_{k_j}}{\partial x_i} \overset{*}{\rightharpoonup} \frac{\partial f}{\partial x_i}$

for $\underline{\underline{p=1}}$ $\|f_k\|_{W^{1,1}} \le C \Rightarrow \exists f_{k_j} \to f$ in $L^1(U)$

but in general $f \notin W^{1,1}(U)$

$f \in L^{1^*}(U)$ $\quad 1^* = \frac{n}{n-1}$

we will see that $f \in BV(U)$

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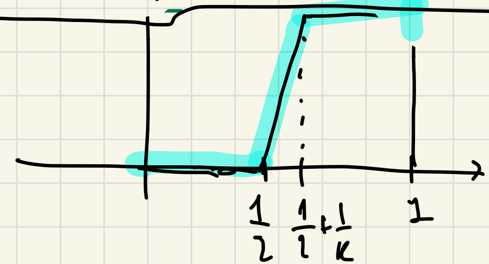
ex: $f_k \in W^{1,1}(0,1)$



$$f_k = \begin{cases} 0 & x \le \frac{1}{2} \\ k(x-\frac{1}{2}) & \frac{1}{2} < x < \frac{1}{2}+\frac{1}{k} \\ 1 & \frac{1}{2}+\frac{1}{k} \le x \le 1 \end{cases}$$

$$\|f_k\|_{L^1} = \left(\frac{1}{2}-\frac{1}{k}\right) + \frac{1}{2k} = \frac{1}{2} - \frac{1}{2k}$$

$$f_k'(x) = \begin{cases} 0 & \\ k & \frac{1}{2} < x < \frac{1}{2}+\frac{1}{k} \\ 0 & \end{cases}$$

$\|f_k'\|_{L^1} = 1$

$$f_k \to \chi_{(\frac{1}{2},1)} = \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases}$$

$f_k \to \chi_{(\frac{1}{2},1)}$ in L^q $\forall q$.

Derivative in the sense of distrib. of $\chi_{(\frac{1}{2},1)}$ is $\delta_{\frac{1}{2}} \Rightarrow \chi_{(\frac{1}{2},1)} \notin W^{1,1}(0,1)$

PROOF OF R-K Take $V \supset\supset U$ and consider the extension $E: W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$, Ef_k is a bounded sequence in $W^{1,p}(\mathbb{R}^n)$ such that $\operatorname{supp}(E f_k) \subseteq \overline{V}$.

NOTE THAT since $C_c^\infty(U) \subseteq C_c^\infty(\mathbb{R}^n)$, $f_k \in W_0^{1,p}(U)$ THEN $f_k \in W_0^{1,p}(\mathbb{R}^n)$, $\operatorname{supp} f_k \subseteq \overline{U}$.

→ We have to prove that up to a subsequence

$$E(f_n) \longrightarrow \tilde{f} \text{ in } L^q(\mathbb{R}^n) \quad \forall\, q < p^* \quad \text{where } \operatorname{supp} \tilde{f} \subset V.$$

then this will imply $f_n \to \tilde{f}$ in $L^q(U)$.

→ Note that by (GNS) if $E(f_k)$ is Cauchy in $L^1(\mathbb{R}^n)$ then it is also Cauchy in $L^q(\mathbb{R}^n)$ $\forall\, q \in (1, p^*)$

$$\|Ef_k - Ef_a\|_{L^q} \underset{\text{INTERPOLATION}}{\leq} \|Ef_k - Ef_a\|_{L^1}^{\theta} \, \|Ef_k - Ef_a\|_{L^{p^*}}^{1-\theta} \quad \text{where } \frac{1}{q} = \theta + \frac{1-\theta}{p^*}$$

$$\overset{(GNS)}{\leq} C \|Ef_k - Ef_a\|_{L^1}^{\theta} \, \|Ef_n - Ef_a\|_{W^{1,p}}^{1-\theta} \leq$$

CONTINUITY of E

$$\leq \overline{C} \, \|Ef_n - Ef_a\|_{L^1}^{\theta} \, \|f_k - f_a\|_{W^{1,p}(U)} \leq \overline{C} \, \|Ef_k - Ef_a\|_{L^1}^{\theta}$$

CLAIM: up to passing to a subsequence Ef_k is Cauchy in $L^1(\mathbb{R}^n)$

If the claim is true the same subsequence Ef_k is Cauchy in $L^q(\mathbb{R}^n)$ $\Rightarrow$ so it converges

$Ef_k \longrightarrow f$ in $L^q(\mathbb{R}^n)$ $\forall\, q < p^*$

($f_k \longrightarrow f$ in $L^q(U)$ $\forall\, q<p^*$).

