X vectorial space on $\mathbb{R}$ (also on $\mathbb{C}$)

closed by sum and multiplication with real constants

distance $d(x,y)$

$x_n \to x$ in the metric space X if

$$\lim_{n\to+\infty} d(x_n, x) = 0$$

X is complete with respect to the distance d

if $\forall\, x_n$ sequence in X which is a Cauchy sequence

$(d(x_n, x_m) \to 0$ as $n, m \to +\infty)$

$\exists\, x \in X$ such that $d(x_n, x) \to 0$.

Among metric spaces (X, d)

vectorial space → distance

we find NORMED SPACES

X is a normed space if $\exists$ a NORM $\|\cdot\| : X \to [0, \infty)$
such that
$\forall$ $d(x,y) = \|x-y\|$

$(X, \|\cdot\|)$ is COMPLETE with respect to the distance associated to the norm

$\to$ $(X, \|\cdot\|)$ is a BANACH SPACE.

$x_n \to x$ in X $\quad d(x_n, x) = \|x_n - x\| \to 0$.

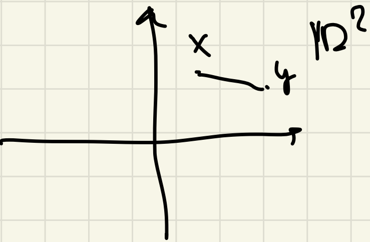
Examples

1) $X = \mathbb{R}^n = \{ (x_1, x_2, \dots x_n) \quad x_i \in \mathbb{R} \}$

$$|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$$\underbrace{|x-y| = \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}}_{\text{length of the segment joining } x \text{ and } y}$$

$x, y \in \mathbb{R}^n$



$(\mathbb{R}^n, |\cdot|)$ is a Banach space.

2) $[a,b] \subseteq \mathbb{R}$ $\quad X = \{ f : [a,b] \to \mathbb{R} \text{ continuous} \}$

$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$$

$(X, \|\cdot\|_\infty)$ is a Banach space.

$f_n : [a,b] \to \mathbb{R}$ $\quad \boxed{\max_{x\in[a,b]} |f_n(x) - f_m(x)| \longrightarrow 0}$

$\boxed{f_n \in X \ \forall n}$

$\forall x \in [a,b] \quad f(x) := \lim_n f_n(x)$

f is continuous in $[a,b]$

(f_n is converging uniformly to f).

3) $X = \{ f : I \to \mathbb{R}$, s. that $\int_I |f(x)| dx < +\infty$ $\quad I \subseteq \mathbb{R}$ interval $\}$

$\boxed{\text{I'm not asking } f \text{ to be CONTINUOUS}}$

$$\|f\|_1 = \int_I |f(x)|\, dx \qquad X = L^1(I)$$

$\lambda \in \mathbb{R}$

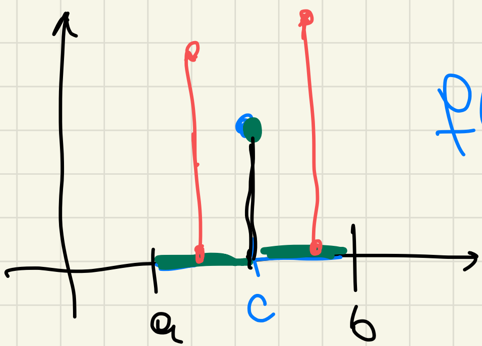
$$\|\lambda f\|_1 = \int_I |\lambda f(x)|\, dx = |\lambda| \, \|f\|_1$$

$$\|f+g\|_1 = \int_I |f(x)+g(x)|\, dx \le \int_I |f(x)| + |g(x)|\, dx$$

$$= \|f\|_1 + \|g\|_1$$

$$\|f\|_1 = 0 \iff \underbrace{\int_I |f(x)|\,dx = 0}$$

$$\boxed{\Rightarrow |f(x)| = 0 \quad \forall x \in I \setminus A \text{ with } \mathcal{L}(A) = 0}$$



$$f(x) = \begin{cases} 0 & \forall x \in [a,b] \setminus \{c\} \\ 1 & x = c \end{cases}$$

