Theorem $f \in W^{1,\infty}_{loc}(U) \iff f$ locally Lipschitz in U

(U open set in $\mathbb{R}^n$)

Last time $f \in W^{1,\infty}_{loc}(U) \Longrightarrow f$ locally Lipschitz

Assume now f locally lipschitz and prove $f \in W^{1,\infty}_{loc}(U)$

$\forall V \subset\subset U \; \exists C_V \quad |f(x)-f(y)| \le C_V |x-y|$

Obviously $f \in L^\infty(V) \quad \forall V \subset\subset U$.

Fix $W \subset\subset V \subset\subset U$, $|h| < \operatorname{dist}(W, \mathbb{R}^n \setminus V) \Rightarrow x + he_i \in V$ if $x \in W$

fix $i = 1 \dots n$ $\quad f_i^h(x) = \dfrac{f(x+he_i) - f(x)}{h} \in L^\infty(W)$

$$|f_i^h(x)| \le \frac{|f(x+he_i) - f(x)|}{|h|} \le \frac{C_V |h|}{h} = C_V$$

$\exists\, h_j \to 0 \quad f_i^{h_j} \to f_i$ weakly* in $L^\infty(W) \Rightarrow \int_W f_i^{h_j} g \to \int_W f_i g$
$\quad \uparrow L^\infty(W)$

$\forall g \in L^1(W)$ (in particular $\forall g \in C_c^\infty(U)$)

Up to a diagonalization argument we may extract a subsequence $f_i^{\bar h_j} \xrightarrow{*} f_i$ weakly in $L^\infty_{loc}(U)$ $\quad f_i \in L^\infty_{loc}(U)$

$$\forall \phi \in C_c^\infty(U) \qquad \lim_{\bar h_j \to 0} \int_U f_i^{\bar h_j} \phi = \int_U f_i \phi \, dx$$

Moreover
$$\lim_{h_j \to 0} \int_U f(x) \cdot \left[\frac{\phi(x+h_j e_i) - \phi(x)}{h_j}\right] \xrightarrow[\text{dominated}]{\text{Lebesgue}} \int_U f(x) \frac{\partial \phi(x)}{\partial x_i}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$= \lim_{h_j \to 0} \int_U \underbrace{\left[\frac{f(x) - f(x+h_j e_i)}{h_j}\right]}_{= -f_i^{h_j}(x)} \cdot \phi(x+h_j e_j) \to \int_U -f_i(x)\phi(x)\,dx$$

f_i is the weak derivative of f and it is in $L^\infty_{loc}(U)$.

$\Rightarrow f \in W^{1,\infty}_{loc}(U)$.

Thm $f \in W^{1,\infty}(\mathbb{R}^n) \Longleftrightarrow f \in \mathcal{C}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$
$|f(x) - f(y)| \leq C|x-y|$

Thm Let U open bdd of class C^1
$f \in W^{1,\infty}(U) \Longleftrightarrow f \in \mathcal{C}^{0,1}(U)$ (up to representative)
f Lipschitz continuous

$\rightarrow$ [apply extension operator and restrict to U)
$E : W^{1,\infty}(U) \longrightarrow W^{1,\infty}(\mathbb{R}^n)$]

$\leftarrow$ if f is Lipschitz in $\bar{U}$ $\quad |f(x) - f(y)| \leq C_f |x-y|$

$\exists\ \tilde{f}$ extension : $\forall x \in \mathbb{R}^n \quad \tilde{f}(x) := \min\limits_{y \in \bar{U}} \left(f(y) + C_f |x-y| \right)$

$\tilde{f} = f$ on $\bar{U}$ and globally Lipschitz.

Take $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \quad \phi \equiv 1$ on $\bar{U} \quad \phi \equiv 0 \quad \mathbb{R}^n \setminus V \quad V \supset\supset U$

$\tilde{f}\phi$ coincide with f on U, bdd and globally Lipschitz.

Embeddings for $W^{k,p}(U)$ $k>1$

Take U open bdd of class C^1 (Case $U=\mathbb{R}^n$, U bdd $W_0^{1,p}(U)$ goes similarly)

$k=2$ ($k>2$ will be done similarly).