Moreover $f \in L^{p^*}(\mathbb{R}^n)$ (so also in $L^{p^*}(U)$). Indeed

$\|Ef_k\|_{L^{p^*}(\mathbb{R}^n)} \overset{(GNS)}{\leq} C\|Ef_k\|_{W^{1,p}(\mathbb{R}^n)} \leq C \Rightarrow$ so up to a

subsequence $Ef_{k_j} \longrightarrow \tilde{f}$ in $L^{p^*}(\mathbb{R}^n)$ $(+\infty > p^* > p \geq 1)$ $\tilde{f} \in L^{p^*}(\mathbb{R}^n)$

$\Rightarrow T_{Ef_{k_j}} \longrightarrow T_{\tilde{f}}$ in the sense of distributions

$T_{Ef_k} \longrightarrow T_f$ in the sense of distr. by L^q convergence

$\Rightarrow T_f = T_{\tilde{f}} \Rightarrow$ FUNDAMENTAL LEMMA of CALC. of VARIATIONS $\Rightarrow$ $f = \tilde{f}$ a.e.

Reduced to prove the claim:

Let $\varepsilon \in (0,1)$ $\quad \eta_\varepsilon(y) = \frac{1}{\varepsilon^n} \eta\left(\frac{y}{\varepsilon}\right)$ (usual mollifiers)

$\boxed{f_\varepsilon^k = E f_k * \eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)}$ $\quad$ $\boxed{\text{supp } f_\varepsilon^k \subseteq V + B(0,1) \ \forall \varepsilon, \forall k.}$

Fix $\varepsilon \in (0,1)$ $\quad \|f_k^\varepsilon\|_\infty \leq \sup_x \int_{\mathbb{R}^n} |E f_k(y)| \, \eta_\varepsilon(x-y) \, dy \leq$

$$\leq \|\eta_\varepsilon\|_{L^\infty} \cdot \int_V |E f_k(y)| \, dy \underset{\text{Hölder}}{\leq} \|\eta_\varepsilon\|_\infty \|E f_k\|_{L^p} |V|^{1-\frac{1}{p}} \leq$$

$$\leq \frac{C}{\varepsilon^n} \cdot \tilde{C} \, |V|^{1-\frac{1}{p}} \leq \overline{C} \frac{1}{\varepsilon^n}$$

$$\| |D f_k^\varepsilon| \|_\infty \leq \| |D \eta_\varepsilon| \|_\infty \cdot \|E f_k\|_{L^p} |V|^{1-\frac{1}{p}} \leq \frac{C}{\varepsilon^{n+1}}$$

f_k^ε are equilipschitz, equibounded, all supported in $V + B(0,1)$.

by ASCOLI ARZELÀ $\exists\, k_i^\varepsilon$ (depending on ε) such that

$$f^\varepsilon_{k_i^\varepsilon} \to \tilde{f}^\varepsilon \text{ UNIFORMLY} \Longrightarrow \boxed{f^\varepsilon_{k_i^\varepsilon} \to \tilde{f}^\varepsilon \text{ in } L^1(\mathbb{R}^n)}$$

Now observe that $\|Ef_k - f_k^\varepsilon\|_{L^1(\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0^+$
UNIFORMLY in k.