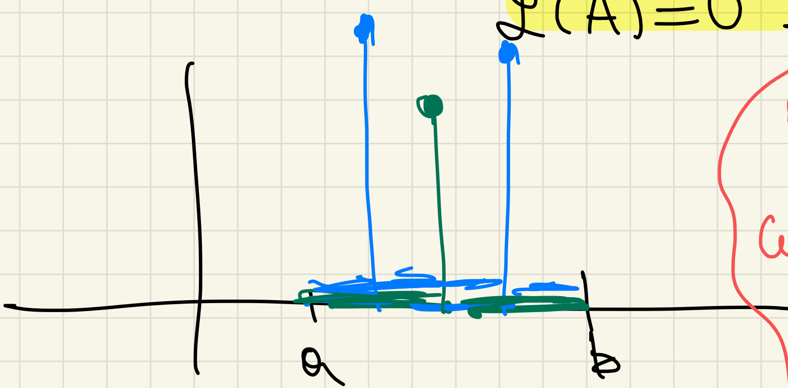
$$\int_a^b |f(x)| = 0$$

$$X = L^1(I) = \left\{ f : I \to \mathbb{R} \text{ such that } \int_I |f(x)|\,dx < +\infty \right\}$$

$f = g$ on $X \iff f(x) = g(x)$ FOR ALMOST EVERY $x \in I$.

$f \in L^1(I) \iff f$ is an equivalence class of functions

$$f = \{ g : I \to \mathbb{R} \text{ such that } \exists A \subseteq I,\ \mathcal{L}(A) = 0 \,]\, g(x) = f(x) \ \forall x \in I \setminus A \}$$



In $L^1(I)$ I identify functions which are equal ALMOST EVERYWHERE (also for random variables we have that

$$X = Y \quad \text{iff} \quad X(\omega) = Y(\omega) \quad \forall \omega \in \Omega' \subseteq \Omega$$

$$\mathbb{P}(\Omega') = 1$$

$(L^1(I), \|\cdot\|_1)$ is a Banach space

$$f_n \to f \text{ in } L^1(I) \iff \|f_n - f\|_1 = \int_I |f_n(x) - f(x)|\, dx \to 0$$

$\boxed{p>1}$ $L^p(I) = \{ f : I \to \mathbb{R}$ such that $\underbrace{\int_I |f(x)|^p \, dx < +\infty \}}$

I identify functions which are equal almost everywhere

$$\int_I |f(x)|^p \, dx = 0 \iff f(x) = 0 \quad \forall x \in I \setminus A$$

for some $A \subseteq I$ with $|A| = 0$

$I = (1, +\infty)$

$f(x) = \frac{1}{x} \notin L^1(1, +\infty)$ $\quad \int_1^{+\infty} \frac{1}{x} \, dx = [\log x]_1^{+\infty} = +\infty$

$f(x) = \frac{1}{x} \in L^2(1, +\infty)$ $\quad \int_1^{+\infty} \left(\frac{1}{x}\right)^2 dx = \left[-\frac{1}{x}\right]_1^{+\infty} = 1 < +\infty$

$$\frac{1}{x} \in L^p(1,+\infty) \quad \forall p>1 \qquad \int_1^{+\infty} \left(\frac{1}{x}\right)^p dx = \frac{1}{p-1}$$

$$\Downarrow$$

$$\int_1^{+\infty} x^{-p}\, dx = \left[\frac{1}{1-p} x^{1-p}\right]_1^{+\infty}$$

$$\frac{1}{\sqrt{x}} \in L^1(0,1) \qquad \int_0^1 \frac{1}{\sqrt{x}}\, dx < +\infty$$

$$\frac{1}{\sqrt{x}} \notin L^2(0,1) \qquad \int_0^1 \left(\frac{1}{\sqrt{x}}\right)^2 dx = \int_0^1 \frac{1}{x}\, dx = +\infty .$$

$$\|f\|_p = \left[\int_I |f(x)|^p dx\right]^{\frac{1}{p}} \qquad \|f\|_2 = \left(\int_I |f(x)|^2 dx\right)^{\frac{1}{2}}$$