$f \in W^{2,p}(U) \iff f \in W^{1,p}(U)$ and $\forall i$ $\frac{\partial f}{\partial x_i} \in W^{1,p}(U)$.

① Case $p=+\infty$ $f, \frac{\partial f}{\partial x_i} \in W^{1,\infty}(U) \to f \in C^{1,1}(U)$

(Lipschitz, with Lipschitz derivatives)

example: E bdd set of class C^1 which satisfies UNIFORM EXTERIOR AND INTERIOR BALL CONDITION

$\to$ then ∂E can be locally written as the graph of a $C^{1,1}$ function.

② $n<p<+\infty$ $\xrightarrow{\text{Morrey}}$ $f, \frac{\partial f}{\partial x_i} \in C^{0,1-\frac{n}{p}}(U) \implies f \in C^{1,1-\frac{n}{p}}(U)$

f Hölder, with Hölder derivatives

③ $p=n$ $\Rightarrow f, \frac{\partial f}{\partial x_i} \in W^{1,n}(U) \hookrightarrow L^q(U) \quad \forall q<+\infty$

$f \in W^{2,n}(U)$

$\Rightarrow f, \frac{\partial f}{\partial x_i} \in L^q(U) \; \forall q<+\infty \Rightarrow f \in W^{1,q}(U)$

$f \in W^{1,q}(U) \quad \forall q<+\infty \Rightarrow f \in C^{0,1-\frac{n}{q}}(U) \quad \forall q<+\infty$

$\Rightarrow f \in C^{0,\alpha}(U) \quad \forall \alpha \in (0,1)$.

(f NOT LIPSCHITZ IN GENERAL!)

④ <u>$p<n$</u> $\quad f, \frac{\partial f}{\partial x_i} \in W^{1,p}(U) \hookrightarrow L^{p^*}(U) \Rightarrow f, \frac{\partial f}{\partial x_i} \in L^{p^*}(U)$

$\rightarrow f \in W^{1,p^*}(U)$.

- $p^* > n \quad (\Leftrightarrow p > \frac{n}{2}) \qquad f \in C^{0,1-\frac{n}{p^*}}(U) = C^{0,2-\frac{n}{p}}(U)$
- $p^* = n \quad (\Leftrightarrow p = \frac{n}{2}) \qquad f \in L^q(U) \quad \forall q<+\infty$
- $p^* < n \quad (\Leftrightarrow p < \frac{n}{2}) \qquad f \in L^q(U) \quad \forall q \le (p^*)^* = \frac{np}{n-2}$

MAKE SENSE for $n>2$

COMPACT EMBEDDINGS

$(W^{1,p}(U), \|\cdot\|_{1,p}) \hookrightarrow (X, \|\cdot\|_X)$ COMPACT if ① $W^{1,p}(U) \subseteq X$,

② $\exists\ C>0\quad \forall f \in W^{1,p}\qquad \|f\|_X \leq C\|f\|_{W^{1,p}(U)}$ (CONTINUOUS EMB)

③ $\forall\ f_k \in W^{1,p}(U)$ bounded, $\|f_k\|_{W^{1,p}} \leq C\ \ \forall k$,
up to passing to a subsequence $\|f_k - f\|_X \to 0$
for some $f \in X$.

OBSERVATION

$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ NEVER COMPACT!

$f \in C_c^\infty(\mathbb{R}^n)\qquad k>0\quad f_k(x) = f(x+ke_1) = f(x_1+k, x_2, \dots, x_n)$

$\|f_k\|_{W^{1,p}} = \|f\|_{W^{1,p}}\ \ \forall k\qquad f_k \to 0$ POINTWISE
$f_k \not\to 0$ in any $L^q(\mathbb{R}^n)$!
$\forall q \in [1, +\infty]$

OBSERVATION for $p > n$ . .