Indeed $\|Ef_k - f_k^\varepsilon\|_{L^1} = \int_{\mathbb{R}^n} \left| Ef_k(x) - \int_{B(0,\varepsilon)} Ef_k(x-y) \frac{1}{\varepsilon^n} \eta\left(\frac{y}{\varepsilon}\right) dy \right| dx$

$z = y/\varepsilon$

$$= \int_{\mathbb{R}^n} \left| Ef_k(x) - \int_{B(0,1)} Ef_k(x-\varepsilon z)\eta(z)dz \right| dx = \quad \left(\text{since } \int_{B(0,1)} \eta(z)dz = 1 \right)$$

$$= \int_{\mathbb{R}^n} \left| \int_{B(0,1)} [Ef_k(x) - Ef_k(x-\varepsilon z)]\eta(z)dz \right| dx$$

$$\leq \int_{\mathbb{R}^n} \int_{B(0,1)} |Ef_k(x) - Ef_k(x-\varepsilon z)|\, \eta(z)\,dz\,dx =$$

Fubini Tonelli $= \int_{B(0,1)} \eta(z) \int_{V+B(0,1)} |Ef_k(x) - Ef_k(x-\varepsilon z)| \, dx \, dz =$

since $\operatorname{supp} Ef_k(\cdot - \varepsilon z) \subseteq V + B(0,1)$
$\operatorname{supp} Ef_k(\cdot) \subseteq V$.

for $h \in \mathbb{R}^n$ $\quad \tau_h g(x) := g(x+h)$

$$= \int_{B(0,1)} \eta(z) \int_{V+B(0,1)} |Ef_k - \tau_{-\varepsilon z} Ef_k| \, dx \, dz \leq \text{Hölder}$$

$$\leq \int_{B(0,1)} \eta(z) \, \| Ef_k - \tau_{-\varepsilon z} Ef_k \|_{L^p} \, |V + B(0,1)|^{1-\frac{1}{p}} \, dz$$

for $g \in W^{1,p}(\mathbb{R}^n)$, for all $p \in [1, +\infty]$

(*) $\| \tau_h g - g \|_{L^p(\mathbb{R}^n)} \leq |h| \, \| |Dg| \|_{L^p(\mathbb{R}^n)}$

$$\leq \int_{B(0,1)} \eta(z) \underbrace{|\varepsilon z|}_{|z| \leq 1} \cdot \underbrace{\| |D\, Ef_k| \|_{L^p}}_{\leq \|Ef_k\|_{W^{1,p}} \leq C.} \cdot |V + B(0,1)|^{1-\frac{1}{p}} \, dz \leq \overline{C} \varepsilon .$$

Proof of ⊛   for $p = +\infty$ is Lipschitz continuity.

take $g \in C_c^\infty(\mathbb{R}^n)$, $1 \le p < +\infty$

scalar product

$$|\tau_h g(x) - g(x)|^p = |g(x+h) - g(x)|^p = \left| \int_0^1 Dg(x+th) \cdot h \, dt \right|^p =$$

$$\le \left[\int_0^1 |Dg(x+th)| \cdot |h| \right]^p \le \boxed{\text{JENSEN for } p>1} \le |h|^p \int_0^1 |Dg(x+th)|^p dt$$

(for $p=1$ OBVIOUS)

$$\|\tau_h g - g\|_{L^p}^p = \int_{\mathbb{R}^n} |\tau_h g - g|^p dx \le |h|^p \int_{\mathbb{R}^n} \int_0^1 |Dg(x+th)|^p dt \, dx$$

$$\underset{\text{FUBINI TONELLI}}{=} |h|^p \int_0^1 \int_{\mathbb{R}^n} |Dg(x+th)|^p dx \, dt = |h|^p \| |Dg| \|_p^p$$

$g \in W^{1,p}(\mathbb{R}^n)$ $\exists g_k \to g$ in $W^{1,p}$ $g_k \in C_c^\infty(\mathbb{R}^n)$

$$\|g - \tau_h g\|_{L^p} \le \|g_k - g\|_{L^p} + \|\tau_h g_k - \tau_h g\|_{L^p} + \|g_k - \tau_h g_k\|_{L^p} \le$$

$$\le 2\|g_k - g\|_{L^p} + |h| \, \| |Dg_k| \|_p \quad \forall k$$

So sending $k \to +\infty$ $\|g - \tau_h g\|_{L^p} \le |h| \, \| |Dg| \|_{L^p}$.