V $\|f\|_p = 0 \iff f(x) = 0$ almost everywhere

$\lambda \in \mathbb{R}$

VV $$\|\lambda f\|_p = \left[\int_I |\lambda f(x)|^p dx\right]^{\frac{1}{p}} = \left[\int_I |\lambda|^p |f(x)|^p dx\right]^{\frac{1}{p}} =$$

$$= \left[|\lambda|^p \int_I |f(x)|^p dx\right]^{\frac{1}{p}} = |\lambda| \left(\int_I |f(x)|^p dx\right)^{1/p}$$

$$= |\lambda| \, \|f\|_p$$

$$\|f+g\|_p \overset{?}{\leq} \|f\|_p + \|g\|_p$$

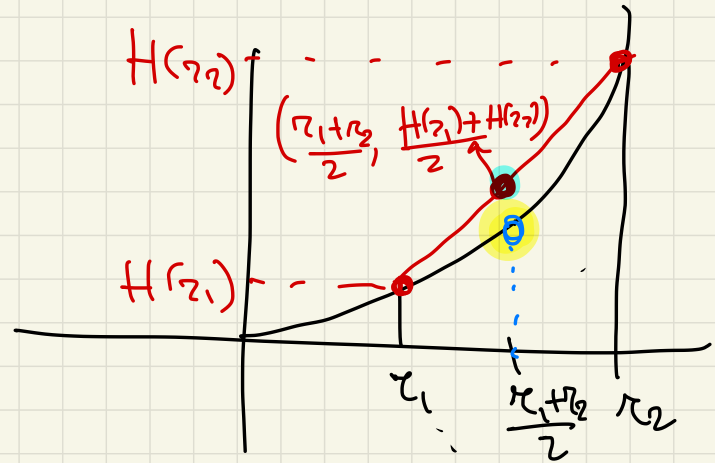
$$\boxed{f \in L^p(I), g \in L^p(I) \Rightarrow f+g \in L^p(I) \, ?}$$

$f \in L^p(I)$ $\qquad \int_I |f(x)|^p \, dx < +\infty$

$g \in L^p(I)$ $\qquad \int_I |g(x)|^p \, dx < +\infty$

$p > 1$

$$\int_I |f(x) + g(x)|^p \, dx$$



$[0, +\infty) \to [0, +\infty)$

$H: \quad r \longmapsto r^p \qquad p > 1 \qquad H(r) = r^p$ CONVEX

$$H\left(\frac{r_1 + r_2}{2}\right) \leq \frac{H(r_1)}{2} + \frac{H(r_2)}{2}$$

$$\boxed{\frac{(r_1 + r_2)^p}{2^p} \leq \frac{r_1^p}{2} + \frac{r_2^p}{2}}$$

$$(x_1+x_2)^p \le \frac{2^p}{2}\left[x_1^p + x_2^p\right] \qquad x_1, x_2 > 0$$

$p \geq 1$

$\forall x_1, x_2 \in \mathbb{R}$

$$(|x_1|+|x_2|)^p \le 2^{p-1}\left[|x_1|^p + |x_2|^p\right]$$

$$\boxed{\int_I |f(x)+g(x)|^p\,dx \le \int_I \left(|f(x)|+|g(x)|\right)^p dx \le 2^{p-1}\int_I \underbrace{|f(x)|^p} + \underbrace{|g(x)|^p}\,dx < +\infty}$$

$$f, g \in L^p(I) \implies f+g \in L^p(I).$$

Now I want to prove that

$$\|f+g\|_p = \left[\int_I |f(x)+g(x)|^p dx\right]^{\frac{1}{p}} \leq \|f\|_p + \|g\|_p =$$

$$= \left[\int_I |f(x)|^p dx\right]^{\frac{1}{p}} + \left[\int_I |g(x)|^p dx\right]^{\frac{1}{p}}$$