U bdd of class C^1.

$$W^{1,p}(U) \xrightarrow[\text{continuously}]{} C^{0,1-\frac{n}{p}}(U) \xrightarrow[\text{compact}]{\forall \alpha < 1-\frac{n}{p}} C^{0,\alpha}(U)$$

$$\xrightarrow[\text{compact}]{} C(\overline{U}), \|\cdot\|_\infty$$

So $W^{1,p}(U) \xrightarrow{\text{COMPACT}} C^{0,\alpha}(U) \quad \forall \alpha < 1-\frac{n}{p}$

$W^{1,p}(U) \xrightarrow{\text{COMPACT}} C(\overline{U}), \|\cdot\|_\infty$

U bdd

$W^{1,p}_0(U) \xrightarrow{\text{COMPACT}} C^{0,\alpha}(U)$

$\xrightarrow{\text{COMPACT}} C(\overline{U}), \|\cdot\|_\infty$

$p \leq n$ Theorem of RELLICH-KONDRACHOV.

U bounded of class C^1

$p < n$. $\quad W^{1,p}(U) \underset{\text{COMPACTLY}}{\hookrightarrow} L^q(U) \quad \forall q \in [1, p^*)$

p^* EXCLUDED!

for $p = n$ $\quad W^{1,n}(U) \underset{\substack{\text{CONTINUOUS} \\ \forall p < n}}{\hookrightarrow} W^{1,p}(U) \xhookrightarrow{\text{COMPACT}} L^q(U) \quad q < p^*$

$W^{1,n}(U) \underset{\text{COMPACT}}{\hookrightarrow} L^q(U) \quad \forall q < +\infty$

U bounded

$p < n \quad W_0^{1,p}(U) \xhookrightarrow{\text{COMPACT}} L^q(U) \quad \forall q \in [1, p^*)$

$p = n \quad W_0^{1,n}(U) \xhookrightarrow{\text{COMPACT}} L^q(U) \quad \forall q < +\infty$

THE EMBEDDING $W^{1,p}(U) \hookrightarrow L^{p^*}(U)$ is NEVER COMPACT !

$W_0^{1,p}(U) \hookrightarrow L^{p^*}(U)$

Take $U = B(0,1)$ $\quad \eta(x) = \begin{cases} c\, e^{\frac{1}{|x|^2-1}} & |x|<1 \\ 0 & |x|>1 \end{cases}$ $\quad \eta \in C^\infty(\mathbb{R}^n)$

$\int_{\mathbb{R}^n} \eta(x) = 1$

$k \in \mathbb{N}$.

$f_k(x) := k^{\frac{n}{p^*}} \eta(kx) \in C_c^\infty(B(0,1))$ if $k > 1$.

$$\|f_k\|_{L^{p^*}(B)} = \left[k^n \int_B |\eta(kx)|^{p^*} dx \right]^{\frac{1}{p^*}} = \|\eta\|_{L^{p^*}}$$

$$\|f_k\|_{L^p(B)} \underset{\text{HÖLDER}}{\leq} \|f\|_{L^{p^*}(B)} \cdot |B|^{\frac{p^*-p}{p^* p}} \leq \|\eta\|_{L^{p^*}} \cdot (\omega_n)^{\frac{p^*-p}{p^* p}}$$

$$|Df_k(x)|^p = k^{\frac{n}{p^*}p} \cdot k^p |D\eta(kx)|^p$$

$$\frac{n}{p^*} + 1 - \frac{n}{p} = 0$$

$$\Rightarrow \| |Df_k| \|_{L^p} = k^{\frac{n}{p^*}+1} k^{-\frac{n}{p}} \| |D\eta| \|_{L^p} = \| |D\eta| \|_{L^p}$$

$\to \|f_k\|_{W^{1,p}} \le C$ uniformly

$$f_k(x) = k^{n/p^*} e^{\frac{1}{k^2|x|^2-1}} \longrightarrow 0 \quad \text{at all } x \neq 0.$$

But $f_k \not\to 0$ in L^{p^*} since $\|f_k\|_{L^{p^*}} \equiv \|\eta\|_{L^{p^*}}$.

$q < p^*$ $\quad \|f_k\|_q^q = \int_B k^{\frac{n}{p^*}q} |\eta(kx)|^q dx = \int_B k^{\frac{nq}{p^*}} k^{-n} |\eta(y)|^q dy$

$$= k^{n\left(\frac{q}{p^*}-1\right)} \|\eta\|_{L^q}^q \xrightarrow[k\to+\infty]{} 0 \qquad \text{since } n\left(\frac{q}{p^*}-1\right) < 0$$