$p=2$ $$\sqrt{\int_I |f(x)+g(x)|^2} \leq \sqrt{\int_I |f(x)|^2 dx} + \sqrt{\int_I |g(x)|^2 dx}$$

to prove this we need to pass through the HÖLDER INEQUALITY. (which will be important on its own)

YOUNG INEQUALITY

p conjugate of q
(q conjugate of p)

$$\Updownarrow$$

$$\frac{1}{p}+\frac{1}{q}=1$$

$p \geq 1 \rightarrow q =$ conjugate of p.

$$\frac{1}{q}+\frac{1}{p}=1$$

$$q=\frac{p}{p-1}$$

$p=2 \longrightarrow$ conjugate $q=2$

$$\frac{1}{2}+\frac{1}{2}=1$$

$p=3 \longrightarrow$ conjugate $q=\frac{3}{2}$

$$\frac{1}{3}+\frac{2}{3}=1$$

$p=1 \longrightarrow$ conjugate $q=+\infty$

$\forall \; a, b \in \mathbb{R} \quad a, b \geq 0 \qquad 1 < p, q < +\infty$

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}$$

where p, q are Conjugate $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$

$p = q = 2$ $\quad \boxed{ab \leq \frac{a^2}{2} + \frac{b^2}{2}} \iff (a-b)^2 \geq 0$

$0 \leq a^2 + b^2 - 2ab$

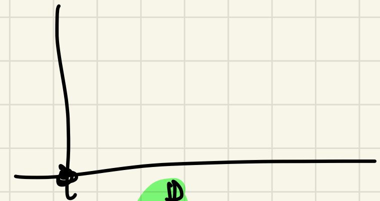
fix p, q conjugate

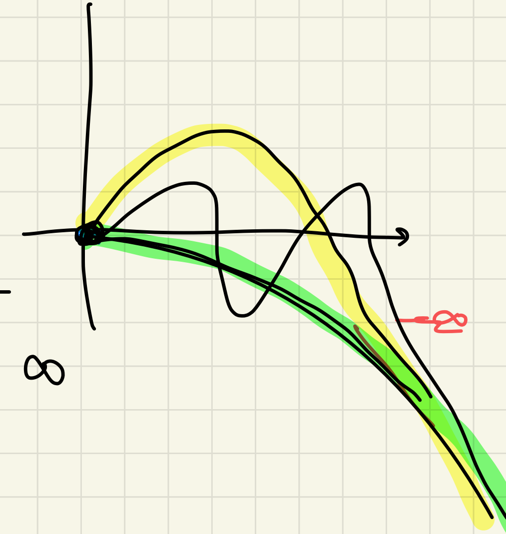
Fix $b \geq 0$

$$\boxed{g(x) = xb - \frac{x^p}{p}}$$

$$g(x) = x \cdot b - \frac{x^p}{p} \qquad x \geq 0$$

$$g(0) = 0$$





$$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} \underbrace{x}_{+\infty} \cdot b - \frac{x^p}{\underbrace{p}_{+\infty}} = -\infty$$

g has a POINT OF MAXIMUM in $[0, +\infty)$

$$g'(x) = b - \frac{\cancel{p} x^{p-1}}{\cancel{p}} = b - x^{p-1}$$

$$g'(x) = 0 \iff x^{p-1} = b \iff \boxed{\bar{x} = b^{\frac{1}{p-1}}}$$

POINT OF MAXIM.

$$g(x) = x \cdot b - \frac{x^p}{p} \leq g(\overline{x}) = \overline{x} \cdot b - \frac{\overline{x}^p}{p} =$$

$\forall x \in [0, +\infty)$

$$= \underbrace{b^{\frac{1}{p-1}} \cdot b}_{} - \frac{1}{p} \cdot b^{\frac{p}{p-1}}$$

$$= b^{\frac{p}{p-1}} \left[\underbrace{1 - \frac{1}{p}}_{} \right] =$$

$q = \frac{p}{p-1}$ $\quad \frac{1}{p} + \frac{1}{q} = 1$

$$= b^q \cdot \frac{1}{q}$$

$\forall a \geq 0 \qquad a \cdot b - \frac{a^p}{p} \leq \frac{b^q}{q}$

$$\boxed{ab \leq \frac{a^p}{p} + \frac{b^q}{q}} \quad \forall a, b \geq 0$$

HÖLDER INEQUALITY

$f \in L^p(I) \qquad g \in L^q(I) \qquad \boxed{\frac{1}{p} + \frac{1}{q} = 1}$

$\Longrightarrow \quad f \cdot g \in L^1(I)$

$$\int_I |f(x)||g(x)|\,dx = \|fg\|_{L^1} \le \|f\|_p \cdot \|g\|_q$$

$$\left[\int_I |f(x)|^p dx\right]^{1/p} \left[\int_I |g(x)|^q dx\right]^{\frac{1}{q}}$$

(IMMEDIATE CONSEQUENCE

$f \in L^2(I),\ g \in L^2(I) \Longrightarrow f \cdot g \in L^1(I)$)

proof $\quad a = \dfrac{|f(x)|}{\|f\|_p} \geq 0 \qquad b = \dfrac{|g(x)|}{\|g\|_q} \geq 0$

I apply YOUNG INEQ.

$$\boxed{a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}}$$

$\forall x \in I$.

$$\int_I \frac{|f(x)|\,|g(x)|}{\|f\|_p \, \|g\|_q}\,dx \leq \int_I \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p}\,dx + \int_I \frac{1}{q} \frac{|g(x)|^q}{\|g\|_q^q}\,dx$$

$$\frac{\int_I |f(x)|\,|g(x)|\,dx}{\|f\|_p \, \|g\|_q} \leq \frac{1}{p} \frac{1}{\cancel{\|f\|_p^p}} \cancel{\int_I |f(x)|^p\,dx} + \frac{1}{q} \frac{1}{\cancel{\|g\|_q^q}} \cancel{\int_I |g(x)|^q\,dx}$$

$$\int_I \frac{|f(x)||g(x)|\,dx}{\|f\|_p \|g\|_q} \le \frac{1}{p} + \frac{1}{q} = 1$$

$$\downarrow$$

$$\int_I |f(x)||g(x)|\,dx = \|fg\|_1 \le \|f\|_p \|g\|_q .$$

Now:

$$f, g \in L^p(I) \Rightarrow \underbrace{\|f+g\|_p} \le \|f\|_p + \|g\|_p. \quad p>1$$

__proof__

$$f, g \in L^p(I) \Rightarrow f+g \in L^p(I)$$

HÖLDER

$$\int_I \underbrace{|f(x)+g(x)|}^p dx \le \|f+g\|_p \, \| |f(x)+g(x)|^{p-1} \|_q$$

$$|f(x)+g(x)|^p = \underbrace{|f(x)+g(x)|}_{\substack{\Uparrow \\ L^p(I)}} \cdot \underbrace{|f(x)+g(x)|^{p-1}}_{\substack{\Uparrow \\ L^q(I)}} \qquad q = \frac{p}{p-1}$$

$$\int_I \left[|f(x)+g(x)|^{p-1} \right]^q = \int_I |f(x)+g(x)|^{\cancel{p-1} \cdot \frac{p}{\cancel{p-1}}} = \int_I \underbrace{|f(x)+g(x)|^p} < +\infty$$

$$|f(x)+g(x)|^p = \underbrace{|f(x)+g(x)|^1}_{\Uparrow \, L^p} \; \underbrace{|f(x)+g(x)|^{p-1}}_{\Uparrow \, L^q}$$

$$\leq \left(|f(x)| + |g(x)|\right) |f(x)+g(x)|^{p-1} =$$

$$= \underbrace{|f(x)|}_{\Uparrow \, L^p} \; \underbrace{|f(x)+g(x)|^{p-1}}_{\Uparrow \, L^q} + \underbrace{|g(x)|}_{\Uparrow \, L^p} \; \underbrace{|f(x)+g(x)|^{p-1}}_{\Uparrow \, L^q}$$

$$\int_I |f(x)+g(x)|^p dx \leq \int_I \underbrace{|f(x)|}_{\Uparrow \, L^p} \; \underbrace{|f(x)+g(x)|^{p-1}}_{\Uparrow \, L^q} dx + \int_I \underbrace{|g(x)|}_{\Uparrow \, L^p} \; \underbrace{|f(x)+g(x)|^{p-1}}_{\Uparrow \, L^q}$$

$$\int_I |f(x)+g(x)|^p\,dx \le \int_I |f(x)|\,|f(x)+g(x)|^{p-1}\,dx + \int_I |g(x)|\,|\;\;|^{p-1}$$

$$\le \|f\|_p \, \big\| |f(x)+g(x)|^{p-1} \big\|_q + \|g\|_p \, \big\| |f(x)+g(x)|^{p-1} \big\|_q$$

$$= \big[\|f\|_p + \|g\|_p\big] \, \underbrace{\big\| |f(x)+g(x)|^{p-1} \big\|_q}_{\parallel}$$

$$q = \frac{p}{p-1}$$

$$\left[\int_I \left[|f(x)+g(x)|^{p-1}\right]^q dx\right]^{\frac{1}{q}} =$$

$$= \left[\int_I |f(x)+g(x)|^{\cancel{p-1}\cdot\frac{p}{\cancel{p-1}}} dx\right]^{\frac{p-1}{p}}$$

$$\frac{\left[\int_I |f(x)+g(x)|^p\,dx\right]^{1}}{\left[\int_I |f(x)+g(x)|^p\,dx\right]^{\frac{p-1}{p}}} \le \frac{\left[\|f\|_p+\|g\|_p\right]\cdot \cancel{\left[\int_I |f(x)+g(x)|^p\,dx\right]^{\frac{p-1}{p}}}}{\cancel{\left[\int_I |f(x)+g(x)|^p\,dx\right]^{\frac{p-1}{p}}}}$$

$$\left[\int_I |f(x)+g(x)|^p\,dx\right]^{\frac{1}{p}} = \|f+g\|_p \le \|f\|_p+\|g\|_p$$

$$1-\left(\frac{p-1}{p}\right)=\frac{1}{p}$$

$\|\cdot\|_p$ is a norm.

$\left(L^p(I), \|\cdot\|_p\right)$ is a Banach space

$f_n \to f$ in $L^p \implies \int_I |f_n(x)-f(x)|^p\,dx \to 0$

So $\left(L^2(I), \|\cdot\|_2\right)$ is a Banach space

but I have also that $f, g \in L^2 \Longrightarrow f \cdot g \in L^1$

$\left(\int_I |f(x)\, g(x)|\, dx \le \|f\|_2 \|g\|_2 \right.$ (Hölder).

ALSO HERE I define A SCALAR PRODUCT

$f, g \longrightarrow f \cdot g = \int_I f(x) g(x)\, dx$

$\mathbb{R}^2$ (x_1, x_2) (y_1, y_2)

SCALAR PRODUCT $(x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 y_2 \in \mathbb{R}$

$$\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$x \quad y \longmapsto x \cdot y \in \mathbb{R}.$$

SCALAR PRODUCT $L^2(I) \times L^2(I) \longrightarrow \mathbb{R}$

$$f \;,\; g \longmapsto (f,g) = \langle f,g \rangle = \int_I f(x)g(x)\,dx$$

$< +\infty$

CONTINUOUS if $\underbrace{f_n \to f}$ in L^2 $\quad \|f_n - f\|_{L^2} \to 0$

$\downarrow$

$$\lim_{n\to+\infty} (f_n, g) = (f,g) \text{ in } \mathbb{R}$$

$$(f_n, g) = \int_I f_n(x) g(x)\, dx =$$

$$= \int_I [f_n(x) - f(x) + f(x)]\, g(x) =$$

$$= \underbrace{\int_I [f_n(x) - f(x)]\, g(x)\, dx}_{\to 0} + \int_I f(x) g(x)\, dx$$

$$\left| \int_I (f_n(x) - f(x))\, g(x)\, dx \right| \leq \int_I |f_n(x) - f(x)|\, |g(x)|\, dx$$

$$\overset{\text{Hölder}}{\leq} \|f_n - f\|_2 \cdot \|g\|_2 \longrightarrow 0$$

$(X, \|\cdot\|)$ Banach space with a SCALAR PRODUCT

$$X \times X \longrightarrow \mathbb{R}$$

$$x, y \longmapsto (x, y)$$

such that $(x, x) = \|x\|^2$

then $\boxed{(X, \|\cdot\|) \text{ is called Hilbert space}}$

$\mathbb{R}^n = X$ $\quad (x, y) = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

$$(x, x) = x_1^2 + x_2^2 + \dots + x_n^2 = |x|^2$$

$X = L^2(I)$ $\qquad (f,g) = \int_I f(x)\,g(x)\,dx$

$$(f,f) = \int_I f(x)\cdot f(x)\,dx =$$

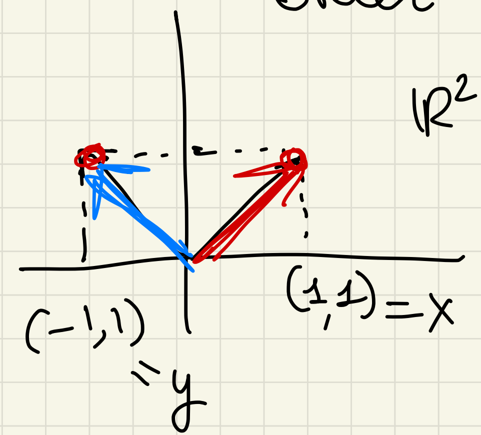
$$= \int_I |f(x)|^2\,dx = \|f\|_2^2$$

Among infinite dimensional spaces the Hilbert spaces are "the most similar" to the finite dimensional case.

$X, \|\cdot\|$ Hilbert space with scalar product (x,y)

$x, y \in X$ are ORTHOGONAL one to the other

$$x \perp y \iff (x,y) = 0$$



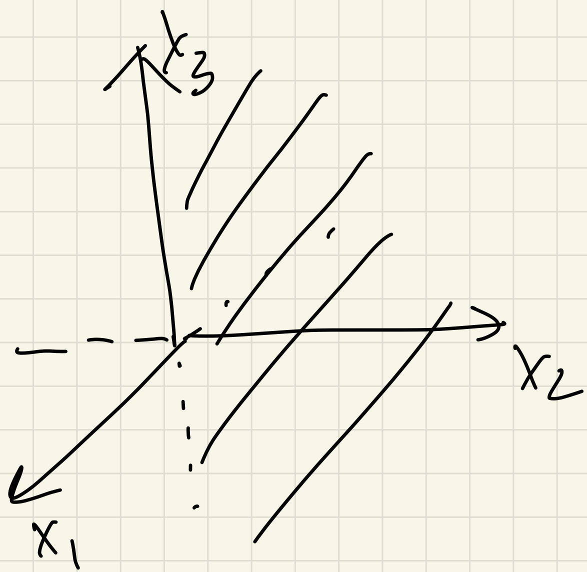
$S \subseteq X$ S subset

$S^\perp$ = orthogonal space =

$$= \{ x \in X \ , \quad (x,s) = 0 \ \ \forall s \in S \}$$

$X = \mathbb{R}^3$ $\qquad S = \{(x_1, 0, 0) \quad x_1 \in \mathbb{R}\} \subseteq \mathbb{R}^3$

$S^\perp = \{(0, y_2, y_3) \quad \text{for } y_2, y_3 \in \mathbb{R}\}$



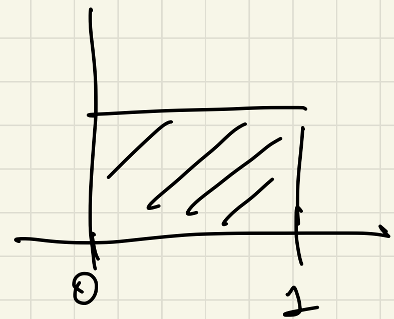
$(S^\perp)^\perp = S$

$X = L^2(0,1)$

$S = \{ \text{constant functions} \}$

$= \{ f \in L^2(0,1) \text{ such that } f(x) = c \text{ for a.e } x \in I \}$

$\int_0^1 c^2 \, dx = c^2 < +\infty$



$S^\perp = \{ f \in L^2(0,1)$ such that $\int_0^1 f(x) \cdot c \, dx = 0 \quad \forall c \}$

$(f(x), c)$

$S^\perp = \{ f \in L^2(0,1) \quad \int_0^1 f(x) \, dx = 0 \}$

ORTHOGONAL PROJECTION THEOREM

X Hilbert space

X has a norm $\|\cdot\|$ is complete

X has a scalar product (x,y) $\quad (x,x) = \|x\|^2$.

$V \subseteq X$ closed subspace

$(v_1, v_2 \in V \implies v_1 + v_2 \in V \quad \forall \lambda \in \mathbb{R} \quad \lambda v_1 \in V$

$v_n \in V \quad v_n \to x \in X \quad \|v_n - x\| \to 0$

$\implies x \in V$ (~~H~~ V contains all the limits of its converging sequences).

Then $\forall x \in X$ there exists a UNIQUE element $v \in V$ (called the orthogonal projection of x in V) and a UNIQUE element $w \in V^\perp$ such that

$$\boxed{x = \underline{\underline{v}} + w}$$

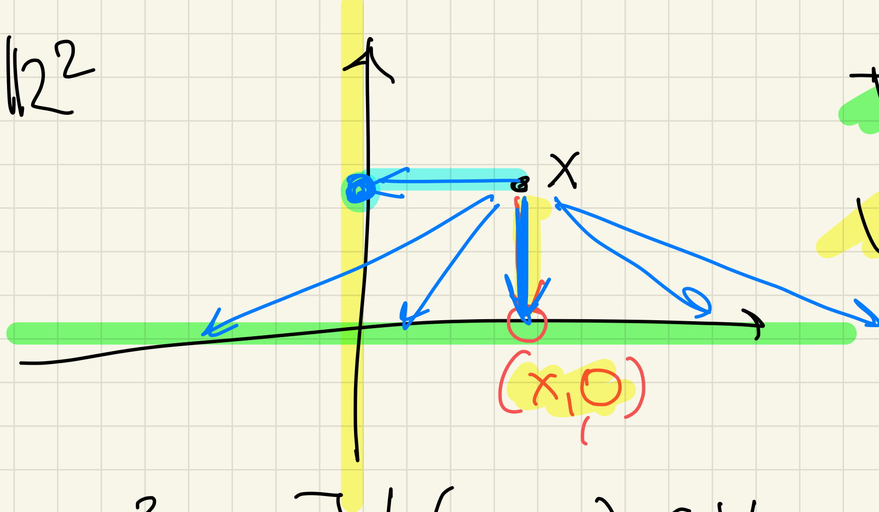
$$\boxed{\begin{aligned} x - v &= w \in V^\perp \\ x - v &\perp V \end{aligned}}$$

v is characterized as the element of V at minimal distance from x

$$d(x,v) = \min_{y \in V} d(x,y)$$

$$d(x,w) = \min_{z \in V^\perp} d(x,z)$$

$\mathbb{R}^2$



$V = \{(x_1, 0) \quad x_1 \in \mathbb{R}\}$

$V^\perp = \{(0, x_2) \quad x_2 \in \mathbb{R}\}$

$\forall \; x \in \mathbb{R}^2 \qquad \exists!\,(x_1, 0) \in V \qquad \exists!\,(0, x_2) \in V^\perp$

$$x = (x_1, 0) + (0, x_2)$$

$$d((x_1, 0), x) = \min_{y_1 \in \mathbb{R}} d(x, (y_1, 0))$$